group theory - week 10

Lie groups, algebras

Georgia Tech PHYS-7143

Homework HW10 due Thursday 2021-07-08

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 10.1	*Conjugacy classes of $SO(3)$*	2 points (+ 2 bonus points, if complete)
Exercise 10.2	*The character of $SO(3)$ 3-dimensional representation*	1 point
Exercise 10.3	*The orthonormality of $SO(3)$ characters*	2 point
Exercise 10.4	*$U(1)$ equivariance of two-modes system for finite angles*	3 points
Exercise 10.6	*$SO(2)$ or harmonic oscillator slice*	2 points

Bonus points

Exercise 10.5	*Integrate the two-modes system*	4 point
Exercise 10.7	*Invariant subspace of the two-modes system*	1 point
Exercise 10.8	*Slicing the two-modes system*	1 point
Exercise 10.9	*The symmetry reduced two-modes flow*	(difficult) 6 points

Total of 10 points = 100 % score.

2021-06-24 Predrag Lecture 19 Lie groups, algebras

The fastest way to watch any week's lecture videos is by letting YouTube run the

lecture playlist

There is way too much material in this week's notes. Watch the main sequence of video clips, that and recommended reading should suffice. The rest is optional.

- Bridging the step from discrete to continuous compact groups: invariant integration measures, characters, character orthonormality and completeness relations:

Rotations in 3 dimensions (30 min)

Lie algebra (21 min)

Birdtracks (6 min)

- ◦ Sect. 10.7 *Lie groups for pedestrians* is advanced material, safely ignored, here only to whet your appetite for things not done in 19th century. It is a very condensed extract of chapters 3 *Invariants and reducibility* and 4 *Diagrammatic notation* from *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [11]. I am usually reluctant to use birdtrack notations in front of graduate students indoctrinated by their professors in the 1890's tensor notation, but now I'm emboldened by the very enjoyable article on *The new language of mathematics* by Dan Silver [22].
- ◦ Ditto for sect. 10.10 *Birdtracks - updated history*. Your professor's notation is as convenient for actual calculations as -let's say- long division using roman numerals. So leave them wallowing in their early progressive rock of 1968, King Crimsons of their youth. You chill to beats younger than Windows 98, to grime, to trap, to hardvapour, to birdtracks.
- ◦ Go to week 16 to learn more.

- OK, I see that formally $SU(2) \simeq SO(3)$, but who ordered "spin?"

Rotations in 2 complex dimensions (42 min)

- ◦ Read sect. 10.3 *SU(2) Pauli matrices*

For overall clarity and pleasure of reading, I like Schwichtenberg [21] (click here) discussion best. If you read anything for this week's lectures, read Schwichtenberg sects. 3.4 to 3.6.

2021-06-29 Predrag Lecture 20 O(2) symmetry sliced

- ◦ sect. 10.4 *Two-modes SO(2)-equivariant flow*

lecture playlist

- *In two real dimensions dynamics is boring* (4:28 min)
- *Symmetries of solutions* (18 min)
- *Symmetry reduction* (1:44 min)
- *Moving frames: with freedom comes responsibility* (8:35 min)
- *Phase of a relative periodic orbit, choice of moving frame* (9:18 min)
- *Comoving frames* (23:09 min)
- *Slice* (4:00 min)
- *How to slice a continuous symmetry* (14:16 min)
  - optional: *Low dimensional slices; 2D flat heart* (1:40 min)
- *Slices are not sections!* (17 sec)
  - optional: *Cross-sections, orbitfolds* (1:18 min)
- *Symmetry reduced equations of motion* (6:12 min)
- *Sections and slices are local, good up to a border* (1:18 min)
- *A spatial Fourier expansion* (5:11 min)
- *First Fourier mode slice* (3:00 min)
- *In-slice time* (1:54 min)

For the two-modes SO(2)-equivariant flow long version, see

- ChaosBook example *Two-modes flow*.
- ChaosBook chapt. *Slice & dice*, sect. 13.1 *Only dead fish go with the flow* to sect. 13.5 *First Fourier mode slice*.

This is difficult material, so it is OK if you do not get it this time around. None of this will be on the final - the main point is that once you face a nonlinear problem, nothing is easy - not even rotations on a circle.

10.1 Other sources (optional)

- You can glance through
  - sect. 10.5 *SO(3) character orthogonality*
  - sect. 10.6 *Linear algebra*

but I do not expect you to master this material.

- C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, works out in full detail the representations and Haar measures for 1-dimensional Lie groups, and explains the difference between rotations and translations.

- Chen, Ping and Wang [7] *Group Representation Theory for Physicists*, Sect 5.3 *Lie algebras* and Sect 5.4 *Finite transformations* work out several $\mathrm{SU}(2)$ and $\mathrm{O}(3)$ examples (click here). Sects 5.5, 5.6 and 5.7 also merit a quick read.

- In his group theory notes D. Vvedensky, chapter 8, sect. 8.3 *Axis–angle representation of proper rotations in three dimensions*, has a very nice discussion of the (10.7) parametrization of the $\mathrm{SO}(3)$ 3-dimensional group manifold: the parameter space corresponds to the interior of a sphere of radius π, and the over the classes of $\mathrm{SO}(3)$ is given by integral over spherical shells. In sect. 8.4 he derives the Haar measure (without calling it so).

  In sect. 8.5 Vvedensky says: "For $\mathrm{SO}(2)$, we were able to determine the characters of the irreducible representations directly, i.e., without having to determine the basis functions of these representations. The structure of $\mathrm{SO}(3)$, however, does not allow for such a simple procedure, so we must determine the basis functions from the outset." That I disagree with; in birdtracks.eu sect. 15.1 *Reps of $SU(2)$* I construct the irreps and label them by their Young tableaus with no recourse to spherical harmonics.

