group theory - week 6

For fundamentalists

Georgia Tech PHYS-7143

Homework HW6

due Tuesday 2021-06-22

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 6.1 *3-disk symbolic dynamics*	2 bonus points
Exercise 6.2 *Reduction of 3-disk symbolic dynamics to binary*	3 bonus points
Exercise 6.3 *3-disk fundamental domain cycles*	2 bonus points
Exercise 6.4 *C_2-equivariance of Lorenz system*	3 points
Exercise 6.5 *Proto-Lorenz system* parts 1.-5.	7 points
Exercise 6.5 *Proto-Lorenz system* parts 6.-8.	6 bonus points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2021-06-08 Predrag Lecture 11 True grit

Lecture 7 (Unedited)

- We work out the symmetry reduction and a breaking of the D_3 symmetry in the $[3 \times 3]$ permutation matrices representation
- A preview; reduction of a cyclic C_N symmetric molecule chain to a $[2 \times 2]$ frequency matrix calculation. It will take a couple of weeks of discrete Fourier lectures to fill in the details. (2:07:51 h)

So far we have covered what any QM fixated Group Theory textbook since 1930's and on covers. Today to turn to what we actually use group theory for *today*, *here*, in Howey, and for that there is no book but ChaosBook.org.

Many fundamental problems of fluid dynamics and more generally non-linear field theories are studied in experimental settings equipped with symmetries. That is the subject of *dynamical systems theory* (of which classical, quantum and stochastic mechanics and field theories are but specialized branches). We start gently, with the famed Lorenz butterfly.

2021-06-10 Predrag Lecture 12 Nonlinear symmetry reduction

Lecture 8 (Unedited) Symmetries and nonlinear systems. First a mini-course on nonlinear dynamics and chaos. I use the symmetry reduction on the Lorenz, and turn it into Van Gogh. Conclusion: can use either a fundamental domain, or invariant polynomial bases to reduce symmetries of a nonlinear system. (2:28:00 h)

- Lorenz flow example. Read ChaosBook Chapter 11. World in a mirror ChaosBook.org Chapter 11 *World in a mirror*. Maybe start with ChaosBook example 10.6 *Equivariance of the Lorenz flow*, example 11.8 *Desymmetrization of Lorenz flow*, and then work your way back if needed.
- example 6.1 *Equivariance of the Lorenz flow*
- example 6.2 *Desymmetrization of Lorenz flow*

The reading and the homework for this week, is augmented by - if you find that helpful - by 'live' online blackboard lectures: click here.

6.1 Other sources (optional)

- *An example: a 1-dimensional system with a symmetry*
- *Fundamental domain*
- *Tiling of state space by a finite group*
- *Make the "fundamental tile" your hood*
- *Symmetry-reduced dynamics*
- *Regular representation of permuting tiles*

6.2 Thoughts (optional)

How I think of the fundamental domain is explained in my online lectures, Week 14, in particular the snippet *Regular representation of permuting tiles*.

Unfortunately - if I had more time, that would have been shorter, this goes on and on, Week 15, lecture *29. Discrete symmetry factorization*, and by the time the dust settles, I do not have a gut feeling for the boundary conditions when it comes to higher-dimensional irreps (see also last week's sect. 6.2 *Discussion*).

The basic insight is that if the symmetry and dynamics commute, one can implement the stratification of the state space by the symmetry first, paying no heed to the dynamics. In arbitrary coordinates, the state space is a stratified by a jumble of group orbits. It is an 'orbitfold', in the sense that in general it contains subspaces on which group orbits are of the dimension of a symmetry subgroup, with the group action on invariant subspaces trivial, and on which group orbits are points.

On the linear level, the natural stratification is implemented by decomposing the state space into irreps of the symmetry group. This is a linear reshuffling of coordinates that makes the action of the symmetry operators as simple as possible. You can think of the new basis vectors as eigenvectors of the symmetry operators (Fourier modes, spherical eigenfunctions, etc.). The nonlinear terms in dynamical equations jumble everything up. They are re-expressed in this basis using Kronecker-product decompositions into sums over products of irreps.

Unfortunately –if I had more time, that would have been shorter– this goes on and on, ChaosBook course 2, Week 15, lecture *29. Discrete symmetry factorization*.

Henriette Roux What do the parameters σ, ρ and b stand for in the Lorenz equations (6.4)?

Predrag The short answer is the truncation of the Navier-Stokes that leads to Lorenz equations is so drastic that they have no longer any physical meaning; in his 1963 paper [14] Lorenz played with the parameters until he empirically found an interesting example of deterministic chaos. Since then, applied mathematicians have reverse-engineered various physical systems to find situations where parameters σ, ρ and b mean something, see remark 6.1 (copied to here from ChaosBook.org). The discrete symmetry of the original Navier-Stokes system ('left' is as good as 'right') happened to survive the drastic truncation from 10^5 Fourier modes (for physically accurate simulations) to 3. I prefer to teach nonlinear dynamics using the Rössler system, precisely because it has no discrete symmetry, just chaos.

