group theory - week 11

SU(2) and SO(3)

Georgia Tech PHYS-7143

Homework HW11 due Thursday, April 7, 2016

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 11.1 *Lie algebra of* $SO(4)$ *and* $SU(2) \otimes SU(2)$ 6 points
Exercise 11.2 *Real and pseudo-real representations of* $SO(3)$ 4 points

Bonus points
Exercise 11.3 *Total spin of* N *particles* 5 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2016-03-29 Predrag Lecture 21 SU(2) and SO(3)

Gutkin notes, Lect. 9 *SU(2), SO(3) and their representations*, Sects. 1-3.1

2016-10-31 Boris Lecture 22 Spinors, Clebsches

Gutkin notes, Lect. 9 *SU(2), SO(3) and their representations*, Sects. 3.2-4

Exercises

11.1. **Lie algebra of SO(4) and SU$(2) \otimes$ SU(2).** One particle Hamiltonian with a central potential has in general SO(3) symmetry group. It turns out, however, that for Coulomb potential the symmetry group is actually larger - SO(4), rather than SO(3). This explains why the energy level degeneracies in the hydrogen atom are anomalously large. So SO(4) and its representations are of a special importance in atomic physics.

(a) Show that the Lie algebra $\mathfrak{so}(4)$ of the group SO(4) is generated by real antisymmetric 4×4 matrices.

(b) What is the dimension of $\mathfrak{so}(4)$?

A natural basis in $\mathfrak{so}(4)$ is provided by antisymmetric matrices $M_{\mu\nu}, \quad \mu, \nu \in 1, 2, 3, 4$, $\mu \neq \nu$, generators of SO(4) rotations which leave invariant the $\mu\nu$-plane. The elements of these matrices are given by

$$(M_{\mu\nu})_{ij} = \delta_{i\mu}\delta_{j\nu} - \delta_{j\mu}\delta_{i\nu} \tag{11.1}$$

(c) Check that these matrices satisfy the following commutation relationship:

$$[M_{ab}, M_{cd}] = M_{ad}\delta_{bc} + M_{bc}\delta_{ad} - M_{ac}\delta_{bd} - M_{bd}\delta_{ac}.$$

(d) Show that Lie algebras of the groups SO(4) and SU$(2) \times$ SU(2) are isomorphic. **Path:**

(d.i) Define matrices

$$J_k = \frac{1}{2}\varepsilon_{kij} M_{i,j}, \qquad K_k = M_{k4}, \quad k = 1, 2, 3$$

and

$$\mathcal{A}_k = \frac{1}{2}(J_k + K_k) \quad \text{and} \quad \mathcal{B}_k = \frac{1}{2}(J_k - K_k)\ .$$

(d.ii) Show that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same commutation relations as two copies of $\mathfrak{su}(2)$.

(e) How does one construct irreps of $\mathfrak{so}(4)$ out of irreps of $\mathfrak{su}(2)$?

(f) Are groups SO(4) and SU$(2) \otimes$ SU(2) isomorphic to each other?

(B. Gutkin)

11.2. **Real and pseudo-real representations of $SO(3)$.** Recall (Gutkin notes, Lect. 4 *Representation Theory II*, Sect. 5 *5. Three types of representations*) that there are exist three types of representation which can be distinguished by the indicator:

$$\int_G d\mu(g)\chi_l(g^2) = \begin{cases} +1 & \text{real} \\ 0 & \text{complex} \\ -1 & \text{pseudo-real} \end{cases} \tag{11.2}$$

Determine for which values of $l = 0, 1/2, 1, 3/2, 2\dots$ the representation D_l of $SO(3)$ is real or pseudo-real.

Reminder: The characters and Haar measure of $SO(3)$ are given by

$$\chi_l(g) = \frac{\sin\left(\left[l+\frac{1}{2}\right]\varphi\right)}{\sin\left(\frac{1}{2}\varphi\right)}, \qquad d\mu(g) = \frac{1}{\pi}\sin^2\varphi d\varphi \tag{11.3}$$

where φ is rotation angle for the group element g.

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11.3. **Total spin of N particles.** Consider a system of four particles with spin $1/2$. Assuming that all (except spin) degrees of freedom are frozen the Hilbert space of the system is given by $V = V_{1/2} \otimes V_{1/2} \otimes V_{1/2} \otimes V_{1/2}$, with $V_{1/2}$ being two-dimensional space for each spin. $V = \oplus V_s$ can be decomposed then into different sectors V_s having the total spin s i.e., $\hat{S}^2 v = s(s+1)v$, for any $v \in V_s$. Here $\hat{S}^2 = (\sum_{i=1}^4 \hat{s}_i)^2$ and $\hat{s}_i = (\hat{s}_i^x, \hat{s}_i^y, \hat{s}_i^z)$ is spin operator for i-th particle.

(a) What are possible values s for the total spin of the system?

(b) Determine dimension of the subspace of V_0 with 0 total spin. In other words: how many times trivial representation enters into product:

$$D = D_{1/2} \otimes D_{1/2} \otimes D_{1/2} \otimes D_{1/2}\,? \tag{11.4}$$

(c) What is the answer to the above questions for N spins?

Hint: it is convenient to use (11.3) to decompose D into irreps.

(B. Gutkin)