- Reading: Chen, Ping and Wang [7] *Group Representation Theory for Physicists*, Sect 5.2 *Definition of a Lie group, with examples* (click here).

- Dirac belt trick applet

- If still anxious, maybe this helps: Mark Staley, *Understanding quaternions and the Dirac belt trick* arXiv:1001.1778.

- ChaosBook Sect 26.1 *Compact groups*

- I have enjoyed reading Mathews and Walker [16] Chap. 16 *Introduction to groups* (click here). Goldbart writes that the book is "based on lectures by Richard Feynman at Cornell University." Very clever. In particular, work through the example of fig. 16.2: it is very cute, you get explicit eigenmodes from group theory alone. The main message is that if you think things through first, you never have to go through using explicit form of representation matrices - thinking in terms of invariants, like characters, will get you there much faster.

- Any book, of 100s available, like Cornwell [8] *Group Theory in Physics: An introduction* that covers group theory might be more to your taste.

- *Hamilton's quaternions* (3:18 min)

- Stone and Goldbart [24] (click here) Chapter 17 Sect 17.6 *Analytic functions and topology* (wherein stereographic projection is revealed to be the geometric origin of the spinor representations of the rotation group)

- This week's lectures are related to AWH Chapter 3 *Vector Analysis* (click here) and Chapter 16 *Angular Momentum* (click here).

10.2 Discussion (optional)

Question 10.1. Henriette Roux asks
Q Why is this complex 2-dimensional vector called a 'spinor'?
A Historical, as Arfken, Weber & Harris [4] explain: "It turns out that half-integral angular momentum states are needed to describe the intrinsic angular momentum of the electron and many other particles. Since these particles also have magnetic moments, an intuitive interpretation is that their charge distributions are spinning about some axis; hence the term spin. It is now understood that the spin phenomena cannot be explained consistently by describing these particles as ordinary charge distributions undergoing rotational motion, [...] "

Schwichtenberg [21]: "[...] spinors have properties that usual vectors do not have. For instance, the factor $1/2$ in the exponent. This factor shows us that a spinor1 is after a rotation by 2π not the same, but gets a minus sign. This is a pretty crazy property, because all objects we deal with in everyday life are exactly the same after a rotation by $360^o = 2\pi$.

Question 10.2. Henriette Roux asks
Q What' relation of Pauli exclusion principle to the spinor 2π rotation amounting to overall minus sign?
A I think of fermion/Grassmann statistics as Archimedes principle + linearity, see my Field Theory [10] *chap. 4 Fermions*. Basically, usually a constraint is imposed by eliminating a variable, for example, given the constraint is $x^2 + y^2 + z^2 = 1$, one gets rid of z by replacing it everywhere with $z \to \sqrt{1 - x^2 - y^2}$. This makes a fully symmetric theory asymmetric and ugly. In linear setting, another option is to keep all the variables and the symmetry, but add a new variable which by construction subtracts a degree of freedom, what I call [12] a "negative dimension". In quantum field theory such variable is called a 'ghost'; it needs to be anti-commuting or Grassmann.

Question 10.3. Henriette Roux asks
Q This course is all about eigenfunctions of symmetry operators. Why are you not teaching us Bessel functions?
A Blame Feynman: On May 2, 1985 my stay at Cornell was to end, and Vinnie of college town *Italian Kitchen* made a special dinner for three of us regulars. Das Wunderkind noticed Feynman ambling down Eddy Avenue, kidnapped him, and here we were, two wunderkinds, two humans.

Feynman was a very smart, forever driven wunderkind. He naturally bonded with our very smart, forever driven wunderkind, who suddenly lurched out of control, and got very competive about at what age who summed which kind of Bessel function series. Something like age twelve, do not remember which one did the Bessels first. At that age I read *Palle Alone in the World*, while my nonwunderkind friend, being from California, watched television 12 hours a day.

When Das Wunderkind taught graduate E&M, he spent hours crafting lectures about symmetry groups and their representations as various eigenfunctions. Students were not pleased.

So, fuggedaboutit! if you have not done your Bessels yet, they are eigenfunctions, just like your Fourier modes, but for a spherical symmetry rather than for a translation symmetry; wiggle like a cosine, but decay radially.

When you need them you'll figure them out. Or sue me.

Question 10.4. Predrag asks
Q You are the best of students now. Are you ready for The Talk?
A Henriette Roux: I'm ready!

10.3 SU(2) Pauli matrices

A lightning, bullet points review.

- $\mathrm{U}(n)$: unitary transformation $U = e^{iH}$
- Unitarity: $U^\dagger U = \mathbf{1} \Rightarrow H^\dagger = H$, the generator is hermitian.
- $\mathrm{SU}(n)$: special unitary transformation $\det U = 1$
- Must know: $\ln\det\ = \mathrm{tr}\ \ln$ for any matrix, so the generator is traceless $\ln\det U = \mathrm{tr}\ \ln U\ = \mathrm{tr}\, H = 0$
- $\mathrm{SU}(2) : H = \begin{pmatrix} a & c \\ e & b \end{pmatrix}, \quad a, b, c, e \in \mathbb{C}$, eight real numbers in all.
- H is hermitian: $H = \begin{pmatrix} a & c+id \\ c-id & b \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}$,
- H is traceless: $0 = \mathrm{tr}\, H \Rightarrow a + b = 0$, three real rotation parameters in all, so

$$
\begin{aligned}
H &= c\sigma_x + d\sigma_y + a\sigma_z \\
&= c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{10.1}
$$

where σ_j are known as Pauli matrices.