6.3 ChaosBook notes

Copied here are a few snippets from this week's lecture notes, needed here just because exercises refer to them - read the full lecture notes instead.

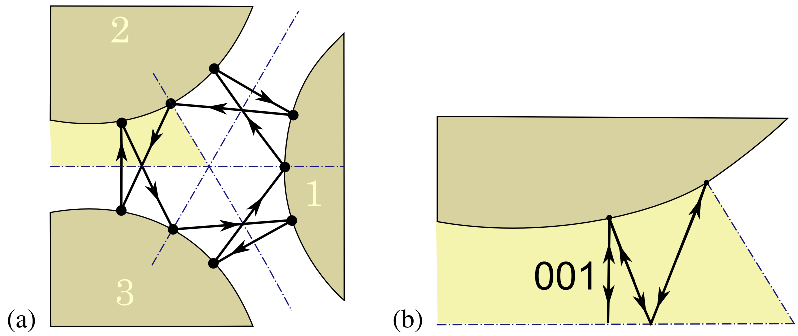

Figure 6.1: (a) The pair of full-space 9-cycles, the counter-clockwise $\overline{121232313}$ and the clockwise $\overline{131323212}$ correspond to (b) one fundamental domain 3-cycle $\overline{001}$.

Definition: Flow invariant subspace. A typical point in fixed-point subspace $\mathcal{M}_H$ moves with time, but, due to equivariance

$$f(gx) = gf(x)\,, \tag{6.1}$$

its trajectory $x(t) = f^t(x)$ remains within $f(\mathcal{M}_H) \subseteq \mathcal{M}_H$ for all times,

$$hf^t(x) = f^t(hx) = f^t(x)\,, \quad h \in H\,, \tag{6.2}$$

i.e., it belongs to a *flow invariant subspace*. This suggests a systematic approach to seeking compact invariant solutions. The larger the symmetry subgroup, the smaller $\mathcal{M}_H$, easing the numerical searches, so start with the largest subgroups H first.

We can often decompose the state space into smaller subspaces, with group acting within each 'chunk' separately:

Definition: Invariant subspace. $\mathcal{M}_\alpha \subset \mathcal{M}$ is an *invariant* subspace if

$$\{\mathcal{M}_\alpha \mid gx \in \mathcal{M}_\alpha \text{ for all } g \in G \text{ and } x \in \mathcal{M}_\alpha\}\,. \tag{6.3}$$

$\{0\}$ and $\mathcal{M}$ are always invariant subspaces. So is any $\mathrm{Fix}\,(H)$ which is point-wise invariant under action of G.

Definition: Irreducible subspace. A space $\mathcal{M}_\alpha$ whose only invariant subspaces under the action of G are $\{0\}$ and $\mathcal{M}_\alpha$ is called *irreducible*.

Example 6.1. ***Equivariance of the Lorenz flow.*** *The velocity field in Lorenz equations [14]*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y-x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ -xz \\ xy \end{bmatrix} \tag{6.4}$$

is equivariant under the action of cyclic group $C_2 = \{e, C^{1/2}\}$ acting on $\mathbb{R}^3$ by a π rotation about the z axis,

$$C^{1/2}(x, y, z) = (-x, -y, z)\,. \tag{6.5}$$

Table 6.1: D_3 correspondence between the binary labeled fundamental domain prime cycles $\tilde{p}$ and the full 3-disk ternary labeled cycles p, together with the D_3 transformation that maps the end point of the $\tilde{p}$ cycle into the irreducible segment of the p cycle. White spaces in the above ternary sequences mark repeats of the irreducible segment; for example, the full space 12-cycle 1212 3131 2323 consists of 1212 and its symmetry related segments 3131, 2323. The multiplicity of p cycle is $m_p = 6n_{\tilde{p}}/n_p$. The shortest pair of fundamental domain cycles related by time reversal (but no spatial symmetry) are the 6-cycles $\overline{001011}$ and $\overline{001101}$.

$\tilde{p}$	p	$\mathbf{g}_{\tilde{p}}$	$\tilde{p}$	p	$\mathbf{g}_{\tilde{p}}$
0	1 2	σ_{12}	000001	121212 131313	σ_{23}
1	1 2 3	C	000011	121212 313131 232323	C^2
01	12 13	σ_{23}	000101	121213	e
001	121 232 313	C	000111	121213 212123	σ_{12}
011	121 323	σ_{13}	001011	121232 131323	σ_{23}
0001	1212 1313	σ_{23}	001101	121231 323213	σ_{13}
0011	1212 3131 2323	C^2	001111	121231 232312 313123	C
0111	1213 2123	σ_{12}	010111	121312 313231 232123	C^2
00001	12121 23232 31313	C	011111	121321 323123	σ_{13}
00011	12121 32323	σ_{13}	0000001	1212121 2323232 3131313	C
00101	12123 21213	σ_{12}	0000011	1212121 3232323	σ_{13}
00111	12123	e	0000101	1212123 2121213	σ_{12}
01011	12131 23212 31323	C	0000111	1212123	e
01111	12132 13123	σ_{23}	$\cdots$	$\cdots$	$\cdots$

