group theory - week 12

Lorentz group; spin

Georgia Tech PHYS-7143

Homework HW12 due Tuesday 2021-07-20

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 12.1 *Lie algebra of* $SO(4)$ *and* $SU(2) \otimes SU(2)$	6 points
Exercise 12.2 SO(n) Clebsch-Gordan series for $V \otimes V$.	3 points
Exercise 12.3 *Lorentz spinology*	5 points
Exercise 12.4 *Lorentz spin transformations*	5 points

Bonus points

Exercise 12.5 *The unbearable lightness of* $SO(4)$ *Lie algebra*	15 points

Total of 19 points = 100 % score.

2021-07-08 Predrag Lecture 23
$\mathrm{SO}(4) = \mathrm{SU}(2) \otimes \mathrm{SU}(2)$; Lorentz group

Lecture 15 (Unedited) SU(2) irreps. $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$. More importantly: Minkowski metric, Lorentz group $\mathrm{SO}(1,3)$ irreps are also labeled by pairs of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ irrep labels. (2:29:20 h)

Gutkin notes sect. 9.2 *Representations of SU(2) and SO(3)*.

Gutkin notes sects. 9.5-9.8 *Product representations of SO(3)*

◦ sect. 12.3 *Spinors and the Lorentz group*

For Lorentz group, read Schwichtenberg [2] Sect. 3.7 (click here).

2021-07-13 Predrag Lecture 24 $\mathrm{SO}(1,3)$; Spin

12.1 Other sources (optional)

◦ sect. 12.4 *Irreps of SO(n)*

14.3A Who ordered J+, J-? (7:37 min)

For $\mathrm{SO}(n)$ see also birdtracks.eu Chapt. 10 *Orthogonal groups*, pp. 121-123.

For $\mathrm{SO}(4) = \mathrm{SU}(2) \otimes \mathrm{SU}(2)$ see also birdtracks.eu sect. 20.3.1 *SO(4) or Cartan $A_1 + A_1$ algebra*.

◦ sect. 12.5 *SO(4) of the Kepler problem*

◦ sect. 12.5.1 *Central force problems*

John Wood's (click here) notes and exercise 12.5 *The unbearable lightness of SO(4) Lie algebra.* The challenge: achieve some elegance in deriving the SO(4) commutator relations.

12.2 Discussion (optional)

Henriette Roux In this course the Levi-Civita tensor appears to be the unique connection for SO(4); but in GR, I learnt that the choice of connection is actually arbitrary and there are theories of gravity which need not use the Levi-Civita tensor. Are these two different concepts which are not necessarily linked?

Predrag Sean Carroll answers your question in arXiv:9712019. He does not understand that the *invariant* tensors are good, as they are what *defines* a given symmetry group:

It is a remarkable property of the above tensors – the metric, the inverse metric, the Kronecker delta, and the Levi-Civita tensor – that, even though they all transform according to the tensor transformation law, their components remain unchanged in any Cartesian coordinate system in flat spacetime. In some sense this makes them bad examples of tensors, since most tensors do not have this property.

However, he then goes on to explain that while in curved spacetime lengths and volumes are measured in the spacetime dependent way, we still need a notion of a volume of a hypercube as a skew product of its edges, ie, the determinant:

The Kronecker tensor can be thought of as the identity map from vectors to vectors (or from dual vectors to dual vectors), which clearly must have the same components regardless of coordinate system. The other tensors (the metric, its inverse, and the Levi-Civita tensor) characterize the structure of spacetime, and all depend on the metric. We shall therefore have to treat them more carefully when we drop our assumption of flat spacetime.

What he then does in his eq. (2.39) is to promote Levi-Civita from 'tensor' to 'symbol' in order to be able to compute determinants, just like we do in flat space SO(n).

See also MathWorld discussion.

Are you happy now?

(A side, nomenclature remark: Levi-Civita is not a 'connection' in the sense the word 'connection' is used in GR.)

