group theory - week 3

Group representations

Georgia Tech PHYS-7143

Homework HW3 due Tuesday 2021-06-01

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 3.1 *1-dimensional representation of anything* 1 point
Exercise 3.2 *2-dimensional representation of S_3* 4 points
Exercise 3.3 *3-dimensional representations of D_3* 5 points

Bonus points
Exercise 3.4 *Abelian groups* 1 point
Exercise 3.5 *Representations of C_N* 1 point

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

3.1 Week's videos, reading

3.1.1 Matrix representations, Schur's Lemma

2021-05-25 Predrag Lecture 3

Irreps, unitary reps and Schur's Lemma.

- Dresselhaus *et al.* [1] Sect. 2.4 *The Unitarity of Representations*.
- Dresselhaus *et al.* [1] Sects. 2.5 and 2.6 *Schur's Lemma*.
- *Lecture 3* (Unedited) Unitarity of reps; Schur's Lemma (2:04:56 h)

3.1.2 Wonderful Orthogonality Theorem

2021-05-27 Predrag Lecture 4

- *Section playlist*
- (extra) *Recap of lect. 3* (5:36 min)
- ◦ Sect. 3.2 It's all about class
- Dresselhaus *et al.* [1] Sect. 2.7 *'Wonderful' Orthogonality Theorem*, sect. 2.8 *Representations and vector spaces*.
- *Whence "orthogonality"?* The ideas: observables are Hermitian; matrix reps are unitary; average over the group to extract invariants. A matrix rep forms a complex unit vector, hence "orthogonality". (9:47 min)
- *Character orthogonality theorem* (X:XX min)
- Dresselhaus *et al.* [1] Sects. 3.1 *Characters and Class* to 3.5 *The number of irreducible representations*.
- *Characters. Character orthogonality.* (X:XX min)
- *Class* (17:56 min)
- *Number of classes equals the number of irreps* (6:24 min)
- (extra) *Discussion:* Irrep dimension; Are classes subgroups, cosets? Week's homework. Classes and irreps of D_3. N-gon intuition. A LaTeX template. (26:39 min)

Tinkham [3] covers the same material in Chapter 3 *Theory of Group Representations*, in a more compact way.

3.2 It's all about class

The essential group theory notions introduced here are the notion of irreducible representations (irreps) and their orthogonality

The Group Orthogonality Theorem: Let D_μ, $D_{\mu'}$ be two irreducible matrix representations of a compact group G of dimensions d_μ, $d_{\mu'}$, where the sum is over all elements of the group, $G = \{g\}$, and $|G|$ is their number, or the order of the group:

$$\frac{1}{|G|}\sum_g^G D^{(\mu)}(g)_a{}^b D^{(\mu')}(g^{-1})^{a'}{}_{b'} = \frac{1}{d_\mu}\delta_{\mu\mu'}\delta_a^{a'}\delta_{b'}^b \,.$$

This is a remarkable formula, one relation for each of the $d_\mu^2 + d_{\mu'}^2$ matrix entries. Still, the explicit matrix entries reflect largely arbitrary coordinate choices - there should a more compact statement of irreducibility, and there is: the "character orthogonality theorem" (3.1).

Consider a reducible representation $D(g)$, i.e., a representation of group element g that after a suitable similarity transformation takes form

$$D(g) = \begin{pmatrix} D^{(a)}(g) & 0 & 0 & 0 \\ 0 & D^{(b)}(g) & 0 & 0 \\ 0 & 0 & D^{(c)}(g) & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix} ,$$

with character for class $\mathcal{C}$ given by

$$\chi(\mathcal{C}) = c_a\,\chi^{(a)}(\mathcal{C}) + c_b\,\chi^{(b)}(\mathcal{C}) + c_c\,\chi^{(c)}(\mathcal{C}) + \cdots ,$$

where c_a, the multiplicity of the ath irreducible representation (colloquially called "irrep"), is determined by the character orthonormality relations,

$$c_a = \overline{\chi^{(a)*}\,\chi} = \frac{1}{h}\sum_k^{class} N_k \chi^{(a)}(\mathcal{C}_k^{-1})\,\chi(\mathcal{C}_k)\,. \qquad (3.1)$$

Knowing characters is all that is needed to figure out what any reducible representation decomposes into! Work out exercise 4.2 as an example.

3.3 Other sources (optional)

3.3.1 Hard work builds character

Irreps, unitary reps, Schur's Lemma.

Section playlist

Chapter 2 *Representation Theory and Basic Theorems*
Dresselhaus *et al.* [1], up to and including
Sect. 2.4 *The Unitarity of Representations* (click here)

This requires character (1:23 min)

Hard work builds character (15:05 min)

The symmetry group of a propeller (6:13 min)

Irreps of C_3 (14:52 min)

Rotation and reflections in a plane: irreps of D_3 (13:38 min)

- (extra) *Discussion: more symmetries, fewer invariant subspaces* (2:10 min)
- (extra) *Discussion: abelian vs. nonabelian* (2:11 min)

This week's Dresselhous exposition (or the corresponding chapter in Tinkham [3]) comes from Wigner's classic [4] *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra*, which is a harder going, but the more group theory you learn the more you'll appreciate it. Eugene Wigner got the 1963 Nobel Prize in Physics, so by mid 60's gruppenpest was accepted in finer social circles.