Example 6.2. ***Desymmetrization of Lorenz flow:*** *(continuation of example 6.1) Lorenz equation (6.4) is equivariant under (6.5), the action of order-2 group $C_2 = \{e, C^{1/2}\}$, where $C^{1/2}$ is $[x, y]$-plane, half-cycle rotation by π about the z-axis:*

$$(x, y, z) \to C^{1/2}(x, y, z) = (-x, -y, z)\,. \tag{6.6}$$

$(C^{1/2})^2 = 1$ condition decomposes the state space into two linearly irreducible subspaces $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$, the z-axis $\mathcal{M}^+$ and the $[x, y]$ plane $\mathcal{M}^-$, with projection operators onto the two subspaces given by

$$\mathbf{P}^+ = \frac{1}{2}(1 + C^{1/2}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}^- = \frac{1}{2}(1 - C^{1/2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.7}$$

so

$$\begin{aligned} \begin{pmatrix} \dot{x}_- \\ \dot{y}_- \end{pmatrix} &= \begin{pmatrix} -\sigma & \sigma \\ \rho & -1 \end{pmatrix} \begin{pmatrix} x_- \\ y_- \end{pmatrix} + \begin{pmatrix} 0 \\ -z\, x_- \end{pmatrix} \\ \dot{z}_+ &= -b\, z_+ + \frac{1}{4}(x_+ + x_-)(y_+ + y_-)\,, \end{aligned} \tag{6.8}$$

where $z_+ = z$. As $(\dot{x}_+, \dot{y}_+) = (0, 0)$, values of (x_+, y_+) are conserved parts of the initial condition. We define the fundamental domain by the (arbitrary) condition $\hat{x}_- \geq 0$, and whenever exits the domain,we replace the function dependence by the corresponding fundamental domain coordinates,

$$(x_-, y_-) = C^{1/2}(\hat{x}_-, \hat{y}_-) = (-\hat{x}_-, -\hat{y}_-) \quad \textit{if} \quad x_- < 0\,,$$

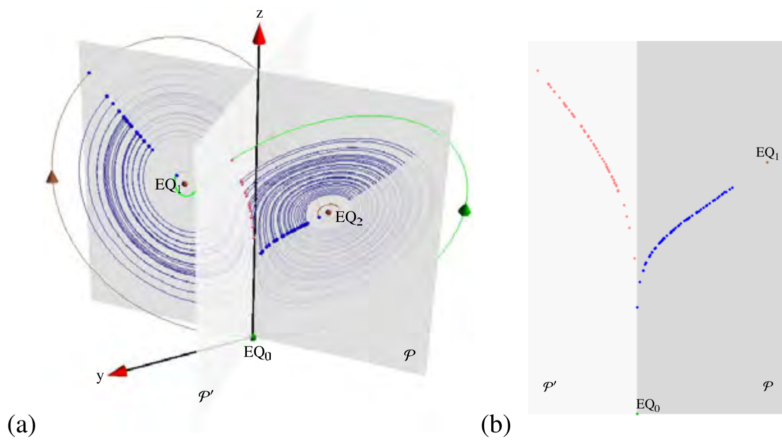

Figure 6.2: (a) Lorenz flow cut by $y = x$ Poincaré section plane $\mathcal{P}$ through the z axis and both $\mathrm{E}_{1,2}$ equilibria. Points where flow pierces into section are marked by dots. To aid visualization of the flow near the E_0 equilibrium, the flow is cut by the second Poincaré section, $\mathcal{P}'$, through $y = -x$ and the z axis. (b) Poincaré sections $\mathcal{P}$ and $\mathcal{P}'$ laid side-by-side. (E. Siminos)

and record that we have applied $C^{1/2}$ (that is the 'reconstruction equation' in the case of a discrete symmetry). When we integrate (6.8), the trajectory coordinates $(\hat{x}_-(t), \hat{y}_-(t))$ are discontinuous whenever the trajectory crosses the fundamental domain border. That, however, we do not care about - the only thing we need are the Poincaré section points and the Poincaré return map in the fundamental domain.

Poincaré section hypersurface can be specified implicitly by a single condition, through a function $U(x)$ that is zero whenever a point x is on the Poincaré section,

$$\hat{x} \in \mathcal{P} \quad \textit{iff} \quad U(\hat{x}) = 0\,. \tag{6.9}$$

In order that there is only one copy of the section in the fundamental domain, this condition has to be invariant, $U(g\hat{x}) = U(\hat{x})$ for $g \in G$, or, equivalently, the normal to it has to be equivariant

$$\partial_j U(g\hat{x}) = g\partial_j U(\hat{x}) \quad \textit{for} \quad g \in G\,. \tag{6.10}$$

There are two kinds of compact (finite-time) orbits. Periodic orbits $x(T_p) = x(T_p)$ are either self dual under rotation $C^{1/2}$, or appear in pairs related by $C^{1/2}$; in the fundamental domain there is only one copy $\hat{x}(T_p) = \hat{x}(T_p)$ of each. Relative periodic orbits (or 'pre-periodic orbits') $\hat{x}(T_p) = C^{1/2}x(T_p)$ they are periodic orbits.

