group theory - week 9

Continuous groups

Georgia Tech PHYS-7143

Homework HW9 due Tuesday 2021-06-29

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 9.1 *Irreps of* $SO(2)$	2 points
Exercise 9.2 *Reduction of product of two* $SO(2)$ *irreps*	1 point
Exercise 9.3 *Irreps of* $O(2)$	2 points
Exercise 9.4 *Reduction of product of two* $O(2)$ *irreps*	1 point

Bonus points

Exercise 9.5 *A fluttering flame front*	4 points
Exercise 9.6 $O(2)$ *fundamental domain for a PDE*	(difficult) 10 points

Total of 6 points = 100 % score.

2021-06-22 Predrag Lecture 17 Continuous groups

2021-06-22 Predrag Lecture 18 Lie groups

The fastest way to watch any week's lecture videos is by letting YouTube run the

lecture playlist

These lectures are about the basic ideas of how one goes from finite groups to the continuous ones. We have worked one example out in week 2, the discrete Fourier transform of example 2.6 *Projection operators for cyclic group C_N*. The cyclic group C_N is generated by the powers of the rotation by $2\pi/N$, and in the $N \to \infty$ limit one only needs to understand the algebra of T_ℓ, generators of infinitesimal transformations, $D(\theta) = 1 + i\sum_\ell \theta_\ell T_\ell$. Applied to functions, they turn out to be partial derivatives.

Continuous symmetries - an introduction (2 min)

They still do not get it! (6 min)

- Lie groups, sect. 9.3: Definition of a Lie group; Cyclic group $C_N \to$ continuous $SO(2)$ plane rotations; Infinitesimal transformations; $SO(2)$ generator of rotations.

What is a symmetry? (8 min)

Group element; transformation generator (8 min)

What is a symmetry group? (7 min)

What is a group orbit? (3 min)

What is dynamics? (2 min)

Group SO(2) (3 min)

Unitary groups are mothers of all finite / compact symmetries.
(1 h 4 min)

- The $N \to \infty$ limit of C_N gets you to the continuous Fourier transform as a representation of $SO(2)$, but from then on this way of thinking about continuous symmetries gets to be increasingly awkward. A fresh restart is afforded by matrix groups, and in particular the unitary group $U(n) = U(1) \otimes SU(n)$, which contains all other compact groups, finite or continuous, as subgroups.

Special orthogonal group SO(n) (9 min)

Symplectic group Sp(n) (9 min)

9.1 Other sources (optional)

Do not get intimidated by this week's lectures notes.

- What's the payback? While for you the geometrically intuitive representation is the set of rotation [2×2] matrices, group theory says no! They split into pairs of 1-dimensional irreps, and the basic building blocks of *our* 2-dimensional rotations on our kitchen table (forget quantum mechanics!) are the $\mathrm{U}(1)$ $[1 \times 1]$ complex unit vector phase rotations.

  - Reading: C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, Sects. 6.1-6.3 Irreps of $\mathrm{SO}(2)$. In particular, note that while geometrically intuitive representation is the set of rotation $[2 \times 2]$ matrices, they split into pairs of 1-dimensional irreps.
  - Reading: C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, Sect. 6.6 completes discussion of Fourier analysis as continuum limit of cyclic groups C_n, compares $\mathrm{SO}(2)$, $\mathrm{O}(2)$, discrete translations group, and continuous translations group.
  - Chen, Ping and Wang [2] *Group Representation Theory for Physicists*, Sect 5.2 *Definition of a Lie group, with examples* (click here).
  - AWH Chapter 17 *Group Theory, Sect. 17.7 Continuous groups* (click here).

- Sect. 9.4 *Character orthogonality theorem*

- *Infinitesimal symmetries: Lie derivative* (8 min)
- *Tell no Lie to plumbers* (39 sec)
- *It's a matter of no small pride for a card-carrying dirt physics theorist to claim* full and total ignorance of group theory (XX min)

9.2 Discussion (optional)

- *Calligraphic M denotes the state space manifold as well as any subspace, such as a group orbit* (3:38 min)
- *Why are continuous transformation group elements represented by exponentials?* (5:39 min)
- *How did we get the Lie algebra? Why is (almost) every symmetry we care about a subgroup of an unitary group?* (9 min)
- *How did we get the SO(2) generator?* (2 min)
- *Orthogonal and unitary transformations* (7 min)
- *Fourier modes are so simple, that no one calls them irreps. But add more symmetries, and there have to be fewer irreps.* (11 min)

