group theory - week 11

SU(2) and SO(3)

Georgia Tech PHYS-7143

Homework HW11 due Tuesday 2021-07-13

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 11.1 *The characters of $SO(3)$ representations* 4 points

Bonus points
Exercise 11.2 *Real and pseudo-real representations of $SO(3)$* 4 points
Exercise 11.3 *Total spin of N particles* 5 points

Total of 4 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2021-07-01 Predrag Lecture 21 SU(2) and SO(3)

The fastest way to watch any week's lecture videos is by letting YouTube run the

- *lecture playlist*
- Gutkin notes, Lect. 9 $SU(2)$, $SO(3)$ *and their representations*, Sects. 1-3.2.
- *14.1 Recap: Irreps of SO(2)* are not what you would have expected (24:37 min)
- *14.2 Defining reps of SO(3) and SU(2)* (5:18 min)
- *14.3 Cartan root lattices; irreps of SO(3)* (22:50 min)
  - Read sect. 11.2 SU(2) $\simeq$ SO(3)
  - For overall clarity and pleasure of reading, I like Schwichtenberg [2] (click here) discussion best. If you read anything for this week's lectures, read Schwichtenberg sects. 3.4 to 3.6.

2021-07-01 Predrag Lecture 22 SO(3) in QM

- *Andrew Scherbakov: Eigenvalues of J^2 and J_z operators* (10:56 min)
- *Andrew Scherbakov: Raising, lowering operators J_+, J_-* (10:40 min)
  - (optional) *14.3A Who ordered J+, J-?* (7:37 min)
- *Andrew Scherbakov: J_x, J_y, and J_z operators for spin-1 particle* (3:58 min)

11.1 Discussion (optional)

- (Still to be uploaded) *14.4 Discussion: How is SU(2) a double cover of SO(3) and what are the physical consequences? For dimensions higher than four, SO(n) and SU(n) get a divorce, and so far quaternions, octonion ideas have not panned out; instead, particle physics has followed ideas of internal SU(2), SU(3) etc symmetries, that currently culminate in the Standard Model. Negative dimensions.* (X:XX min)

11.1.1 Recap of the course, so far (optional)

Predrag This course is all about *class* (physically distinct symmetry operations) and *character* (mining numbers from symmetries).

Here are some question by the dream student Henriette Roux (pseudonym) that I have answered in part in class discussions, but still have to write up:

Henriette Roux Why is it that the Fourier transformation works? The presence of a discrete but infinite translational symmetry in a system calls for its use of it to diagonalize the matrix and thus make calculations easier, but exactly *why* is the Fourier transform able to do this?

Henriette Roux How is this Fourier transform as we have studied in the space/point groups section related to that which we have derived from the projection operators?

Henriette Roux As an extension of the Fourier transform, are there any equivalent of Fourier transforms for rotations or other infinite but discrete symmetries as well? So for example, if there is a system with a discrete but infinite rotational symmetry, is there a "rotational" transform where the representing matrix is diagonalized? Are there whole classes of such transformations?

Henriette Roux You say that position and momentum are "dual" to each other, and so is the real space and reciprocal space (I guess it's the same thing as position and momentum but just for argument sake). The commonality between these are the fact that they can be Fourier transformed from one space to another. Does this mean that unitary operations, e^{iHt}, suggest a Fourier transform from the "energy" or "frequency" space to "time" space as well?

Henriette Roux This seems very closely related to Noether's theorem as well, is there a way to explain this similarity?

Henriette Roux The special thing about Lie groups is that there exist analytic functions which link $g(a)$ and $c = f(b, a)$ for $g(c) = g(a)g(b)$. Does this need for analytic functions come from the fact that to construct a group manifold, the maps relating different "local" Euclidean spaces need to be C^∞, or smooth? If so, is there a reference we can refer to which explains how the Lie groups satisfy all the other conditions of a manifold (establishing an open ball, building an atlas and so on) as well? Just as an extension, how do you even study groups which do not fall under the realm of a manifold? Don't common functions like differentials and integrations not apply in spaces outside a manifold?

Henriette Roux Why is that we Taylor expand the group in the first place? How is this connected to the shift to left/right group operators?

The next few questions are about General Relativity, and how is what is covered in this course applicable to GR:

Henriette Roux We keep to the first order in the expansion for $g(\theta)$ as we are considering the tangent space to the manifold. In the context of the GR, the tangent space was defined as the space of directional derivatives at a point. In our case, we are studying groups, which are not, in general, vectors (well I guess they can be [1×1] vectors/matrices but that's only specific irreps, so how do we understand the concept of tangent space as you have define it?

- Or does it work out since Lie groups are always Abelian and thus have an infinite number of 1D irreps?