10.4 Two-modes SO(2)-equivariant flow

Consider the pair of $\mathrm{U}(1)$-equivariant complex ODEs

$$
\begin{aligned}
\dot{z}_1 &= (\mu_1 - \mathrm{i}\, e_1)\, z_1 + a_1\, z_1|z_1|^2 + b_1\, z_1|z_2|^2 + c_1\, \overline{z}_1\, z_2 \\
\dot{z}_2 &= (\mu_2 - \mathrm{i}\, e_2)\, z_2 + a_2\, z_2|z_1|^2 + b_2\, z_2|z_2|^2 + c_2\, z_1^2\,,
\end{aligned} \tag{10.2}
$$

with $z_1,\ z_2$ complex, and all parameters real valued.

This system is a generic example of a few-modes truncation of a Fourier representation of some physical flow, such as fluid dynamics convection flow, truncated in such a way that the model exhibits the same symmetries as the full original problem, while being drastically simpler to study. It is a merely a toy model with no physical interpretation, just like the iconic Lorenz flow. We use it to illustrate the effects of continuous symmetry on chaotic dynamics.

We refer to this toy model as the *two-modes* system. It belongs to the family of simplest ODE systems that we know that (a) have a continuous $\mathrm{U}(1) \simeq \mathrm{SO}(2)$, but no discrete symmetry (if at least one of $e_j \neq 0$). (b) models ‘weather’, in the same sense that Lorenz equation models ‘weather’, (c) exhibits chaotic dynamics, (d) can be easily visualized, in the dimensionally lowest possible setting required for chaotic dynamics, with the full state space of dimension $d = 4$, and the $\mathrm{SO}(2)$-reduced dynamics taking place in 3 dimensions, and (e) for which the method of slices reduces the symmetry by a single global slice hyperplane.

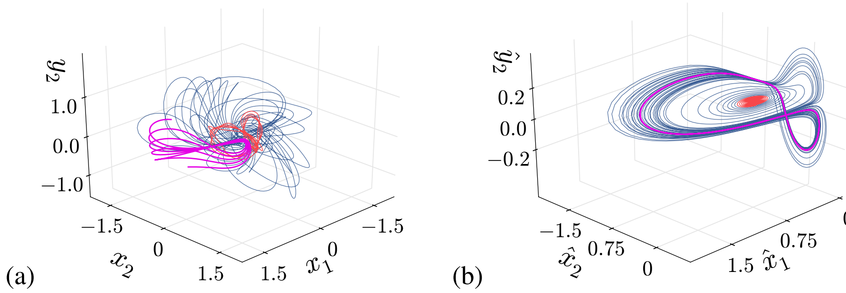


Figure 10.1: Two-modes flow before (a) and after (b) symmetry reduction by first Fourier mode slice. Here a long trajectory (red and blue) starting on the unstable manifold of the TW_1 (red), until it falls on to the strange attractor (blue) and the shortest relative periodic orbit $\overline{1}$ (magenta). Note that the relative equilibrium becomes an equilibrium, and the relative periodic orbit becomes a periodic orbit after the symmetry reduction.

The model has an unreasonably high number of parameters. After some experimentation we fix or set to zero various parameters, and in the numerical examples that follow, we settle for parameters set to

$$\begin{aligned} \mu_1 &= -2.8\,,\ \mu_2 = 1\,,\ e_1 = 0\,,\ e_2 = 1\,, \\ a_1 &= -1\,,\ a_2 = -2.66\,,\ b_1 = 0\,,\ b_2 = 0\,,\ c_1 = -7.75\,,\ c_2 = 1\,, \end{aligned} \tag{10.3}$$

unless explicitly stated otherwise. For these parameter values the system exhibits chaotic behavior. Experiment! If you find a more interesting behavior for some other parameter values, please let us know. The simplified system of equations can now be written as a 3-parameter $\{\mu_1, c_1, a_2\}$ two-modes system,

$$\begin{aligned} \dot{z}_1 &= \mu_1\, z_1 - z_1|z_1|^2 + c_1\, \overline{z}_1\, z_2 \\ \dot{z}_2 &= (1-\mathrm{i})\, z_2 + a_2\, z_2|z_1|^2 +\, z_1^2\,. \end{aligned} \tag{10.4}$$

In order to numerically integrate and visualize the flow, we recast the equations in real variables by substitution $z_1 = x_1 + i\, y_1$, $z_2 = x_2 + i\, y_2$. The two-modes system (10.2) is now a set of four coupled ODEs

exercise 10.5

$$\begin{aligned} \dot{x}_1 &= (\mu_1 - r^2)\, x_1 + c_1\, (x_1 x_2 + y_1 y_2)\,, \qquad r^2 = x_1^2 + y_1^2 \\ \dot{y}_1 &= (\mu_1 - r^2)\, y_1 + c_1\, (x_1 y_2 - x_2 y_1) \\ \dot{x}_2 &= x_2 + y_2 + x_1^2 - y_1^2 + a_2 x_2 r^2 \\ \dot{y}_2 &= -x_2 + y_2 + 2\, x_1 y_1 + a_2 y_2 r^2\,. \end{aligned} \tag{10.5}$$