12.3 Spinors and the Lorentz group

A Lorentz transformation is any invertible real $[4 \times 4]$ matrix transformation Λ,

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} \tag{12.1}$$

which preserves the Lorentz-invariant Minkowski bilinear form $\Lambda^T \eta \Lambda = \eta$,

$$x^{\mu} y_{\mu} = x^{\mu} \eta_{\mu\nu} y^{\nu} = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

with the metric tensor $\eta = diag(1, -1, -1, -1)$.

A contravariant four-vector $x^{\mu} = (x^0, x^1, x^2, x^3)$ can be arranged [3] into a Hermitian $[2\times 2]$ matrix in $Herm(2, \mathbb{C})$ as

$$\underline{x} = \sigma_{\mu} x^{\mu} = \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix} \tag{12.2}$$

in the hermitian matrix basis

$$\sigma_{\mu} = \bar{\sigma}^{\mu} = (\mathbb{1}_2, \boldsymbol{\sigma}) = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)\,, \quad \bar{\sigma}_{\mu} = \sigma^{\mu} = (\mathbb{1}_2, -\boldsymbol{\sigma})\,, \tag{12.3}$$

with $\boldsymbol{\sigma}$ given by the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12.4}$$

With the trace formula for the metric

$$\frac{1}{2}\mathrm{tr}\,(\sigma_\mu \bar{\sigma}_\nu) = \eta_{\mu\nu}\,, \tag{12.5}$$

the covariant vector x^μ can be recovered by

$$\frac{1}{2}\mathrm{tr}\,(\underline{x}\bar{\sigma}^\mu) = \frac{1}{2}\mathrm{tr}\,(x^\nu \sigma_\nu \bar{\sigma}^\mu) = x^\nu \eta_\nu{}^\mu = x^\mu \tag{12.6}$$

The Minkowski norm squared is given by

$$\det \underline{x} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = x_\mu x^\mu\,, \tag{12.7}$$

and with (12.3)

$$\overline{x} = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}, \tag{12.8}$$

the Minkowski scalar product is given by

$$x^\mu y_\mu = \frac{1}{2}\mathrm{tr}(\underline{x}\,\overline{y})\,. \tag{12.9}$$

The *special linear group* $SL(2,\mathbb{C})$ in two complex dimensions is given by the set of all matrices Λ such that

$$SL(2,\mathbb{C}) = \{\Lambda \in GL(2,\mathbb{C}) \,|\, \det\Lambda = +1\}. \tag{12.10}$$

Let a matrix $\Lambda \in SL(2,\mathbb{C})$ act on $\underline{x} \in Herm(2,\mathbb{C})$ as

$$\underline{x} \mapsto \underline{x}' = \Lambda \underline{x} \Lambda^\dagger \tag{12.11}$$

where † denotes Hermitian conjugation. The Minkowski scalar product is preserved, $\det\,\underline{x}' = \det\,\underline{x}$. Thus $\underline{x}'$ can also be represented by a real linear combination of generalized Pauli matrices

$$\underline{x}' = \sigma_\mu x'^\mu \quad \text{with } x'_\mu x'^\mu = x_\mu x^\mu \tag{12.12}$$

and Λ explicitly acts as a Lorentz transformation (12.1), with $\Lambda^\mu{}_\nu = \frac{1}{2}\,\mathrm{tr}\,(\bar{\sigma}^\mu \Lambda \sigma_\nu \Lambda^\dagger)$. The mapping is two-to-one, as two matrices $\pm\Lambda \in SL(2,\mathbb{C})$ generate the same Lorentz transformation $\Lambda\underline{x}\Lambda^\dagger = (-\Lambda)\underline{x}(-\Lambda)^\dagger$. This Λ belong to the proper orthochronous Lorentz group $SO^+(1,3)$, and it can be shown that $SL(2,\mathbb{C})$ is simply connected and is the double universal cover of the $SO^+(1,3)$.