- *Why did we move from orthogonal group O(n) to special orthogonal group SO(n)?* (3:32 min)
- *Why SU(n) rather than U(n)?* (6:30 min)
- *Why is SU(n) dimension $n^2 - 1$?* (56 sec)
- *What are these "characters"? And why is there a* Journal of Linear Algebra*, today? Inconclusive blah blah.* (12 min)
- *Rant 1 - Is beauty symmetry? The first piece of art found in China is a perfect disk carved out of jade. All of Bach is symmetries.* (9 min)
- *Rant 2 - students find letter A beautifully symmetric, but Predrag finds zero 'O' the most beautiful grade.* (1 min)
- *Rant 3 - SO(3) & SU(2) preview and a long rant - listen to it at your own risk. Roger Penrose thoughts on quantum spacetime and quantum brain. Are laws of physics time invariant? Waiting for dark energy to go away. Arrow of time.* (17 min)
- *Rant 4 - SO(3) & SU(2) preview and a long rant - listen to it at your own risk. Get this: math uses 2d complex vectors (spinors) to build our real 3d space. And all we see - starlight, graphene, greenhouse effect, helioseismography, gravitational wave detectors - it is all irreps!* (12 min)
- *Rant 5 - Help me, I'm bullied by a mathematician.* (3 min)
- *Rant 6 - you can always count on Prof. Z.* (1/2 min)

Question 9.1. Henriette Roux, pondering exercise 4.2, writes

Q I want to make sure I understand the concept of irreducible representations.

1. If a representation (which can be thought of as a sort of basis) is reducible, all group element matrices can be simultaneously diagonalized. I want to be able to see how this definition of reducibility matches with the notion of block diagonalizability of an overall representation D(g).
2. AWH p. 822-823 has a discussion of this, but I'm wondering if there's an intuitive way to connect these two definitions or if it's just linear algebra.
3. We have familiarized ourselves with the concept of (conjugacy) classes. Here, we now add in the concept of character, which is just the trace of any matrix in a given class (and every matrix of the same class will have the same trace b/c of the properties of classes/traces).
4. So to find the characters for a given representation, we just need to find the classes and then take the trace of a matrix representation in each class?
5. My next and related question then concerns what character means conceptually. Does it relate classes to other classes within a given representation, or different representations (whether reducible or not), or both? AWH says that "the set of characters for all elements and irreducible representations of a finite group defines an orthogonal finite-dimensional vector space."

6. How does a vector space come about from a set of traces, each of which I normally think of as just a number, like the determinant? And finally,
7. How can we use our knowledge of classes/character to find irreducible representations, since that seems to be an important goal in examining a group.
8. Exercise 4.2 (c) says to find the characters for this representation, which seems to imply that character depends on representation. But I would've thought that character, which is a trace of a matrix, is invariant under any similarity transform, which is how you get from a reducible representation to an irreducible representation.
9. Do the multiplicities of irreducible representations correspond to the multiplicity of characters (i.e. the number of elements in each class)? If so, why? (Or if not, why not?)
10. The same thing for classes, correct? Classes shouldn't depend on representation b/c they can be thought of as corresponding to a physical operation (e.g. transposition or cyclic permutation), something which is independent of basis.

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A Great framing for a discussion, thanks! I'll probably reedit this post several times, everybody's input is very welcome. Items numbered as in above:

(2) My favorite step-by-step, pedagogical exposition are the chapters 2 *Representation Theory and Basic Theorems* and 3 *Character of a Representation* of Dresselhaus *et al.* [4]. There is too much material for our course, but if you want to understand it once for all times, it's worth your time.

(3) Correct.

(4) Correct. Note, however, that while every matrix representation has a trace, and thus a character, you want to decompose this character into the sum of irrep characters, as it is obvious after the block diagonalization has been attained.

(5) The unitary diagonalization matrix, whose entries are characters, takes character-weighted sums of classes in order to project them onto irreps, just like what the Fourier representation does. The result (as we know from projection operators analysis), are mutually orthogonal sub-spaces.

(6) Whenever you do not understand something about finite groups, ask yourself - how does it work for finite lattice Fourier representation?

There the vector space comes via a unitary transformation from the configuration coordinates (where each group element is represented by a full matrix) to the diagonalized, irreducible subspaces coordinates (Fourier modes).

The unitary $\mathcal{F}$ matrix is full of ω^{ij}, ie, characters of the cyclic group C_n. That's where the characters come from.

Now mess up C_3 by adding a reflection. Dihedral group D_3, the group of rotations and reflections, has more symmetry constrains, it cannot have 6 irreps, as reflection invariance mixes together the two senses of rotation. Now there are 3 classes, ie, kinds of things the group does: nothing, flip, rotate. The unitary transformation that diagonalizes group element matrices is now morally a smaller unitary $[3 \times 3]$ matrix from 'classes' in configuration space to 'irreps' in the diagonalized representation, where some sub-spaces must have dimension higher than one.

The surprise, for me, is that the entries in the unitary diagonalization matrix can still be written as traces of irreps, ie, characters. For me it is a calculation, a beautiful example of mathematics leading us somewhere where our intuition falls short. If you find a good intuitive explanation somewhere, please let us all know.