Try integrating (10.5) with random initial conditions, for long times, times much beyond which the initial transients have died out. What is wrong with this picture? Figure 10.4 (a) is a mess. As we show here, the attractor is built up by a nice 'stretch & fold' action, hidden from the view by the continuous symmetry induced drifts. That is fixed by 'quotienting' model's SO(2) symmetry, and reducing the dynamics to a 3-dimensional symmetry-reduced state space, figure 10.4 (b).

exercise 10.6
exercise 10.7
exercise 10.8

10.5 SO(3) character orthogonality (optional)

In 3 Euclidean dimensions, a rotation around z axis is given by the SO(2) matrix

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp\varphi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{10.6}$$

An arbitrary rotation in $\mathbb{R}^3$ can be represented by

$$R_{\boldsymbol{n}}(\varphi) = e^{-i\varphi\, \boldsymbol{n}\cdot\boldsymbol{L}} \qquad \boldsymbol{L} = (L_1, L_2, L_3)\,, \tag{10.7}$$

where the unit vector $\boldsymbol{n}$ determines the plane and the direction of the rotation by angle φ. Here L_1, L_2, L_3 are the generators of rotations along x, y, z axes respectively,

$$L_1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_2 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10.8}$$

with Lie algebra relations

$$[L_i, L_j] = i\varepsilon_{ijk} L_k\,. \tag{10.9}$$

All SO(3) rotations by the same angle θ around different rotation axis $\boldsymbol{n}$ are conjugate to each other,

$$e^{i\phi\boldsymbol{n}_2\cdot\boldsymbol{L}} e^{i\theta\boldsymbol{n}_1\cdot\boldsymbol{L}} e^{-i\phi\boldsymbol{n}_2\cdot\boldsymbol{L}} = e^{i\theta\boldsymbol{n}_3\cdot\boldsymbol{L}}\,, \tag{10.10}$$

with $e^{i\phi\boldsymbol{n}_2\cdot\boldsymbol{L}}$ and $e^{-i\theta\boldsymbol{n}_2\cdot\boldsymbol{L}}$ mapping the vector $\boldsymbol{n}_1$ to $\boldsymbol{n}_3$ and back, so that the rotation around axis $\boldsymbol{n}_1$ by angle θ is mapped to a rotation around axis $\boldsymbol{n}_3$ by the same θ. The conjugacy classes of SO(3) thus consist of rotations by the same angle about all distinct rotation axes, and are thus labelled the angle θ. As the conjugacy class depends only on θ, the characters can only be a function of θ. For the 3-dimensional special orthogonal representation, the character is

exercise 10.2

$$\chi = 2\cos(\theta) + 1\,. \tag{10.11}$$

For an irrep labeled by j, the character of a conjugacy class labeled by θ is

$$\chi^{(j)}(\theta) = \frac{\sin(j+1/2)\theta}{\sin(\theta/2)} \tag{10.12}$$

To check that these characters are orthogonal to each other, one needs to define the group integration over a parametrization of the SO(3) group manifold. A group element is parametrized by the rotation axis $\boldsymbol{n}$ and the rotation angle $\theta \in (-\pi, \pi]$, with $\boldsymbol{n}$ a unit vector which ranges over all points on the surface of a unit ball. Note however, that a π rotation is the same as a $-\pi$ rotation ($\boldsymbol{n}$ and $-\boldsymbol{n}$ point along the same direction), and the $\boldsymbol{n}$ parametrization of SO(3) is thus a 2-dimensional surface of a unit-radius ball with the opposite points identified.

The Haar measure for SO(3) requires a bit of work, here we just note that after the integration over the solid angle (characters do not depend on it), the Haar measure is

$$dg = d\mu(\theta) = \frac{d\theta}{2\pi}(1-\cos(\theta)) = \frac{d\theta}{\pi}\sin^2(\theta/2)\,. \tag{10.13}$$

exercise 10.3

With this measure the characters are orthogonal, and the character orthogonality theorems follow, of the same form as for the finite groups, but with the group averages replaced by the continuous, parameter dependant group integrals

$$\frac{1}{|G|}\sum_{g\in G} \to \int_G dg\,.$$

The good news is that, as explained in ChaosBook.org Chap. *Relativity for cyclists* (and in *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [11]), one never needs to actually explicitly construct a group manifold parametrizations and the corresponding Haar measure.

10.6 Linear algebra (optional)

In this section we collect a few basic definitions. A sophisticated reader might prefer skipping straight to the definition of the Lie product (10.21), the big difference between the group elements product used so far in discussions of finite groups, and what is needed to describe continuous groups.

Vector space. A set V of elements $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ is called a *vector* (or *linear*) *space* over a field $\mathbb{F}$ if

(a) *vector addition* "+" is defined in V such that V is an abelian group under addition, with identity element $\mathbf{0}$;

(b) the set is *closed* with respect to *scalar multiplication* and vector addition

$$\begin{aligned} a(\mathbf{x}+\mathbf{y}) &= a\mathbf{x}+a\mathbf{y}\,, \qquad a,b\in\mathbb{F}\,, \quad \mathbf{x},\mathbf{y}\in V \\ (a+b)\mathbf{x} &= a\mathbf{x}+b\mathbf{x} \\ a(b\mathbf{x}) &= (ab)\mathbf{x} \\ 1\,\mathbf{x} &= \mathbf{x}\,, \qquad 0\,\mathbf{x}=\mathbf{0}\,. \end{aligned} \tag{10.14}$$

Here the field $\mathbb{F}$ is either $\mathbb{R}$, the field of reals numbers, or $\mathbb{C}$, the field of complex numbers. Given a subset $V_0 \subset V$, the set of all linear combinations of elements of V_0, or the *span* of V_0, is also a vector space.

