group theory - week 13

Simple Lie algebras; SU(3)

Georgia Tech PHYS-7143

Homework HW13 due Tuesday 2021-07-20

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 13.1 *Root systems of simple Lie algebras* 5 points
Exercise 13.2 *Meson octet* 5 points

Bonus points
Exercise 13.3 *SU*(3) *symmetry in 3D harmonic oscillator* 5 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2021-07-13 Predrag Lecture 25 Reps of simple Lie algebras

In week 4 we learned that for finite groups there is one very special matrix rep, the *regular representation* constructed from the group multiplication table, that is intrinsic to the abstract group itself, and whose reduction yields all irreps of a given group.

In week 9 we saw that for continuous groups we need to study the Lie algebra of the finite number of generators T_j, rather than the infinity of group elements $g = \exp(i\phi \cdot T)$. Here the finite group multiplication table is replaced by the 'Lie product', i.e., the table of Lie commutators' fully antisymmetric structure constants iC_{ijk}.

So far we have chosen the hermitian basis T_j. But non-hermitian bases are also OK, as we know from raising / lowering operators of SU(2) of quantum mechanics irrep constructions.

- (Unedited 34:45 min) *16.1 Adjoint representation and Killing form.*
- Gutkin notes, sect. 10.1 *Adjoint representation and Killing form*.
- (Unedited 15:13 min) *16.2 Cartan sub-algebra and roots.*
- Gutkin notes, Sect. 10.2 *Cartan sub-algebra and roots*.
- (Unedited 19:02 min) *16.3 Root systems.*
- Gutkin notes, Sect. 10.3 *Main properties of root systems*.
- (Unedited 21:30 min) *16.4 Construction of representations.* SU(2) example; analogy to presentations of finite groups, such as D_n (4.1); 'grand circles' on Cartan lattices.
- Gutkin notes, Sect. 10.4 *Building up representations of* g.

2021-07-15 Predrag Lecture 26 Cartan's SU(3) irreps

- (Unedited 2:03:14 h) *Representations of SU*(3).
- Gutkin notes, Sect. 10.5 *Representations of SU*(3).

13.1 Group theory news (optional)

Turns out, applications of group theory go way beyond what is covered in this course:

- Mathematicians map E_8, and it is bigger than the human genome.
- **Group theory of defamation:** The officers argued Sawant's statements impugned them individually even though she only spoke about the police department as a whole. The court says suing as individuals and advancing a group theory of defamation takes far more than the officers showed in their complaint.

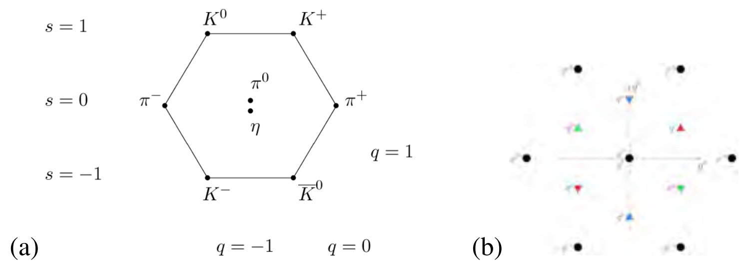

Figure 13.1: (a) The meson (pseudoscalars) octet. (b) The quark triplet, the anti-quark triplet and the gluon octet. (Wikipedia).

> [W]hether proceeding under an individual or group theory, Plaintiffs must plead that the statements "specifically" identified or singled them out, or was understood as "referring to [them] in particular." Sims, 20 Wn. App. at 236.

Exercises

13.1. **Root system of simple Lie algebras.**

a) Determine dimensions of Lie algebras $\mathfrak{so}(N), \mathfrak{su}(N)$ and dimensions of their Cartan subalgebras. What is the number of the positive roots for these Lie algebras?

b) Show that $N \times N$ diagonal matrices H_i with zero traces and uper/lower corner $N \times N$ matrices $E^{(a,b)}$ with the elements $E^{(a,b)}_{i,j} = \delta_{ia}\delta_{ib}$ provide Cartan-Weyl basis of $\mathfrak{su}(N)$. To put it differently, show that $E^{(a,b)}$ are eigenstates for adjoint representation of H_i's.

(B. Gutkin)

13.2. **Meson octet.** In Gutkin lecture notes, Lect. 11 *Strong interactions: flavor SU*(3), the meson octet, figure 13.1 (a)

$$\begin{aligned}\Phi &= \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K^0} & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ & 0 \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & K^+ \\ 0 & 0 & K^0 \\ K^- & \overline{K^0} & 0 \end{pmatrix} + \frac{\eta}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned} \tag{13.1}$$

is interpreted as arising from the adjoint representation of SU(3), i.e., the traceless part of the quark-antiquark $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$ outer product (see figure 13.1 (b)),

$$\begin{pmatrix} u\overline{u} & u\overline{d} & u\overline{s} \\ d\overline{u} & d\overline{d} & d\overline{s} \\ s\overline{u} & s\overline{d} & s\overline{s} \end{pmatrix} . \tag{13.2}$$

where we have replaced in (13.1) the constituent $q \otimes \overline{q}$ combinations by the names of the elementary particles they build.

Given the quark quantum numbers

	Q	I	I_3	Y	B
u	2/3	1/2	1/2	1/3	1/3
d	-1/3	1/2	-1/2	1/3	1/3
s	-1/3	0	0	-2/3	1/3

verify the strangeness and charge assignments of figure 13.1 (a).

13.3. **SU(3) symmetry in 3D harmonic oscillator.** The Hamiltonian of 3D isotropic harmonic oscillator is given by

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + \frac{m\omega^2}{2} x_i^2 = \hbar\omega \sum_{i=1}^{3} (a_i^\dagger a_i + 1/2),$$

where $a_i = \sqrt{\frac{m\omega}{2\hbar}} x_i + i\sqrt{\frac{1}{2m\omega\hbar}} p_i$ is creation ($a_i^\dagger$ resp. annihilation) operator satisfying $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = 0$.

a) Show that $a_i \to U_{i,j} a_j$, with $U \in U(3)$ is a symmetry of the Hamiltonian. In other words isotropic 3D harmonic oscillator has $U(3)$ rather than $O(3)$ symmetry!

b) Calculate degeneracy of the n-th level $E_n = \omega\hbar(n + 3/2)$ of the oscillator.

c) By comparison of dimensions find out which representations of SU(3) appear in the spectrum of harmonic oscillator.

(B. Gutkin)