(7) That's automatic, now. Each irrep has the projection operator associated with it; we construct it as a sub-product of factors in Hamilton-Cayley formula. Now we know we can write it -just as we did with the Fourier representation- as sum over all class group actions, each weighted by a the irrep's character.

(8) Characters are elements of the unitary matrix with one index running over classes, the other over irreps. So you expect character to differ from representation to representation; very clear from D_3 character table. As always, you already know that from the Fourier representation example.

(9) They do not. Dresselhaus *et al.* [4] has the answer - enter it here once you understand it.

(10) Correct.

Question 9.2. Henriette Roux, digesting sect. 10.7.1, asks

Q Please explain when one keeps track of the order of tensorial indices?

A In a tensor, upper, lower indices are separately ordered - and that order matters. The simplest example: if some indices form an antisymmetric pair, writing them in wrong order gives you a wrong sign. In a matrix representation of a group action, one has to distinguish between the "in" set of indices – the ones that get contracted with the initial tensor, and the "out" set of indices that label the tensor after the transformation. If you understand Eq. (3.22) in birdtracks.eu, you get it. Does that answer your question?

Question 9.3. Henriette Roux asks

Q Please explain the $M_{\mu\nu,\delta\rho}$ generators of $\mathrm{SO}(n)$.

A Let me know if you understand the derivation of Eqs. (4.51) and (4.52) in birdtracks.eu. Does that answer your question?

9.3 Continuous symmetries: unitary and orthogonal

This week's lectures are not taken from any particular book, they are about basic ideas of how one goes from finite groups to the continuous ones that any physicist should know. We have worked one example out earlier, in week 9 and ChaosBook Sect. A24.4. It gets you to the continuous Fourier transform as a representation of $\mathrm{U}(1) \simeq \mathrm{SO}(2)$, but from then on this way of thinking about continuous symmetries gets to be increasingly awkward. So we need a fresh restart; that is afforded by matrix groups, and in particular the unitary group $\mathrm{U}(n) = \mathrm{U}(1) \otimes \mathrm{SU}(n)$, which contains all other compact groups, finite or continuous, as subgroups.

The main idea in a way comes from discrete groups: the cyclic group C_N is generated by the powers of the smallest rotation by $\Delta\theta = 2\pi/N$, and in the $N \to \infty$ limit one only needs to understand the commutation relations among T_ℓ, generators of infinitesimal transformations,

$$D(\Delta\theta) = 1 + i \sum_\ell \Delta\theta_\ell T_\ell + O(\Delta\theta^2)\,. \qquad (9.1)$$

These thoughts are spread over chapters of my book *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [3] that you can steal from my website, but the book itself is too sophisticated for this course.

9.4 Character orthogonality theorem

You might like my intuitive derivation [3] of the character orthogonality theorem for continuous compact lie groups, birdtracks.eu sect. 8.2 *Characters*.

Note that the replacement of an irrep matrix representation $D^{(\mu)}(g)_a{}^b$ by its character $\chi^{(\mu)}(g)$ (a single scalar quantity) does not mean that any of the matrix indices structure is lost; the full $D^{(\mu)}(g)_a{}^b$ can be recovered by differentiation, as in birdtracks.eu eq. (8.27).

References

[1] N. B. Budanur and P. Cvitanović, "Unstable manifolds of relative periodic orbits in the symmetry-reduced state space of the Kuramoto-Sivashinsky system", J. Stat. Phys. **167**, 636–655 (2015).

[2] J.-Q. Chen, J. Ping, and F. Wang, *Group Representation Theory for Physicists* (World Scientific, Singapore, 1989).

[3] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2008).

[4] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

Exercises

9.1. **Irreps of SO(2).** Matrix

$$T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{9.2}$$

is the generator of rotations in a plane.

(a) Use the method of projection operators to show that for rotations in the kth Fourier mode plane, the irreducible $1D$ subspaces orthonormal basis vectors are

$$\mathbf{e}^{(\pm k)} = \frac{1}{\sqrt{2}} \left(\pm \mathbf{e}_1^{(k)} - i \, \mathbf{e}_2^{(k)} \right) .$$

How does T act on $\mathbf{e}^{(\pm k)}$?

(b) What is the action of the $[2\times 2]$ rotation matrix

$$D^{(k)}(\theta) = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} , \qquad k = 1, 2, \cdots$$

on the $(\pm k)$th subspace $\mathbf{e}^{(\pm k)}$?

(c) What are the irreducible representations characters of SO(2)?

9.2. **Reduction of a product of two SO(2) irreps.** Determine the Clebsch-Gordan series for SO(2). Hint: Abelian group has 1-dimensional characters. Or, you are just multiplying terms in Fourier series.