A basis. $\{\mathbf{e}^{(1)}, \cdots, \mathbf{e}^{(d)}\}$ is any linearly independent subset of V whose span is V. The number of basis elements d is the *dimension* of the vector space V.

Dual space, dual basis. Under a general linear transformation $g \in GL(n, \mathbb{F})$, the row of basis vectors transforms by right multiplication as $\mathbf{e}^{(j)} = \sum_k (\mathbf{g}^{-1})^j{}_k \, \mathbf{e}^{(k)}$, and the column of x_a's transforms by left multiplication as $x' = \mathbf{g}x$. Under left multiplication the column (row transposed) of basis vectors $\mathbf{e}_{(k)}$ transforms as $\mathbf{e}_{(j)} = (\mathbf{g}^\dagger)_j{}^k \mathbf{e}_{(k)}$, where the *dual rep* $\mathbf{g}^\dagger = (\mathbf{g}^{-1})^\top$ is the transpose of the inverse of $\mathbf{g}$. This observation motivates introduction of a *dual* representation space $\bar{V}$, the space on which $GL(n, \mathbb{F})$ acts via the dual rep $\mathbf{g}^\dagger$.

Definition. If V is a vector representation space, then the *dual space* $\bar{V}$ is the set of all linear forms on V over the field $\mathbb{F}$.

If $\{\mathbf{e}^{(1)}, \cdots, \mathbf{e}^{(d)}\}$ is a basis of V, then $\bar{V}$ is spanned by the *dual basis* $\{\mathbf{e}_{(1)}, \cdots, \mathbf{e}_{(d)}\}$, the set of d linear forms $\mathbf{e}_{(k)}$ such that

$$\mathbf{e}_{(j)} \cdot \mathbf{e}^{(k)} = \delta_j^k \,,$$

where δ_j^k is the Kronecker symbol, $\delta_j^k = 1$ if $j = k$, and zero otherwise. The components of dual representation space vectors $\bar{y} \in \bar{V}$ will here be distinguished by upper indices

$$(y^1, y^2, \ldots, y^n) \,. \tag{10.15}$$

They transform under $GL(n, \mathbb{F})$ as

$$y'^a = (\mathbf{g}^\dagger)^a{}_b y^b \,. \tag{10.16}$$

For $GL(n, \mathbb{F})$ no complex conjugation is implied by the † notation; that interpretation applies only to unitary subgroups $U(n) \subset GL(n, \mathbb{C})$. In the index notation, $\mathbf{g}$ can be distinguished from $\mathbf{g}^\dagger$ by keeping track of the relative ordering of the indices,

$$(\mathbf{g})_a^b \to g_a{}^b \,, \qquad (\mathbf{g}^\dagger)_a^b \to g^b{}_a \,. \tag{10.17}$$

Algebra. A set of r elements $\mathbf{t}_\alpha$ of a vector space $\mathcal{T}$ forms an algebra if, in addition to the vector addition and scalar multiplication,

(a) the set is *closed* with respect to multiplication $\mathcal{T} \cdot \mathcal{T} \to \mathcal{T}$, so that for any two elements $\mathbf{t}_\alpha, \mathbf{t}_\beta \in \mathcal{T}$, the product $\mathbf{t}_\alpha \cdot \mathbf{t}_\beta$ also belongs to $\mathcal{T}$:

$$\mathbf{t}_\alpha \cdot \mathbf{t}_\beta = \sum_{\gamma=0}^{r-1} \tau_{\alpha\beta}{}^\gamma \mathbf{t}_\gamma \,, \qquad \tau_{\alpha\beta}{}^\gamma \in \mathbb{C} \,; \tag{10.18}$$

(b) the multiplication operation is *distributive*:

$$\begin{aligned} (\mathbf{t}_\alpha + \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma &= \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma + \mathbf{t}_\beta \cdot \mathbf{t}_\gamma \\ \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta + \mathbf{t}_\gamma) &= \mathbf{t}_\alpha \cdot \mathbf{t}_\beta + \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma \,. \end{aligned}$$

The set of numbers $\tau_{\alpha\beta}{}^\gamma$ are called the *structure constants*. They form a matrix rep of the algebra,

$$(\mathbf{t}_\alpha)_\beta{}^\gamma \equiv \tau_{\alpha\beta}{}^\gamma \,, \tag{10.19}$$

whose dimension is the dimension r of the algebra itself.

Depending on what further assumptions one makes on the multiplication, one obtains different types of algebras. For example, if the multiplication is associative

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma = \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta \cdot \mathbf{t}_\gamma)\,,$$

the algebra is *associative*. Typical examples of products are the *matrix product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c\,, \qquad \mathbf{t}_\alpha \in V \otimes \bar{V}\,, \tag{10.20}$$

and the *Lie product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c - (t_\alpha)_c^b (t_\beta)_b^a\,, \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \tag{10.21}$$

which defines a *Lie algebra*.

10.7 Lie groups for pedestrians (optional)

[...] which is an expression of consecration of angular momentum.
— Mason A. Porter's student

Definition: A Lie group is a topological group G such that (i) G has the structure of a smooth differential manifold, and (ii) the composition map $G \times G \to G : (g, h) \to gh^{-1}$ is smooth, i.e., $\mathbb{C}^\infty$ differentiable.

Do not be mystified by this definition. Mathematicians also have to make a living. The compact Lie groups that we will deploy here are a generalization of the theory of SO$(2) \simeq$ U(1) rotations, i.e., Fourier analysis. By a 'smooth differential manifold' one means objects like the circle of angles that parameterize continuous rotations in a plane, figure 10.2, or the manifold swept by the three Euler angles that parameterize SO(3) rotations.