- What happens if we keep the expansion to the 2nd order? Does the mathematics change in any way? Is there a good reason to ignore the 2nd and higher order expansions, not just in the physics sense (keeping to largest order of significance) but in the mathematical way of understanding things?

Question 11.1. Henriette Roux asks

Q Why is this complex 2-dimensional vector called a 'spinor'?

A Historical, as Arfken, Weber & Harris [1] explain: "It turns out that half-integral angular momentum states are needed to describe the intrinsic angular momentum of the electron and many other particles. Since these particles also have magnetic moments, an intuitive interpretation is that their charge distributions are spinning about some axis; hence the term spin. It is now understood that the spin phenomena cannot be explained consistently by describing these particles as ordinary charge distributions undergoing rotational motion, [...] "

Schwichtenberg [2]: "[...] spinors have properties that usual vectors do not have. For instance, the factor $1/2$ in the exponent. This factor shows us that a spinor1 is after a rotation by 2π not the same, but gets a minus sign. This is a pretty crazy property, because all objects we deal with in everyday life are exactly the same after a rotation by $360^o = 2\pi$.

11.2 SU(2) and SO(3)

K. Y. Short

An element of $\mathrm{SU}(2)$ can be written as

$$U_{\mathbf{n}}(\phi) = e^{i\phi\,\sigma\cdot\hat{\mathbf{n}}/2} \tag{11.1}$$

where σ_j is a Pauli matrix and ϕ is a real number. What is the significance of the $1/2$ factor in the argument of the exponential?

Consider a generic position vector $\boldsymbol{x} = (x, y, z)$ and construct a Hermitian matrix of the form

$$\begin{aligned}\sigma\cdot\boldsymbol{x} &= \sigma_x x + \sigma_y y + \sigma_z z \\ &= \begin{pmatrix}0 & x\\ x & 0\end{pmatrix} + \begin{pmatrix}0 & -iy\\ iy & 0\end{pmatrix} + \begin{pmatrix}z & 0\\ 0 & -z\end{pmatrix} \\ &= \begin{pmatrix}z & x-iy\\ x+iy & -z\end{pmatrix}\end{aligned} \tag{11.2}$$

Its determinant

$$\det\begin{pmatrix}z & x-iy\\ x+iy & -z\end{pmatrix} = -(x^2+y^2+z^2) = -\boldsymbol{x}^2 \tag{11.3}$$

gives the length of a vector. Consider a $\mathrm{SU}(2)$ transformation (11.1) of this matrix, $U^\dagger(\sigma\cdot\boldsymbol{x})U$. Taking the determinant, we find the same expression as before:

$$\det U(\sigma\cdot\boldsymbol{x})U^\dagger = \det U\,\det(\sigma\cdot\boldsymbol{x})\,\det U^\dagger = \det(\sigma\cdot\boldsymbol{x})\,. \tag{11.4}$$

Just as $\mathrm{SO}(3)$, $\mathrm{SU}(2)$ preserves the lengths of vectors.

To make the correspondence between $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ more explicit, consider a $\mathrm{SU}(2)$ transformation on a complex two-component *spinor*

$$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{11.5}$$

related to $\boldsymbol{x}$ by

$$x = \frac{1}{2}(\beta^2 - \alpha^2), \quad y = -\frac{i}{2}(\alpha^2 + \beta^2), \quad z = \alpha\beta \tag{11.6}$$

Check that a $\mathrm{SU}(2)$ transformation of ψ is equivalent to a $\mathrm{SO}(3)$ transformation on $\boldsymbol{x}$. From this equivalence, one sees that a $\mathrm{SU}(2)$ transformation has three real parameters that correspond to the three rotation angles of $\mathrm{SO}(3)$. If we label the "angles" for the $\mathrm{SU}(2)$ transformation by α, β, and γ, we observe, for a "rotation" about $\hat{x}$

$$U_x(\alpha) = \begin{pmatrix} \cos\alpha/2 & i\sin\alpha/2 \\ i\sin\alpha/2 & \cos\alpha/2 \end{pmatrix}, \tag{11.7}$$

for a "rotation" about $\hat{y}$,

$$U_y(\beta) = \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix}, \tag{11.8}$$

and for "rotation" about $\hat{z}$,

$$U_z(\gamma) = \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix}. \tag{11.9}$$

Compare these three matrices to the corresponding $\mathrm{SO}(3)$ rotation matrices:

$$R_x(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\zeta & \sin\zeta \\ 0 & -\sin\zeta & \cos\zeta \end{pmatrix}, \quad R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$
$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.10}$$

They're equivalent! Result: *Half the rotation angle generated by SU(2) corresponds to a rotation generated by SO(3).*

What does this mean? At this point, probably best to switch to Schwichtenberg [2] (click here) who explains clearly that $\mathrm{SU}(2)$ is a simply-connected group, and thus the "mother" or covering group, or the double cover of $\mathrm{SO}(3)$. This means there is a two-to-one map from $\mathrm{SU}(2)$ to $\mathrm{SO}(3)$; an $\mathrm{SU}(2)$ turn by 4π corresponds to an $\mathrm{SO}(3)$ turn by 2π. So, the building blocks of your 3-dimensional world are not 3-dimensional real vectors, but the 2-dimensional complex spinors! Quantum mechanics chose electrons to be spin 1/2, and there is nothing Fox Channel can do about it.