As the flow is C_2-invariant, so is its linearization $\dot{x} = Ax$. Evaluated at E_0, A commutes with $C^{1/2}$, and the E_0 stability matrix A decomposes into $[x, y]$ and z blocks.

The 1-dimensional $\mathcal{M}^+$ subspace is the fixed-point subspace, with the z-axis points left point-wise invariant *under the group action*

$$\mathcal{M}^+ = \mathrm{Fix}\,(C_2) = \{x \in \mathcal{M} \mid g\,x = x \text{ for } g \in \{e, C^{1/2}\}\} \tag{6.11}$$

(here $x = (x, y, z)$ is a 3-dimensional vector, not the coordinate x). A C_2-fixed point $x(t)$ in $\mathrm{Fix}\,(C_2)$ moves with time, but according to (6.2) remains within $x(t) \in \mathrm{Fix}\,(C_2)$ for all times; the subspace $\mathcal{M}^+ = \mathrm{Fix}\,(C_2)$ is flow invariant. *In case at hand this jargon is a bit*

of an overkill: clearly for $(x, y, z) = (0, 0, z)$ the full state space Lorenz equation (6.4) is reduced to the exponential contraction to the E_0 equilibrium,

$$\dot{z} = -b\,z\,. \tag{6.12}$$

However, for higher-dimensional flows the flow-invariant subspaces can be high-dimensional, with interesting dynamics of their own. Even in this simple case this subspace plays an important role as a topological obstruction: the orbits can neither enter it nor exit it, so the number of windings of a trajectory around it provides a natural, topological symbolic dynamics.

The $\mathcal{M}^-$ subspace is, however, not *flow-invariant, as the nonlinear terms $\dot{z} = xy - bz$ in the Lorenz equation (6.4) send all initial conditions within $\mathcal{M}^- = (x(0), y(0), 0)$ into the full, $z(t) \neq 0$ state space $\mathcal{M}/\mathcal{M}^+$.*

By taking as a Poincaré section any $C^{1/2}$-equivariant, non-self-intersecting surface that contains the z axis, the state space is divided into a half-space fundamental domain $\tilde{\mathcal{M}} = \mathcal{M}/C_2$ and its 180^o rotation $C^{1/2}\tilde{\mathcal{M}}$. An example is afforded by the $\mathcal{P}$ plane section of the Lorenz flow in figure 6.3. Take the fundamental domain $\tilde{\mathcal{M}}$ to be the half-space between the viewer and $\mathcal{P}$. Then the full Lorenz flow is captured by re-injecting back into $\tilde{\mathcal{M}}$ any trajectory that exits it, by a rotation of π around the z axis.

As any such $C^{1/2}$-invariant section does the job, a choice of a 'fundamental domain' is here largely mater of taste. For purposes of visualization it is convenient to make the double-cover nature of the full state space by $\tilde{\mathcal{M}}$ explicit, through any state space redefinition that maps a pair of points related by symmetry into a single point. In case at hand, this can be easily accomplished by expressing (x, y) in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, and then plotting the flow in the 'doubled-polar angle representation:'

$$(\hat{x}, \hat{y}, z) \;=\; (r\cos 2\theta, r\sin 2\theta, z) \;=\; ((x^2 - y^2)/r, 2xy/r, z)\,, \tag{6.13}$$

as in figure 6.4 (a). In contrast to the original G-equivariant coordinates $[x, y, z]$, the Lorenz flow expressed in the new coordinates $[\hat{x}, \hat{y}, z]$ is G-invariant. *In this representation the $\tilde{\mathcal{M}} = \mathcal{M}/C_2$ fundamental domain flow is a smooth, continuous flow, with (any choice of) the fundamental domain stretched out to seamlessly cover the entire $[\hat{x}, \hat{y}]$ plane.*

(E. Siminos and J. Halcrow)

6.4 Chaotic 3-spring, integrable 3-vortex systems (optional)

Continued from sect. 4.1.

Simon Berman According to the 2019 Phys. Rev. Lett., of Katz-Saporta and Efrati [12], *Self-driven fractional rotational diffusion of the harmonic three-mass system*, a system of three masses connected by harmonic springs might be the simplest mechanical system (homonuclear triatomic molecule, such as ozone, except the three couplings are not the same) that exhibits a *geometric phase*. Away from its resting configuration the system is nonlinear, and once its rotational SO(2) symmetry is reduced, and as its energy is increased, it exhibits all kinds of shape-dependent chaotic geometric phases. Katz and Efrati [12] mostly do numerical simulations and plot displacement vs. time diffusion plots in its 6D phase space, like this is still early 1960's. The earlier arXiv:1706.09868 version has more information than the PRL (no doubt thanks to impatience of the referees, plus space constraints of a PRL). One suspects that a bit of thinking along periodic orbit theory lines could yield some insight into the diffusive properties of its shape-changing dynamics.

