group theory - week 15

Many particle systems. Young tableaux

Georgia Tech PHYS-7143

Homework HW15 due 2021-08-04 - optional

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Bonus points

Exercise 15.1 *Representations of* $SU(3)$	5 points
Exercise 15.2 *Young tableaux for* S_5	3 points
Exercise 15.3 *Young tableaux for* $SU(3)$	3 points
Exercise 15.4 *Irrep projection operators for unitary groups*	5 points

All points are bonus.

(no videos) Predrag Lecture 28 Many particle systems. Young tableaux

Gutkin notes, Lect. 12 *Many particle systems*.

Excerpt from Predrag's monograph [5], fetch it here: Sect. 9.3 *Young tableaux*.

(no videos) Predrag Lecture 29 Young tableaux

Excerpts from Predrag's monograph [5], fetch them here:
Sect. 2.2 *First example: SU*(n) (skim over casimirs and beyond: this example gives you a flavor of birdtracks computations, you do not need to work it out in detail),
Sect. 6.1 *Symmetrization*,
Sect. 6.2 *Antisymmetrization*,
Sect. 9.1 *Two-index tensors*,
Sect. 9.2 *Three-index tensors*, and Table 9.1.

Reading for this week: Sect. 9.3 *Young tableaux*.

Young tableaux for SU(3) and SU(n) have not yet been covered in the lectures, but you can easily learn them yourself, from, for example, Gutkin notes, Lect. 12 *Young tableaux*. Boris Gutkin is a professor, beyond learning new stuff, so he follows old fashioned references such as Fulton and Harris [6]. The resulting simple recipe with 0 explanation can be found, for example, here: *Young diagrams* by C.G. Wohl.

A modern exposition is given in *Group Theory – Birdtracks, Lie's, and Exceptional Groups,* birdtracks.eu Chapt. 9 *Unitary groups*. Currently I am a fan of the Alcock-Zeilinger algorithm [1–3], based on the simplification rules of ref. [2], which leads to explicitly Hermitian and compact expressions for the projection operators.

Probably best to read Alcock-Zeilinger course *The Special Unitary Group, Birdtracks, and Applications in QCD* notes [4]. Alcock-Zeilinger fully supersedes Cvitanović's formulation, and any future full exposition of birdtracks reduction of SU(N) tensor products into irreducible representations should be based on the Alcock-Zeilinger algorithm.

15.1 Other sources (optional)

The clearest current exposition and the most powerful irrep reduction of SU(n) is given in the triptych of papers by Judith Alcock-Zeilinger and her thesis adviser H. Weigart, University of Cape Town:

Simplification rules for birdtrack operators [3],
Compact Hermitian Young projection operators [2], and
Transition operators [1].

Probably best to read Alcock-Zeilinger course *The Special Unitary Group, Birdtracks, and Applications in QCD* notes [4]. You want to study these in detail if your research leads you to study of multiparticle states.

References

[1] J. Alcock-Zeilinger and H. Weigert, "Transition operators", J. Math. Phys. **58**, 051702 (2016).

[2] J. Alcock-Zeilinger and H. Weigert, "Compact Hermitian Young projection operators", J. Math. Phys. **58**, 051702 (2017).

[3] J. Alcock-Zeilinger and H. Weigert, "Simplification rules for birdtrack operators", J. Math. Phys. **58**, 051701 (2017).

[4] J. M. Alcock-Zeilinger, The Special Unitary Group, Birdtracks, and Applications in QCD, tech. rep. (Univ. Tübingen, 2018).

[5] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2008).

[6] W. Fulton and J. Harris, *Representation Theory* (Springer, New York, 1991).

Exercises

15.1. **Representations of SU(3).** Any irrep of SU(3) can be labeled $D(p,q)$ by its highest weight $\lambda = p\lambda_1 + q\lambda_2$, where $\lambda_{1,2}$ are the two fundamental weights.

(a) Find all irreps $D(p,q)$ of SU(3) with the dimensions less then 20 (see lecture notes for the dimensions of $D(p,q)$).

(b) Draw the lattice Λ generated by $\lambda_{1,2}$ and mark there all the weights v (i.e., lattice nodes) which belong to irrep. $D(3,0)$. Is $D(3,0)$ a real irrep?

(c) Consider product (reducible) representation $3 \otimes 3$, where $3 = D(1,0)$ is the fundamental irrep. Mark all the weights v on Λ which belong to $3 \otimes 3$. Using this find out decomposition of $3 \otimes 3$ into irreps:

$$3 \otimes 3 = \Box \oplus \triangle, \qquad \Box = ?, \qquad \triangle = ?$$

Hint: see lecture notes for similar exercise on $3 \otimes \bar{3}$.

(d) Using previous results find decomposition of $3 \otimes 3 \otimes 3$ into irreps.

(B. Gutkin)

15.2. **Young tableaux for S_5.**

(a) Draw all Young diagrams for the symmetric group S_5. How many irreducible representations has it? Which of the diagrams correspond to one-dimensional irreps?

(b) Find Young diagram corresponding to the irrep of S_5 with the largest dimension? Draw Young tableaux corresponding to this irrep/Young diagram. What is the dimension of this irrep?

(c) What are the dimensions of the remaining irreps?

(B. Gutkin)

15.3. **Young tableaux for SU(3).** Solve exercise 15.1 (c,d) by using Young tableaux.

Remark: If Young tableaux for SU(3) are not covered in the lectures, learn them yourself from, for example, birdtracks.eu *Group Theory Birdtracks, Lie's, and Exceptional Groups.* The resulting simple recipe with 0 explanation can be found, for example, here *C.G. Wohl*.

(B. Gutkin)

15.4. **Irrep projection operators for unitary groups.** Derive projection operators and dimensions for irreps of the Kronecker product of the defining and the adjoint reps of SU(n) listed in *Group Theory Birdtracks, Lie's, and Exceptional Groups*, birdtracks.eu table 9.3. (Ignore "indices," we have not defined them here.)