References

[1] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*, 7th ed. (Academic, New York, 2013).

[2] J. Schwichtenberg, *Physics from Symmetry* (Springer, Berlin, 2015).

Exercises

11.1. **The characters of SO(3) representations:** Show that for an irrep labeled by j, the character of a conjugacy class labeled by θ

$$\chi^{(j)}(\theta) = \frac{\sin(j+1/2)\theta}{\sin(\theta/2)} \tag{11.11}$$

can be obtained by taking the trace of $R_z^j(\theta)$. Verify that for $j = 1$ this character is the three dimensional special orthogonal representation character (10.11).

11.2. **Real and pseudo-real representations of SO(3).** Recall (Gutkin notes, sect. 4.5 *Representation Theory II*, Sect. 5 *5. Three types of representations*) that there are exist three types of representation which can be distinguished by the indicator (4.6):

$$\int_G d\mu(g)\chi^{(\ell)}(g^2) = \begin{cases} +1 & \text{real} \\ 0 & \text{complex} \\ -1 & \text{pseudo-real} \end{cases} . \tag{11.12}$$

Determine for which values of $\ell = 0, 1/2, 1, 3/2, 2 \ldots$ the representation D_ℓ of SO(3) is real or pseudo-real.

Hint: The characters and Haar measure (10.13) of SO(3) are given by

$$\chi^{(\ell)}(g) = \frac{\sin\left(\left[\ell+\frac{1}{2}\right]\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}, \qquad d\mu(g) = \frac{d\theta}{\pi}\sin^2(\theta/2) \tag{11.13}$$

where θ is rotation angle for the group element g.

(B. Gutkin)

11.3. **Total spin of N particles.** Consider a system of four particles with spin $1/2$. Assuming that all (except spin) degrees of freedom are frozen the Hilbert space of the system is given by $V = V_{1/2} \otimes V_{1/2} \otimes V_{1/2} \otimes V_{1/2}$, with $V_{1/2}$ being two-dimensional space for each spin. $V = \oplus V_s$ can be decomposed then into different sectors V_s having the total spin s i.e., $\hat{S}^2 v = s(s+1)v$, for any $v \in V_s$. Here $\hat{S}^2 = (\sum_{i=1}^4 \hat{s}_i)^2$ and $\hat{s}_i = (\hat{s}_i^x, \hat{s}_i^y, \hat{s}_i^z)$ is spin operator for i-th particle.

(a) What are possible values s for the total spin of the system?

(b) Determine dimension of the subspace of V_0 with 0 total spin. In other words: how many times trivial representation enters into product:

$$D = D_{1/2} \otimes D_{1/2} \otimes D_{1/2} \otimes D_{1/2}\,? \tag{11.14}$$

(c) What is the answer to the above questions for N spins?

Hint: it is convenient to use (11.13) to decompose D into irreps.

(B. Gutkin)

11.4. **Splitting of degeneracies in a central potential.** Hamiltonian H_0 has rotational symmetry of SO(3).

(a) What are the possible energy level degeneracies of H_0?

A weak perturbation V with a symmetry T_d of full tetrahedron group is added (e.g., V is a potential created by lattice of atoms with a symmetry of T_d).

(b) What will be the degeneracies of new Hamiltonian $H_0 + V$?

(c) Assuming that the total angular momentum of the system before the perturbation is $l = 2$. How the degeneracies of the corresponding energy level will be split after the perturbation is applied?

(B. Gutkin)

11.5. **Quadrupole transitions.**
a) Write $Q_1 = xy$, $Q_2 = zy$, $Q_3 = x^2 - y^2$ and $Q_4 = 2z^2 - x^2 - y^2$ as components of spherical tensor of rank 2. *Hint:* use spherical harmonics $Y_l^m(\theta, \varphi)$.
b) The last quantity Q_4 is known as quadrupole moment. What are the selection rules for transitions induced by Q_4 in a system with SO(3) symmetry? In other words, for which m, l and k, j the transition rates:

$$P_{m,l\to k,j} \sim |\langle m\, l|Q_4|j\, k\rangle|^2$$

are non-zero?
c) By using Wigner-Eckart theorem write down the relationship between $|\langle m\, l|Q_4|j\, k\rangle|^2$ and $|\langle m\, l|Q_1|j\, k\rangle|^2$ in terms of Clebsch-Gordan coefficients.

(B. Gutkin)