In the symmetry-reduced or the 'shape' state space there is a D_3 symmetry. One sees it in their [12] Hamiltonian (2): the b^{ij} vectors can be viewed as the three coordinates of an equilateral triangle in the $w_1 - w_2$ plane. Since the Hamiltonian only depends on $|w|$ and in a symmetric way on $w \cdot b^{ij}$, it has a D_3 symmetry for (w_1, w_2) components of the w vector, and a reflection symmetry for w_3. So the total symmetry group is $D_3 \times C^{1/2}$.

Predrag As the system is D_3 symmetric, the symmetry should be quotiented as in (this week's lectures) and ChaosBook.org. The students from Weizmann (as well as all our local plumber apprentices) believe they have been born knowing everything, and thus they do not need to take ChaosBook.org/course1, so they would have no idea that

- they are supposed to quotient the symmetry
- probability densities (eigenfunctions of the evolution operator; Perron--Frobenius and its generalizations) block diagonalize as irreps of $D_3 = C_{3v}$, and
- that makes all calculations, numerical and periodic orbit-type more transparent and more convergent.

By going to relative w's coordinates, one has quotiented only the $2D$ Euclidean translations and SO(2) rotations, no discrete symmetries, so D_3 still remains. Now, anyone who has taken ChaosBook.org/course1 knows that the next step is to quotient D_3, and do the calculation in the 1/6th of the phase space, i.e., the fundamental domain.

I'm curious whether I'm right, because soon we'll look at space groups (infinite lattices with discrete symmetries) and there I have confused understanding of how to quotient the space group, but that is related to diffusion in space, rather than the angular diffusion, as in this 3-springs system.

We can make this a course project for a student in this course (a project instead of taking the final). To be especially pedagogical, we'll ask them to do it in Julia (there is one potential candidate on Piazza).

Predrag proposal: 2-body, 3-spring system We need the *simplest* illustration of a geometric phase, and its diffusion along the continuous symmetry direction induced by chaotic ("turbulent") shape-changing dynamics. So let's take one of the masses infinite. Still 3 springs, but only 2 bodies moving in a plane. We still have SO(2) continuous symmetry to reduce. What remains is the $D_2 = \{e, \sigma\}$ symmetry of exchanging the two particles, with two irreps, the symmetric and the antisymmetric normal modes. There is shape-changing dynamics, with the potential a nonlinear function of w_j's, so for larger energies we expect angular geometric phase diffusion, but in a lower-dimensional phase space than that of the free 3-springs system. Easier to work out and look at Poincaré sections, search for relative equilibria and relative periodic orbits, compute the angular diffusion constant from its cycle expansion formulation.

Predrag: N **vortex system** Went to hear Tomoki Ohsawa (Google Scholar), talk about *Symplectic reduction and the Lie–Poisson dynamics of point vortices on the plane*, arXiv:1808.01769 .

I had previously written to Tomoki's friend Molei about how much I had already suffered through Weinstein, Marsden, etc. moment maps, for decades. We all have to do symmetry reductions, but with Marsden it is always the moment map, and then the climax is the rigid 3D body example which is the end-all of every article and book. Perhaps due to my pleas, Ohsawa gave us a gentle, sensible seminar, Weinstein-Marsden for humans, where he explained why moment map is called 'moment,' etc.. As nice a birthday present one could hope for, see the slides here.

Ohsawa develops a Hamiltonian formulation of the dynamics of the "shape" of N point vortices on the plane and the sphere. If $N = 3$, it is the dynamics of the shape of the triangle formed by three point vortices, regardless of the position and orientation of the triangle on the plane/sphere.

For the planar case, reducing the basic equations of point vortex dynamics by the special Euclidean group SE(2) yields a Lie-Poisson equation for relative configurations of the vortices. The shape dynamics is periodic in certain cases. The approach can be extended to the spherical case by first lifting the dynamics from the two-sphere to C^2 and then performing reductions by symmetries.

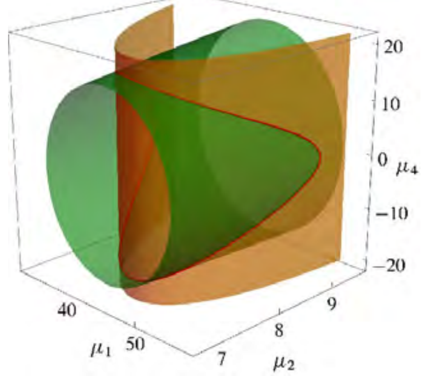

Figure 6.3: (green) Level set of quadratic Casimir C_2 ellipsoid. (orange) Level set of Hamiltonian h. The intersection is the unique periodic orbit of the symmetry-reduced $N = 3$ vortex system. See also the corresponding figure for 3 vortices on the sphere on p. 46 of the slide presentation. T. Ohsawa

I think Ohsawa discovery is that the system has a previously un-noted quartic Casimir, whose invariance reduces the dimension of the symmetry reduced phase space by one degree of freedom (dof). The implies that the symmetry-reduced dynamics in the $N = 3$ case is 1 dof, i.e., integrable, see figure 6.3. In addition, if the sum of the vortex circulations is 0, the $N = 4$ case is integrable. This fact is not yet explained - my intuition is that the zero total circulation implies extra rotational symmetry. For more vortices I expect the usual Hamiltonian mixed phase space.