By 'compact' one means that these parameters run over finite ranges, as opposed to parameters in hyperbolic geometries, such as Minkowsky SO$(3, 1)$. The groups we focus on here are compact by default, as their representations are linear, finite-dimensional matrix subgroups of the unitary matrix group U(d).

Example 1. Circle group. A circle with a direction, figure 10.2, is invariant under rotation by any angle $\theta \in [0, 2\pi)$, and the group multiplication corresponds to composition of two rotations $\theta_1 + \theta_2 \mod 2\pi$. The natural representation of the group action is by a complex numbers of absolute value 1, i.e., the exponential $e^{i\theta}$. The composition rule is then the complex multiplication $e^{i\theta_2} e^{i\theta_1} = e^{i(\theta_1+\theta_2)}$. The circle group is a *continuous group*, with infinite number of elements, parametrized by the continuous parameter $\theta \in [0, 2\pi)$. It can be thought of as the $n \to \infty$ limit of the cyclic group C$_n$. Note that the circle divided into n segments is *compact*, in distinction to the infinite lattice of integers $\mathbb{Z}$, whose limit is a *line* (noncompact, of infinite length).

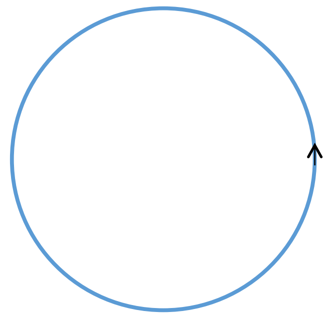


Figure 10.2: Circle group $S^1 = \mathrm{SO}(2)$, the symmetry group of a circle with directed rotations, is a compact group, as its natural parametrization is either the angle $\phi \in [0, 2\pi)$, or the perimeter $x \in [0, L)$.

An element of a $[d \times d]$ -dimensional matrix representation of a *Lie group* continuously connected to identity can be written as

$$g(\boldsymbol{\phi}) = e^{i\boldsymbol{\phi}\cdot T}\,, \qquad \boldsymbol{\phi}\cdot T = \sum_{a=1}^{N} \phi_a T_a\,, \tag{10.22}$$

where $\boldsymbol{\phi} \cdot T$ is a *Lie algebra* element, T_a are matrices called 'generators', and $\boldsymbol{\phi} = (\phi_1, \phi_2, \cdots, \phi_N)$ are the parameters of the transformation. Repeated indices are summed throughout, and the dot product refers to a sum over Lie algebra generators. Sometimes it is convenient to use the Dirac bra-ket notation for the Euclidean product of two real vectors $x, y \in \mathbb{R}^d$, or the product of two complex vectors $x, y \in \mathbb{C}^d$, i.e., indicate complex x-transpose times y by

$$\langle x|y\rangle = x^\dagger y = \sum_i^d x_i^* y_i\,. \tag{10.23}$$

Finite unitary transformations $\exp(i\boldsymbol{\phi}\cdot T)$ are generated by sequences of infinitesimal steps of form

$$g(\boldsymbol{\delta\phi}) \simeq 1 + i\boldsymbol{\delta\phi}\cdot T\,, \quad \boldsymbol{\delta\phi} \in \mathbb{R}^N\,, \quad |\boldsymbol{\delta\phi}| \ll 1\,, \tag{10.24}$$

where T_a, the *generators* of infinitesimal transformations, are a set of linearly independent $[d \times d]$ hermitian matrices (see figure 10.3 (b)).

The reason why one can piece a global transformation from infinitesimal steps is that the choice of the "origin" in coordinatization of the group manifold sketched in figure 10.3 (a) is arbitrary. The coordinatization of the tangent space at one point on the group manifold suffices to have it everywhere, by a coordinate transformation g, i.e., the new origin y is related to the old origin x by conjugation $y = g^{-1}xg$, so all tangent spaces belong the same class, they are geometrically equivalent.

Unitary and orthogonal groups are defined as groups that preserve 'length' norms, $\langle gx|gx\rangle = \langle x|x\rangle$, and infinitesimally their generators (10.24) induce no change in the

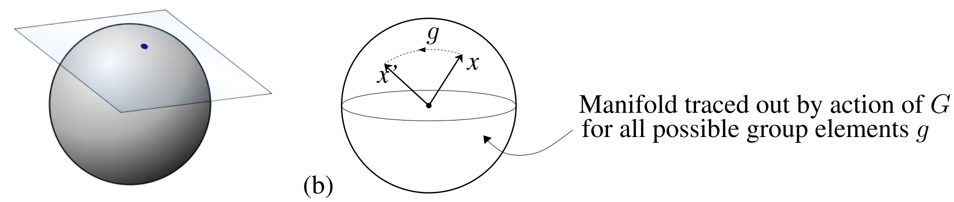


Figure 10.3: (a) Lie algebra fields $\{t_1, \cdots, t_N\}$ span the tangent space of the group orbit $\mathcal{M}_x$ at state space point x, see (10.26) (figure from WikiMedia.org). (b) A global group transformation $g : x \to x'$ can be pieced together from a series of infinitesimal steps along a continuous trajectory connecting the two points. The group orbit of state space point $x \in \mathbb{R}^d$ is the N-dimensional manifold of all actions of the elements of group G on x.

norm, $\langle T_a x|x\rangle + \langle x|T_a x\rangle = 0$, hence the Lie algebra generators T_a are hermitian for,

$$T_a^\dagger = T_a\,. \tag{10.25}$$

The flow field at the state space point x induced by the action of the group is given by the set of N tangent fields

$$t_a(x)_i = (T_a)_{ij} x_j\,, \tag{10.26}$$

which span the d-dimensional *group tangent space* at state space point x, parametrized by $\boldsymbol{\delta\phi}$.