In this course, we learn about full reducibility of finite and compact continuous groups in two parallel ways. On one hand, I personally find the multiplicative *projection operators* (1.25), coupled with the notion of class algebras (Harter [2] (click here) appendix C) most intuitive - a block-diagonalized subspace for each distinct eigenvalue of a given all-commuting matrix.

On the other hand, the character weighted sums (here related to the multiplicative projection operators as in ChaosBook Example A24.2 *Projection operators for discrete Fourier transform*) offer a deceptively 'simple' and elegant formulation of full-reducibility theorems, preferred by all standard textbook expositions.

3.3.2 History (optional)

The structure of finite groups was understood by late 19th century. A full list of finite groups was another matter. The complete proof of the classification of all finite groups takes about 3 000 pages, a collective 40-years undertaking by over 100 mathematicians, read the wiki.

Alex Kontorovich, Rutgers MAT 640:503 Complex Analysis. A wonderful lecturer, here he diverges into the story of Cardano and cubics. They are *cube*-ic for a reason. Did you know people learned to use $\sqrt{-1}$ before they understood that a number can be negative, like -1? Listen to his first lecture. Oh no! He just made me solve the cubic, something I had avoided my entire life. So far. You'll love it.

According to Kevin Hartnett, *The 'Useless' Perspective That Transformed Mathematics*: Representation theory was initially dismissed. Today, it's central to much of mathematics.

Groups are complicated collections of mathematical objects – like numbers or symmetries – that stand in a particular structured relationship with each other. Representation theory is a way of taking such complicated objects and "representing" them with simpler objects. It converts the sometimes mysterious world of groups into the well-trammeled territory of linear algebra, the study of simple transformations performed on objects called vectors, which are effectively directed line segments. These objects are defined by coordinates, which can be displayed in the form of a matrix, the core element of linear algebra, an array of numbers. While groups are abstract and often difficult to get a handle on, matrices and linear algebra are elementary.

Geordie Williamson, *Mathematics in light of representation theory*. October 16, 2015 at Urania: Symmetry is all around us. The mathematical study of symmetry becomes simpler when we linearize, and in doing so we enter the realm of representation theory. Representation theory has applications throughout mathematics (the Fourier transform, monstrous moonshine, the Langlands program, the proof of Fermat's last theorem, ...) and science (crystallography and spectroscopy in chemistry, signal processing in engineering, the standard model in physics, ...). The lecture is an introduction to the representation theory of finite groups, both over the complex numbers and over fields of positive characteristic (so-called modular representation theory). Williamson discusses Frobenius' discovery of the character table in Berlin in 1896, Brauer's first steps in modular representation theory in the 1930s, and the role of the character table in the discovery of the monster simple group in the 1980s. Williamson finishes with a discussion of recent developments in the modular representations of symmetric and finite general linear groups.

From Emory Math Department: A pariah is real! The simple finite groups fit into 18 families, except for the 26 sporadic groups. 20 sporadic groups AKA the Happy Family are parts of the Monster group. The remaining six loners are known as the pariahs. (Check the notes sect. 5.2 *Literature* for links to the Ree group and the whole classification.)

References

[1] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

[2] W. G. Harter and N. dos Santos, "Double-group theory on the half-shell and the two-level system. I. Rotation and half-integral spin states", Amer. J. Phys. **46**, 251–263 (1978).

[3] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

[4] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1931).

Exercises

3.1. **1-dimensional representation of anything.** Let $D(g)$ be a representation of a group G. Show that $d(g) = \det D(g)$ is one-dimensional representation of G as well.

(B. Gutkin)

3.2. **2-dimensional representation of S_3.**

(i) Show that the group S_3 can be generated by two permutations:

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

(ii) Show that matrices:

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ D(d) = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix},$$

with $z = e^{i2\pi/3}$, provide proper (faithful) representation for these elements and find representation for the remaining elements of the group.

(iii) Is this representation irreducible?

(B. Gutkin)

3.3. **3-dimensional representations of D_3.** The dihedral group D_3 is the symmetry group of the equilateral triangle. It has 6 elements

$$D_3 = \{E, C, C^2, \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}\},$$

where C is rotation by $2\pi/3$ and $\sigma^{(i)}$ is reflection along one of the 3 symmetry axes.

(i) Prove that this group is isomorphic to S_3

(ii) Show that matrices

$$D(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(C) = \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^2 \end{pmatrix}, D(\sigma^{(1)}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.2}$$

generate a 3-dimensional representation $D(g)$ of D_3. Hint: Calculate products for representations of group elements and compare with the group table (see lecture).

(iii) Show that this is a reducible representation which can be split into one dimensional A and two-dimensional representation Γ. In other words find a matrix R such that

$$RD(g)R^{-1} = \begin{pmatrix} A(g) & 0 \\ 0 & \Gamma(g) \end{pmatrix}$$

for all elements g of D_3. (Might help: D_3 has only one (non-equivalent) 2-dim irreducible representation).

(B. Gutkin)

3.4. **Abelian groups.** Let G be a group with only one-dimensional irreducible representations. Show that G is Abelian.

(B. Gutkin)

3.5. **Representations of C_N.** Find all irreducible representations of C_N.

(B. Gutkin)