Parenthetically, statement that there is quartic order Casimir that is invariant under the symmetry group can probably be written as a syzygy constraint on the invariant polynomial basis in (Hilbert's) theory of invariant polynomial bases.

In the Katz-Saporta and Efrati [12] example there is no quartic Casimir, so one ends with a generic chaotic system. Ohsawa's geometric technique works because of the simple symplectic structure on the point-vortex problem (there is no 'momentum'), whereas Katz-Saporta and Efrati problem is a standard classical-mechanical one on the cotangent bundle of a configuration space, with momentum there. Ohsawa believes that one can apply the techniques developed by Richard Montgomery to this setting as well. (Montgomery's paper motivated him to work on the point-vortex problem).

There are also examples in cardiac (!) dynamics where one must reduce 2D Euclidean symmetry first, with similar outcome to yours, but no moment maps, as such PDEs have no variational formulation (that I am aware of). Googling "Barkley model" might do the trick. I do not think there is a variational (Lagrangina) formulation.

But that is the whole point - *any* flow with a symmetry has to have the symmetry quotiented out. It's easier to understand this for flows which are not symplectic - in that case, every continuous symmetry parameter reduces the dimension of the symmetry-reduced state space by one. The Hamiltonin case is a pain (or bliss, if you love moment maps) because every continuous symmetry reduces the dimension of the phase space by one degree of freedom (ie, by 2). Also variational problems obey Noether's theorem, our (dissipative) problems usually do not. If I understand this right...

6.5 Eigenfunctions (optional)

What follows is an inconclusive discussion of eigenfunctions over fundamental domains - feel free to ignore...

Predrag Heilman and Strichartz [10] *Homotopies of Eigenfunctions and the Spectrum of the Laplacian on the Sierpinski Carpet*, arXiv:0908.2942, is not an obvious read for us, but they compute a spectrum on a square domain, and we might have to be mindful of it: " Since all of our domains are invariant under the D_4 symmetry group, we can simplify the eigenfunction computations by reducing to a fundamental domain. On this domain we impose appropriate boundary

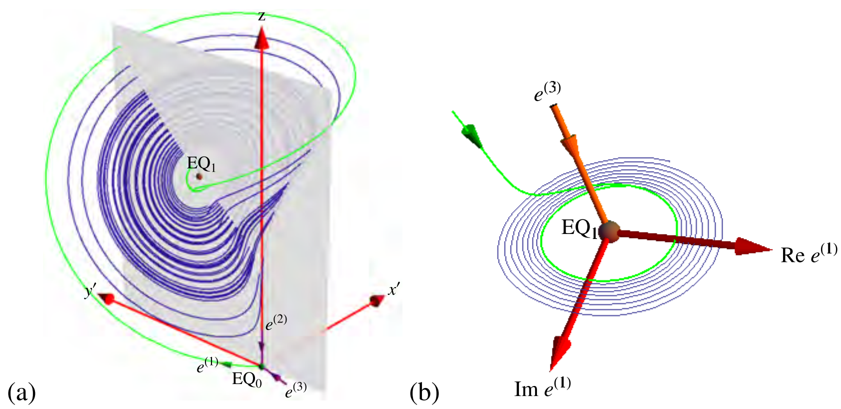

Figure 6.4: (a) Lorenz attractor plotted in $[\hat{x}, \hat{y}, z]$, the doubled-polar angle coordinates (6.13), with points related by π-rotation in the $[x, y]$ plane identified. Stable eigenvectors of E_0: $\mathbf{e}^{(3)}$ and $\mathbf{e}^{(2)}$, along the z axis (6.12). Unstable manifold orbit $W^u(\mathrm{E}_0)$ (green) is a continuation of the unstable $\mathbf{e}^{(1)}$ of E_0. (b) Blow-up of the region near E_1: The unstable eigenplane of E_1 defined by $\mathrm{Re}\,\mathbf{e}^{(2)}$ and $\mathrm{Im}\,\mathbf{e}^{(2)}$, the stable eigenvector $\mathbf{e}^{(3)}$. The descent of the E_0 unstable manifold (green) defines the innermost edge of the strange attractor. As it is clear from (a), it also defines its outermost edge. (E. Siminos)

conditions according to the rep-resentation type. For the 1-dimensional representation, we restrict to the sector $0 \leq \theta \leq \pi/4$. Recall that the functions will extend evenly when reflected about $\theta = 0$ in the 1++ and 1– cases, and oddly in the 1-+ and 1+- cases. Note that performing an even extension across a ray is equivalent to imposing Neumann boundary conditions on that ray. Similarly, the odd extension is equivalent to Dirichlet conditions. For the 2-dimensional representation our fundamental domain is the sector $0 \leq \theta \leq \pi/2$, and we impose Neumann boundary conditions on the ray $\theta = 0$ and Dirichlet conditions on the ray $\theta = \pi/2$. Note that our fundamental domains are simply connected. "

This seems to be saying that one gets the 2-dimensional representation by doubling the fundamental domain and mixing boundary conditions. Do you understand that?