For continuous groups the Lie algebra, i.e., the algebra spanned by the set of N generators T_a of infinitesimal transformations, takes the role that the $|G|$ group elements play in the theory of discrete groups (see figure 10.3).

10.7.1 Invariants

One constructs the irreps of finite groups by identifying matrices that commute with all group elements, and using their eigenvalues to decompose arbitrary representation of the group into a unique sum of irreps. The same strategy works for the compact Lie groups, (10.30), and is indeed the key idea that distinguishes the invariance groups classification developed in *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [11] from the 19th century Cartan-Killing classification of Lie algebras.

Definition. The vector $q \in V$ is an *invariant vector* if for any transformation $g \in \mathcal{G}$

$$q = Gq\,. \tag{10.27}$$

Definition. A tensor $x \in V^p \otimes \bar{V}^q$ is an *invariant tensor* if for any $g \in G$

$$x^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q} = G^{a_1}{}_{c_1} G^{a_2}{}_{c_2} \ldots G_{b_1}{}^{d_1} \ldots G_{b_q}{}^{d_q} x^{c_1 c_2 \ldots c_p}_{d_1 \ldots d_q}\,. \tag{10.28}$$

If a bilinear form $m(\bar{x}, y) = x^a M_a{}^b y_b$ is invariant for all $g \in \mathcal{G}$, the matrix

$$M_a{}^b = G_a{}^c G^b{}_d M_c{}^d \tag{10.29}$$

is an *invariant matrix*. Multiplying with ${G_b}^e$ and using the unitary, we find that the invariant matrices *commute* with all transformations $g \in \mathcal{G}$:

$$[G, \mathbf{M}] = 0\,. \tag{10.30}$$

Definition. An *invariance group* $\mathcal{G}$ is the set of all linear transformations (10.28) that preserve the primitive invariant relations (and, by extension, *all* invariant relations)

$$\begin{aligned} p_1(x, \bar{y}) &= p_1(Gx, \bar{y}G^\dagger) \\ p_2(x, y, z, \ldots) &= p_2(Gx, Gy, Gz \ldots)\,, \qquad \ldots\,. \end{aligned} \tag{10.31}$$

Unitarity guarantees that all contractions of primitive invariant tensors, and hence all composed tensors $h \in H$, are also invariant under action of $\mathcal{G}$. As we assume unitary $\mathcal{G}$, it follows that the list of primitives must always include the Kronecker delta.

Example 2. If $p^a q_a$ is the only invariant of $\mathcal{G}$

$$p'^a q'_a = p^b (G^\dagger G)_b{}^c q_c = p^a q_a\,, \tag{10.32}$$

then $\mathcal{G}$ is the full *unitary group* $U(n)$ (invariance group of the complex norm $|x|^2 = x^b x_a \delta^a_b$), whose elements satisfy

$$G^\dagger G = 1\,. \tag{10.33}$$

Example 3. If we wish the z-direction to be invariant in our 3-dimensional space, $q = (0, 0, 1)$ is an invariant vector (10.27), and the invariance group is $O(2)$, the group of all rotations in the x-y plane.

10.7.2 Infinitesimal transformations, Lie algebras

A unitary transformation G infinitesimally close to unity can be written as

$$G_a{}^b = \delta^b_a + iD^b_a\,, \tag{10.34}$$

where D is a hermitian matrix with small elements, $|D^b_a| \ll 1$. The action of $g \in \mathcal{G}$ on the conjugate space is given by

$$(G^\dagger)_b{}^a = G^a{}_b = \delta^a_b - iD^a_b\,. \tag{10.35}$$

D can be parametrized by $N \le n^2$ real parameters. N, the maximal number of independent parameters, is called the *dimension* of the group (also the dimension of the Lie algebra, or the dimension of the adjoint rep).

Here we shall consider only infinitesimal transformations of form $G = 1 + iD$, $|D^a_b| \ll 1$. We do not study the entire group of invariant transformation, but only the transformations connected to the identity. For example, we shall not consider invariances under coordinate reflections.

The generators of infinitesimal transformations (10.34) are hermitian matrices and belong to the $D^a_b \in V \otimes \bar{V}$ space. However, not any element of $V \otimes \bar{V}$ generates an allowed transformation; indeed, one of the main objectives of group theory is to define the class of allowed transformations.

This subspace is called the *adjoint* space, and its special role warrants introduction of special notation. We shall refer to this vector space by letter A, in distinction to the defining space V. We shall denote its dimension by N, label its tensor indices by $i, j, k \ldots$, denote the corresponding Kronecker delta by a thin, straight line,

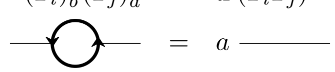
$\delta_{ij} =$  $, \qquad i, j = 1, 2, \ldots, N\,,$ (10.36)

and the corresponding transformation generators by

$(C_A)_{i,}{}^{a}_{b} = \frac{1}{\sqrt{a}}(T_i)^a_b =$  $\qquad a, b = 1, 2, \ldots, n \qquad i = 1, 2, \ldots, N\,.$

Matrices T_i are called the *generators* of infinitesimal transformations. Here a is an (uninteresting) overall normalization fixed by the orthogonality condition

$$(T_i)^a_b (T_j)^b_a = \text{tr}\,(T_i T_j) = a\,\delta_{ij}$$

 (10.37)