Consider the fully antisymmetric Levi-Civita tensor $\varepsilon = -\varepsilon^{-1} = -\varepsilon^{\mathrm{T}}$ in two dimensions

$$\varepsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{12.13}$$

This defines a *symplectic* (i.e., *skew-symmetric*) bilinear form $\langle u, v\rangle = -\langle v, u\rangle$ on two *spinors* u and v, elements of the two-dimensional complex vector (or spinor) space $\mathbb{C}^2$

$$u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \quad v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \tag{12.14}$$

equipped with the symplectic form

$$\langle u, v\rangle = u^1 v^2 - u^2 v^1 = u^{\mathsf{T}} \varepsilon v\,. \tag{12.15}$$

This symplectic form is $SL(2, \mathbb{C})$-invariant

$$\langle u, v\rangle = u^{\mathsf{T}} \varepsilon v = \langle \Lambda u, \Lambda v\rangle = u^{\mathsf{T}} \Lambda^{\mathsf{T}} \varepsilon \Lambda v\,, \tag{12.16}$$

so one can interpret the group acting on spinors as $SL(2, \mathbb{C}) \cong \mathrm{Sp}(2, \mathbb{C})$, the complex symplectic group in two dimensions

$$\mathrm{Sp}(2, \mathbb{C}) = \{\Lambda \in GL(2, \mathbb{C}) \,|\, \Lambda^{\mathsf{T}} \varepsilon \Lambda = \varepsilon\}\,. \tag{12.17}$$

Summary. The group of Lorentz transformations of spinors is the group SL(2,C) of $[2 \times 2]$ complex matrices with determinant 1, i.e., the invariant tensor is the 2-index Levi-Civita ε_{AB}. The SL(2,C) matrices are parametrized by three complex dimensions and therefore six real ones (the matrices have four complex numbers and one complex constraint on the determinant). This matches the 6 dimensions of the group manifold associated with the Lorentz group $\mathrm{SO}(1, 3)$.

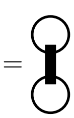
Andrew M. Steane writes "A spinor is the most basic mathematical object that can be Lorentz-transformed." His *An introduction to spinors*, arXiv:1312.3824, might help you develop intuition about spinors.

Andrzej Trautman tracks the origin of spinors to Euclid, and General Relativity to Clifford. He includes a letter from Hades saying, inter alia, "Unfortunately, it appears that there is now in your world a race of vampires, called referees, who clamp down mercilessly upon mathematicians unless they know the right passwords."

12.4 Irreps of SO(n) (optional)

The dimension of the defining representation of is $\mathrm{SO}(n)$ is given by the trace of the adjoint projection operator:

$$N = \mathrm{tr}\, \mathbf{P}_A = \quad = \frac{n(n-1)}{2}\,. \tag{12.18}$$



Dimensions of the other reps are listed in table 12.1.

12.5 SO(4) of the Kepler problem (optional)

One of "hidden" symmetries of quantum mechanics is the $\mathrm{SO}(4)$ of the Kepler problem.

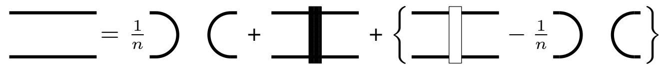
Young tableaux	□ × □	=	•	+	⊟	+	□□
Dimensions	n^2	=	1	+	$\frac{n(n-1)}{2}$	+	$\frac{(n+2)(n-1)}{2}$
Projectors							

Table 12.1: SO(n) Clebsch-Gordan series for $V \otimes V$, worked out in detail in *Group Theory – Birdtracks, Lie's, and Exceptional Groups,* birdtracks.eu Chapt. 10 *Orthogonal groups*.