Boris Here is my present understanding of the fundamental domains issue: If you want simple boundary conditions like Dirichlet or Neumann you stick to 1d representations only. They connect eigenfunction to itself at the fundamental domain boundaries – otherwise you would need to connect pair of functions (would be something like boundary conditions for spinor in case of 2d representations.) So what you do is the following: take the largest abelian subgroup $Z_2 \times Z_2$ (for D_4) and split its spectrum with respect to its fundamental domain defined as 1/4 of the square (twice the fundamental domain of the full group). Then your see that Dirichlet-Dirichlet and Neumann-Neumann Hamiltonians still have Z_2 symmetry so your split them further into the Hamiltonians of the 1/8 fundamental domain. But Dirichlet-Neumann remains 1/4th of the square.

Predrag Your argument is in the spirit of Harter's class operators construction (see week 5) of higher-dimensional representations by using particular chains of subgroups, but I am not able to visualize how that larger fundamental domain (of the lower-order subgroup) folds back into the small fundamental domain of the whole group. By the time the dust settles, I have the symmetry factorization of the determinants that we need, but I do not have a gut feeling for the boundary conditions that you do, when it comes to higher-dimensional irreps.

Commentary

Remark 6.1. Lorenz equation. The Lorenz equation (6.4) is the most celebrated early illustration of "deterministic chaos" [14] (but not the first - that honor goes to Dame Cartwright [2] in 1945. Amusingly, Denisov and Ponomarev [5] argue that Ben F. Laposky might have been the first to observe chaotic attractors as early as 1953, which, strictly speaking falls after 1945, even in Russia). Lorenz's 1963 paper, which can be found in reprint collections refs. [4, 9], is a pleasure to read, and it is still one of the best introductions to the physics motivating such models (read more about Lorenz here). The equations, a set of ODEs in $\mathbb{R}^3$, exhibit strange attractors. W. Tucker [21–23] has proven rigorously (via interval arithmetic) that the Lorenz attractor is strange for the original parameters (no stable orbits) and that it has a long stable periodic orbit for slightly different parameters. In contrast to the hyperbolic strange attractors such as the weakly perturbed cat map [3], the Lorenz attractor is structurally unstable. Frøyland [6] has a nice brief discussion of Lorenz flow. Frøyland and Alfsen [7] plot many periodic and heteroclinic orbits of the Lorenz flow; some of the symmetric ones are included in ref. [6]. Guckenheimer-Williams [8] and Afraimovich-Bykov-Shilnikov [1] offer an in-depth discussion of the Lorenz equation. The most detailed study of the Lorenz equation was undertaken by Sparrow [19]. For a geophysics derivation, see Rothman course notes [17]. For a physical interpretation of ρ as "Rayleigh number," see Jackson [11] and Seydel [18]. The Lorenz truncation to 3 modes, however, is so drastic that the model bears no relation to the geophysical hydrodynamics problem that motivated it. Just for fun, as Lorentz was such a lovable weatherman, in 1972 Willem Malkus constructed [15], by a feat of reverse engineering, a physical system, a "water wheel", popularized by Strogatz [20], that is described by Lorentz equations. You can see it simulated on wolfram.com, and tested experimentally at http://www.ace.gatech.edu. There is no deep physics in this lovely game, it is but a cute distraction. For detailed pictures of Lorenz invariant manifolds consult Vol II of Jackson [11] and "Realtime visualization of invariant manifolds" by Ronzan. The Lorenz attractor is a very thin fractal – as we shall see, stable manifold thickness is of the order 10^{-4} – whose fractal structure has been accurately resolved by D. Viswanath [24, 25]. If you wonder what analytic function theory has to say about Lorenz, check ref. [26]. Modular flows are your thing? E. Ghys and J. Leys have a beautiful tale for you. Refs. [13, 16] might also be of interest.

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Exercises

6.1. **3-disk symbolic dynamics.** As periodic trajectories will turn out to be our main tool to breach deep into the realm of chaos, it pays to start familiarizing oneself with them now by sketching and counting the few shortest prime cycles. Show that the 3-disk pinball has $3 \cdot 2^{n-1}$ itineraries of length n. List periodic orbits of lengths 2, 3, 4, 5, $\cdots$. Verify that the shortest 3-disk prime cycles are 12, 13, 23, 123, 132, 1213, 1232, 1323, 12123, $\cdots$. Try to sketch them. (continued in exercise 6.3)

A comment about exercise 6.1, exercise 6.2, and exercise 6.3: If parts of these problems seem trivial - they are. The intention is just to check that you understand what these symbolic dynamics codings are - the main message is that the really smart coding (fundamental domain) is 1-to-1 given by the group theory operations that map a point in the fundamental domain to where it is in the full state space. This observation you might not find deep, but it is - instead of *absolute* labels, in presence of symmetries one only needs to keep track of *relative* motions, from domain to domain, does not matter which domain in absolute coordinates - that is what group actions do. And thus the word '*relative*' creeps into this exposition.

