group theory - week 13

Flavor SU(3)

Georgia Tech PHYS-7143

Homework HW13 due Tuesday, April 19, 2016

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 13.1 *Root systems of simple Lie algebras* 5 points
Exercise 13.2 *Meson octet* 5 points

Bonus points
Exercise 13.3 *Gell-Mann–Okubo mass formula* 8 points
Exercise 13.4 $SU(3)$ *symmetry in 3D Harmonic Oscillator* 5 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

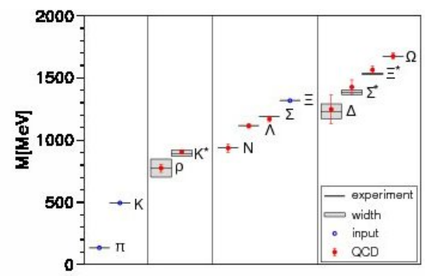

Figure 13.1: A lattice gauge theory calculation of the light QCD spectrum. Horizontal lines and bands are the experimental values with their decay widths. The π, K and and Ξ have no error bars because they are used to set the light and strange quark masses and the overall scale respectively. From Scholarpedia.

2016-04-12 Boris Lecture 25 Cartan construction of SU(3) irreps

Gutkin notes, Lect. 10 *Representations of simple algebras, general construction. Application to SU(3)*, Sect. 5.

2016-04-14 Predrag Lecture 26 Flavor SU(3)

Gutkin notes, Lect. 11 *Strong interactions: flavor SU(3)*. Heisenberg isospin SU(2). Gell-Mann flavor SU(3). Gell-Mann-Okubo mass formula.

13.1 Literature

The Gell-Mann-Okubo mass sum rules [1–3] are an easy consequence of the approximate SU(3) flavor symmetry. Determination of quark masses is much harder - they are parameters of the standard model, determined by optimizing the spectrum of particle masses obtained by lattice QCD calculations as compared to the experimental baryon and meson masses. The best determination of the mass spectrum as of 2012 is given in figure 13.1. Up, down quarks are about 3 and 6 MeV, respectively, with strange quark mass about 100 MeV, all with large error brackets.

References

[1] M. Gell-Mann, *The eightfold way: a theory of strong interaction symmetry*, Synchrotron Laboratory Report CTSL-20 (CalTech, 1961).

[2] M. Gell-Mann, "Symmetries of baryons and mesons", Phys. Rev. **125**, 1067–1084 (1962).

[3] S. Okubo, "Note on unitary symmetry in strong interactions", Prog. Theor. Phys. **27**, 949–966 (1962).

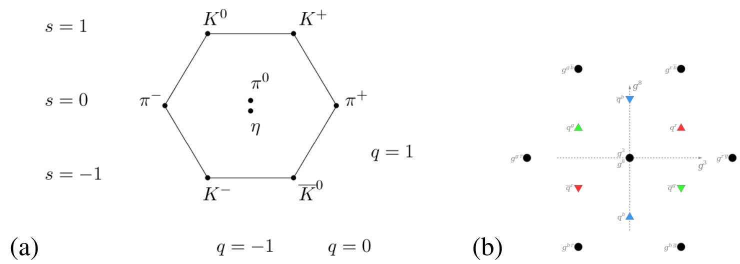

Figure 13.2: (a) The meson (pseudoscalars) octet. (b) The quark triplet, the anti-quark triplet and the gluon octet. (Wikipedia).

Exercises

13.1. **Root system of simple Lie algebras.**

a) Determine dimensions of Lie algebras $\mathfrak{so}(N), \mathfrak{su}(N)$ and dimensions of their Cartan subalgebras. What is the number of the positive roots for these Lie algebras?

b) Show that $N \times N$ diagonal matrices H_i with zero traces and uper/lower corner $N \times N$ matrices $E^{(a,b)}$ with the elements $E^{(a,b)}_{i,j} = \delta_{ia}\delta_{ib}$ provide Cartan-Weyl basis of $\mathfrak{su}(N)$. To put it differently, show that $E^{(a,b)}$ are eigenstates for adjoint representation of H_i's.

(B. Gutkin)

13.2. **Meson octet.** In Gutkin lecture notes, Lect. 11 *Strong interactions: flavor SU(3)*, the meson octet, figure 13.2 (a)

$$\begin{aligned}
\Phi &= \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K^0} & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ & 0 \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & K^+ \\ 0 & 0 & K^0 \\ K^- & \overline{K^0} & 0 \end{pmatrix} + \frac{\eta}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad (13.1)
\end{aligned}$$

is interpreted as arising from the adjoint representation of SU(3), i.e., the traceless part of the quark-antiquark $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$ outer product (see figure 13.2 (b)),

$$\begin{pmatrix} u\overline{u} & u\overline{d} & u\overline{s} \\ d\overline{u} & d\overline{d} & d\overline{s} \\ s\overline{u} & s\overline{d} & s\overline{s} \end{pmatrix} . \qquad (13.2)$$

where we have replaced in (13.1) the constituent $q \otimes \overline{q}$ combinations by the names of the elementary particles they build.

Given the quark quantum numbers

	Q	I	I_3	Y	B
u	2/3	1/2	1/2	1/3	1/3
d	-1/3	1/2	-1/2	1/3	1/3
s	-1/3	0	0	-2/3	1/3

verify the strangeness and charge assignments of figure 13.2 (a).

13.3. **Gell-Mann–Okubo mass formula.** The mass symmetry-breaking interaction for an isospin multiplet is proportional to the 3rd component of the isospin operator, I_3. Similarly, the symmetry-breaking interaction of SU(3) for the meson octet is given by the 8th component of the octet operator $Y = \lambda_8$. Derive the GMO mass formula for mesons

$$m_\eta^2 = \frac{4\,m_K^2 - m_\pi^2}{3}\,. \tag{13.3}$$

by eliminating the parameter for the strength of this interaction, as in Gutkin lecture notes, Lect. 11 *Strong interactions: flavor SU(3)* .

13.4. $SU(3)$ **symmetry in 3D Harmonic Oscillator.** The Hamiltonian of 3D isotropic harmonic oscillator is given by

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + \frac{m\omega^2}{2} x_i^2 = \hbar\omega \sum_{i=1}^{3} (a_i^\dagger a_i + 1/2),$$

where $a_i = \sqrt{\frac{m\omega}{2\hbar}} x_i + i\sqrt{\frac{1}{2m\omega\hbar}} p_i$ is creation ($a_i^\dagger$ resp. annihilation) operator satisfying $[a_i, a_j^\dagger] = \delta_{i,j}$, $[a_i, a_j] = 0$.

a) Show that $a_i \to U_{i,j} a_j$, with $U \in U(3)$ is a symmetry of the Hamiltonian. In other words isotropic 3D harmonic oscillator has $U(3)$ rather than $O(3)$ symmetry!

b) Calculate degeneracy of the n-th level $E_n = \omega\hbar(n + 3/2)$ of the oscillator.

c) By comparison of dimensions find out which representations of $SU(3)$ appear in the spectrum of harmonic oscillator.

(B. Gutkin)