John Baez discusses it in a fun read here: "if we take the angular momentum together with the Runge–Lenz vector, we get 6 conserved quantities—and these turn out to come from the group of rotations in 4 dimensions, SO(4), which is itself 6-dimensional. The obvious symmetries in this group just rotate a planet's elliptical orbit, while the unobvious ones can also squash or stretch it, changing the eccentricity of the orbit. [...] wavefunctions for bound states of hydrogen can be reinterpreted as functions on the 3-sphere, S^3. The sneaky SO(4) symmetry then becomes obvious: it just rotates this sphere! And the Hamiltonian of the hydrogen atom is closely connected to the Laplacian on the 3-sphere. The Laplacian has eigenspaces of dimensions n^2 where $n = 1, 2, 3, \ldots$, and these correspond to the eigenspaces of the hydrogen atom Hamiltonian. "

When the energy is fixed, the symmetry becomes Lie algebra SO$(3, 1)$ for positive-energy, scattering states, or SO(4) for negative-energy, bound states.

Michele Cini's lecture notes, p. 18 gives hydrogen as an example of why we don't believe in miracles such as "accidental" eigenvalue degeneracies, but assume that we must have missed a "hidden" symmetry. Cini writes: "Wolfgang Pauli in 1926 first solved [...] the H atom using the SO(4) symmetry." I didn't know that it was Pauli...

To dig deeper, skim through Baez *Mysteries of the gravitational 2-body problem*.

Bander and Itzykson [1] *Group theory and the hydrogen atom (I)* might be OK, but I have not read it.

12.5.1 Central force problems (optional)

For another way of looking at the H atom (and all solvable central force problems) download John Wood's chapter (click here) from *Quantum Mechanics for Nuclear Structure: I. A Primer,* IOP science series.

exercise 12.5

The SO$(2, 1)$ method can be extended to solve relativistic central force problems (one of my students did his Ph.D. thesis on this 20 years ago).

Q: Is the geometry associated with these algebraic structures, as applied to central force problems, explored?

References

[1] M. Bander and C. Itzykson, "Group theory and the hydrogen atom (I)", Rev. Mod. Phys. **38**, 330–345 (1966).

[2] J. Schwichtenberg, *Physics from Symmetry* (Springer, Berlin, 2015).

[3] E. Wigner, "On unitary representations of the inhomogeneous Lorentz group", Ann. Math. **40**, 149–204 (1939).

Exercises

12.1. **Lie algebra of SO(4) and SU(2) ⊗ SU(2).** One particle Hamiltonian with a central potential has in general SO(3) symmetry group. It turns out, however, that for Coulomb potential the symmetry group is actually larger - SO(4), rather than SO(3). This explains why the energy level degeneracies in the hydrogen atom are anomalously large. So SO(4) and its representations are of a special importance in atomic physics.

(a) Show that the Lie algebra $\mathfrak{so}(4)$ of the group SO(4) is generated by real antisymmetric 4×4 matrices.

(b) What is the dimension of $\mathfrak{so}(4)$?

A natural basis in $\mathfrak{so}(4)$ is provided by antisymmetric matrices $M_{\mu\nu}$, $\mu, \nu \in 1, 2, 3, 4$, $\mu \neq \nu$, generators of SO(4) rotations which leave invariant the $\mu\nu$-plane. The elements of these matrices are given by

$$(M_{\mu\nu})_{ij} = \delta_{i\mu}\delta_{j\nu} - \delta_{j\mu}\delta_{i\nu} \tag{12.19}$$

(c) Check that these matrices satisfy the commutation relationship

$$[M_{ab}, M_{cd}] = M_{ad}\delta_{bc} + M_{bc}\delta_{ad} - M_{ac}\delta_{bd} - M_{bd}\delta_{ac}. \tag{12.20}$$

(d) Show that Lie algebras of the groups SO(4) and SU(2) × SU(2) are isomorphic. **Path:**

(d.i) Define matrices

$$J_k = \frac{1}{2}\varepsilon_{kij}M_{i,j}, \qquad K_k = M_{k4}, \quad k = 1, 2, 3$$

and

$$\mathcal{A}_k = \frac{1}{2}(J_k + K_k) \quad \text{and} \quad \mathcal{B}_k = \frac{1}{2}(J_k - K_k)\ .$$

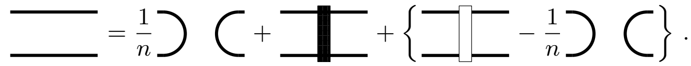
(d.ii) Show that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same commutation relations as two copies of $\mathfrak{su}(2)$.