6.2. **Reduction of 3-disk symbolic dynamics to binary.** (continued from exercise 6.1)

(a) Verify that the 3-disk cycles
$\{\overline{1\,2}, \overline{1\,3}, \overline{2\,3}\}$, $\{\overline{1\,2\,3}, \overline{1\,3\,2}\}$, $\{\overline{12\,13} + 2 \text{ perms.}\}$,
$\{\overline{121\,232\,313} + 5 \text{ perms.}\}$, $\{\overline{121\,323} + 2 \text{ perms.}\}$, $\cdots$,
correspond to the fundamental domain cycles $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$ respectively.

(b) Check the reduction for short cycles in table 6.1 by drawing them both in the full 3-disk system and in the fundamental domain, as in figure 6.3.

(c) Optional: Can you see how the group elements listed in table 6.1 relate irreducible segments to the fundamental domain periodic orbits?

(continued in exercise 6.3)

6.3. **3-disk fundamental domain cycles.** Try to sketch $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$. in the fundamental domain, and interpret the symbols $\{0, 1\}$ by relating them to topologically distinct types of collisions. Compare with table 6.1. Then try to sketch the location of periodic points in the Poincaré section of the billiard flow. The point of this exercise is that while in the configuration space longer cycles look like a hopeless jumble, in the Poincaré section they are clearly and logically ordered. The Poincaré section is always to be preferred to projections of a flow onto the configuration space coordinates, or any other subset of state space coordinates which does not respect the topological organization of the flow.

6.4. **C_2-equivariance of Lorenz system.** Verify that the vector field in Lorenz equations (6.4)

$$\dot{x} = v(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y - x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix} \tag{6.14}$$

is equivariant under the action of cyclic group $C_2 = \{e, C^{1/2}\}$ acting on $\mathbb{R}^3$ by a π rotation about the z axis,

$$C^{1/2}(x, y, z) = (-x, -y, z)\,,$$

as claimed in example 6.1.

6.5. **Proto-Lorenz system.** Here we quotient out the C_2 symmetry by constructing an explicit "intensity" representation of the desymmetrized Lorenz flow.

1. Rewrite the Lorenz equation (6.4) in terms of variables

$$(u, v, z) = (x^2 - y^2, 2xy, z)\,, \tag{6.15}$$

show that it takes form

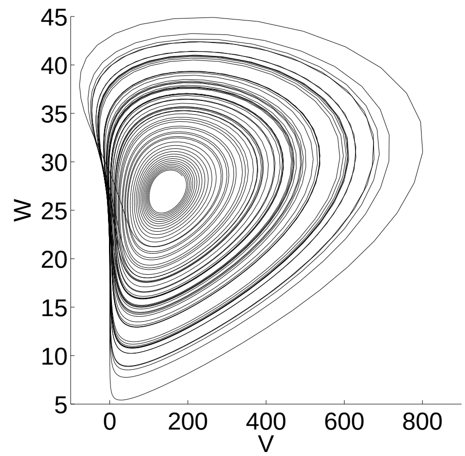
$$\begin{aligned} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} -(\sigma+1)u + (\sigma - r)v + (1-\sigma)N + vz \\ (r-\sigma)u - (\sigma+1)v + (r+\sigma)N - uz - Nz \\ v/2 - bz \end{bmatrix} \\ N &= \sqrt{u^2 + v^2}\,. \end{aligned} \tag{6.16}$$

2. Show that this is the (Lorenz)/C_2 quotient map for the Lorenz flow, i.e., that it identifies points related by the π rotation (6.6).

3. Show that (6.15) is invertible. Where does the inverse not exist?

4. Compute the equilibria of proto-Lorenz and their stabilities. Compare with the equilibria of the Lorenz flow.

5. Plot the strange attractor both in the original form (6.4) and in the proto-Lorenz form (6.16)



for the Lorenz parameter values $\sigma = 10$, $b = 8/3$, $\rho = 28$. Topologically, does it resemble more the Lorenz, or the Rössler attractor, or neither? (plot by J. Halcrow)

6. Show that a periodic orbit of the proto-Lorenz is either a periodic orbit or a relative periodic orbit of the Lorenz flow.

7. Show that if a periodic orbit of the proto-Lorenz is also periodic orbit of the Lorenz flow, their Floquet multipliers are the same. How do the Floquet multipliers of relative periodic orbits of the Lorenz flow relate to the Floquet multipliers of the proto-Lorenz?

8. Show that the coordinate change (6.15) is the same as rewriting

$$\begin{aligned} \dot{r} &= \frac{r}{2}(-\sigma - 1 + (\sigma + \rho - z)\sin 2\theta \\ &\quad +(1-\sigma)\cos 2\theta) \\ \dot{\theta} &= \frac{1}{2}(-\sigma + \rho - z + (\sigma - 1)\sin 2\theta \\ &\quad +(\sigma + \rho - z)\cos 2\theta) \\ \dot{z} &= -bz + \frac{r^2}{2}\sin 2\theta\,. \end{aligned} \tag{6.17}$$

in variables

$$(u, v) = (r^2 \cos 2\theta, r^2 \sin 2\theta)\,,$$

i.e., squaring a complex number $z = x + iy$, $z^2 = u + iv$.