9.3. **Irreps of O(2).** O(2) is a group, but not a Lie group, as in addition to continuous transformations generated by (9.2) it has, as a group element, a parity operation

$$\sigma = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right]$$

which cannot be reached by continuous transformations.

(a) Is this group Abelian, i.e., does T commute with $R(k\theta)$? Hint: evaluate first the $[T, \sigma]$ commutator and/or show that $\sigma D^{(k)}(\theta)\sigma^{-1} = D^{(k)}(-\theta)$.

(b) What are the equivalence (i.e., conjugacy) classes of this group?

(c) What are irreps of O(2)? What are their dimensions?
Hint: O(2) is the $n \to \infty$ limit of D_n, worked out in exercise 4.4 *Irreducible representations of dihedral group D_n*. Parity σ maps an SO(2) eigenvector into another eigenvector, rendering eigenvalues of any O(2) commuting operator degenerate. Or, if you really want to do it right, apply Schur's first lemma to improper rotations

$$R^{'}(\theta) = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} \sigma = \begin{pmatrix} \cos k\theta & \sin k\theta \\ \sin k\theta & -\cos k\theta \end{pmatrix}$$

to prove irreducibility for $k \neq 0$.

(d) What are irreducible characters of O(2)?

(e) Sketch a fundamental domain for O(2).

9.4. **Reduction of a product of two O(2) irreps.** Determine the Clebsch-Gordan series for O(2), i.e., reduce the Kronecker product $D^{(k)} \bigotimes D^{(\ell)}$.

9.5. **A fluttering flame front.**

(a) Consider a linear partial differential equation for a real-valued field $u = u(x,t)$ defined on a periodic domain $u(x,t) = u(x+L,t)$:

$$u_t + u_{xx} + \nu u_{xxxx} = 0\,, \qquad x \in [0, L]\,. \tag{9.3}$$

In this equation $t \geq 0$ is the time and x is the spatial coordinate. The subscripts x and t denote partial derivatives with respect to x and t: $u_t = \partial u/d\partial$, u_{xxxx} stands for the 4th spatial derivative of $u = u(x,t)$ at position x and time t. Consider the form of equations under coordinate shifts $x \to x + \ell$ and reflection $x \to -x$. What is the symmetry group of (9.3)?

(b) Expand $u(x,t)$ in terms of its SO(2) irreducible components (hint: Fourier expansion) and rewrite (9.3) as a set of linear ODEs for the expansion coefficients. What are the eigenvalues of the time evolution operator? What is their degeneracy?

(c) Expand $u(x,t)$ in terms of its O(2) irreducible components (hint: Fourier expansion) and rewrite (9.3) as a set of linear ODEs. What are the eigenvalues of the time evolution operator? What is their degeneracy?

(d) Interpret $u = u(x,t)$ as a 'flame front velocity' and add a quadratic nonlinearity to (9.3),

$$u_t + \tfrac{1}{2}(u^2)_x + u_{xx} + \nu u_{xxxx} = 0\,, \qquad x \in [0, L]\,. \tag{9.4}$$

This nonlinear equation is known as the Kuramoto-Sivashinsky equation, a baby cousin of Navier-Stokes. What is the symmetry group of (9.4)?

(e) Expand $u(x,t)$ in terms of its $\mathrm{O}(2)$ irreducible components (see exercise 9.3) and rewrite (9.4) as an infinite tower of coupled nonlinear ODEs.

(f) What are the degeneracies of the spectrum of the eigenvalues of the time evolution operator?

9.6. **O(2) fundamental domain for Kuramoto-Sivashinsky equation.** You have C_2 discrete symmetry generated by flip σ, which tiles the space by two tiles.

- Is there a subspace invariant under this C_2? What form does the tower of ODEs take in this subspace?
- How would you restrict the flow (the integration of the tower of coupled ODEs) to a fundamental domain?

This problem is indeed hard, a research level problem, at least for me and the grad students in our group. Unlike the beautiful full-reducibility, character-orthogonality representation theory of linear problems, in nonlinear problems symmetry reduction currently seems to require lots of clever steps and choices of particular coordinates, and we am not at all sure that our solution is the optimal one. Somebody looking at the problem with a fresh eye might hit upon a solution much simpler than ours. Has happened before :)

Burak Budanur's solution is written up in Budanur and Cvitanović [1] *Unstable manifolds of relative periodic orbits in the symmetry-reduced state space of the Kuramoto-Sivashinsky system* sect. 3.2 *$O(2)$ symmetry reduction*, eq. (17) (get it here).

9.7. **Lie algebra from invariance.** Derive the Lie algebra commutator and the Jacobi identity as particular examples of the invariance condition, using both index and birdtracks notations. The invariant tensors in question are "the laws of motion," i.e., the generators of infinitesimal group transformations in the defining and the adjoint representations.