(e) How does one construct irreps of $\mathfrak{so}(4)$ out of irreps of $\mathfrak{su}(2)$?

(f) Are groups SO(4) and SU(2) ⊗ SU(2) isomorphic to each other?

(B. Gutkin)

12.2. **SO(n) Clebsch-Gordan series for $V \otimes V$.**
(a) Show that the product of two n-dimensional reps of SO(n) decomposes into three irreps:



(12.21)

(b) Compute the dimensions of the three irreps.
(c) Which one is the adjoint one, and why? Hint: check the invariance condition. (This is worked out in detail in *Group Theory – Birdtracks, Lie's, and Exceptional Groups,* birdtracks.eu Chapt. 10 *Orthogonal groups*.)

12.3. **Lorentz spinology.**
Show that

(a)
$$x^2 = x_\mu x^\mu = \det \underline{x} \tag{12.22}$$

(b)
$$x_\mu y^\mu = \frac{1}{2}(\det(\underline{x}+\underline{y}) - \det(\underline{x}) - \det(\underline{y})) \tag{12.23}$$

(c)
$$x_\mu y^\mu = \frac{1}{2}\mathrm{tr}\ (\underline{x}\,\overline{y})\,, \tag{12.24}$$

where $\overline{y} = \bar{\sigma}_\mu y^\mu$

12.4. **Lorentz spin transformations.**
Let a matrix $\Lambda \in SL(2,\mathbb{C})$ act on hermitian matrix $\underline{x}$ as

$$\underline{x} \mapsto \underline{x}' = \Lambda \underline{x} \Lambda^\dagger\,. \tag{12.25}$$

(a) Check that $\underline{x}'$ is Hermitian, and the Minkowski scalar product (12.23) is preserved.
(b) Show that Λ explicitly acts as a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.
(c) Show that the mapping from a $\Lambda \in SL(2,\mathbb{C})$ to the Lorentz transformation in SO$(1,3)$ is two-to-one.
(d) Consider the Levi-Civita tensor $\epsilon = -\epsilon^{-1} = -\epsilon^{\mathrm{T}}$ in two dimensions,

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{12.26}$$

and the associated symplectic form

$$\langle u, v\rangle = u^{\mathrm{T}}\varepsilon v = u^1 v^2 - u^2 v^1\,. \tag{12.27}$$

Show that this symplectic form is $SL(2,\mathbb{C})$-invariant

$$\langle u, v\rangle = u^{\mathrm{T}}\varepsilon v = \langle \Lambda u, \Lambda v\rangle = u^{\mathrm{T}}\Lambda^{\mathrm{T}}\varepsilon\Lambda v\,. \tag{12.28}$$

12.5. **The unbearable lightness of SO(4) Lie algebra.** Download John Wood's (click here) notes. The challenge: achieve some elegance in deriving the SO(4) commutator bracket relations, for example reduce the number of steps in the calculation by 30% or 50%.
The prize: a case of beer, details to be negotiated with John.

The challenges start on p. 9-8, following eq. (9.21), i.e., “(i)”, “(iv)”, and “(v)”. For instance, on p. 9-11 John indicates all of the cancellations. These suggest that his solution is “calculating zero” unnecessarily. One could take linear combinations of the operators that possess these commutator bracket relations; but the combinations do not seem a priori warranted on the basis of the dynamics of the problem.

(J. Wood)