For every invariant tensor q, the infinitesimal transformations $G = 1 + iD$ must satisfy the invariance condition (10.27). Parametrizing D as a projection of an arbitrary hermitian matrix $H \in V \otimes \bar{V}$ into the adjoint space, $D = \mathbf{P}_A H \in V \otimes \bar{V}$,

$$D^a_b = \frac{1}{a}(T_i)^a_b \epsilon_i\,, \tag{10.38}$$

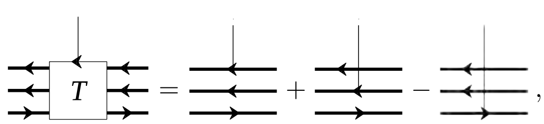
we obtain the *invariance condition* which the *generators* must satisfy: they *annihilate* invariant tensors:

$$T_i q = 0\,. \tag{10.39}$$

To state the invariance condition for an arbitrary invariant tensor, we need to define the action of generators on the tensor reps. By substituting $G = 1 + i\epsilon \cdot T + O(\epsilon^2)$ and keeping only the terms linear in ϵ, we find that the generators of infinitesimal transformations for tensor reps act by touching one index at a time:

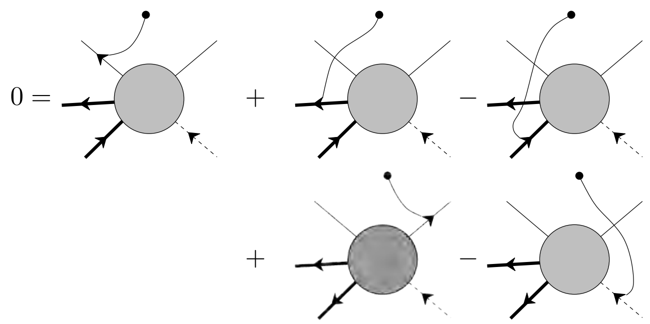
$$\begin{aligned} (T_i)_{b_1 \ldots b_q}^{a_1 a_2 \ldots a_p}{}_{,\,c_p \ldots c_2 c_1}^{d_q \ldots d_1} &= (T_i)^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} \\ &+ \delta^{a_1}_{c_1} (T_i)^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} + \ldots + \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots (T_i)^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} \\ &- \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} (T_i)^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} - \ldots - \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots (T_i)^{d_q}_{b_q}\,. \end{aligned} \tag{10.40}$$

This forest of indices vanishes in the birdtrack notation, enabling us to visualize the formula for the generators of infinitesimal transformations for any tensor representation:

 (10.41)

with a relative minus sign between lines flowing in opposite directions. The reader will recognize this as the Leibnitz rule.

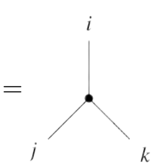
The invariance conditions take a particularly suggestive form in the birdtrack notation. Equation (10.39) amounts to the insertion of a generator into all external legs of the diagram corresponding to the invariant tensor q:



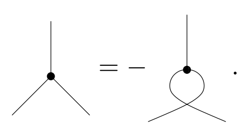
(10.42)

The insertions on the lines going into the diagram carry a minus sign relative to the insertions on the outgoing lines.

As the simplest example of computation of the generators of infinitesimal transformations acting on spaces other than the defining space, consider the adjoint rep. Where does the ugly word "adjoint" come from in this context is not obvious, but remember it this way: this is the one distinguished representation, which is intrinsic to the Lie algebra, with the explicit matrix elements $(T_i)_{jk}$ of the adjoint rep given by the fully antisymmetric structure constants iC_{ijk} of the algebra (i.e., its multiplication table under the commutator product). It's the continuous groups analogue of the multiplication table, or the regular representation for the finite groups. The factor i ensures their reality (in the case of hermitian generators T_i), and we keep track of the overall signs by always reading indices *counterclockwise* around a vertex:

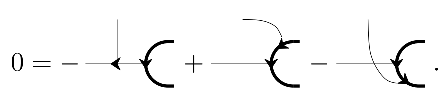
$-iC_{ijk} =$ 

(10.43)

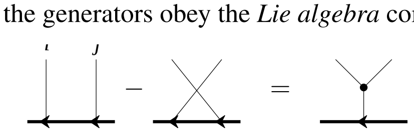


(10.44)

As all other invariant tensors, the generators must satisfy the invariance conditions (10.42):



Redrawing this a little and replacing the adjoint rep generators (10.43) by the structure constants, we find that the generators obey the *Lie algebra* commutation relation

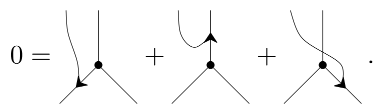


(10.45)

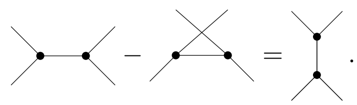
In other words, the Lie algebra commutator

$$T_i T_j \quad - \quad T_j T_i \quad = \quad iC_{ijk} T_k \,. \tag{10.46}$$

is simply a statement that T_i, the generators of invariance transformations, are themselves invariant tensors. Now, honestly, do you prefer the three-birdtracks equation (10.45), or the mathematician's page-long definition of the adjoint rep? It's a classic example of bad notation getting into way of understanding a relation of beautiful simplicity. The invariance condition for structure constants C_{ijk} is likewise



Rewriting this with the dot-vertex (10.43), we obtain



(10.47)

This is the Lie algebra commutator for the adjoint rep generators, known as the *Jacobi relation* for the structure constants

$$C_{ijm} C_{mkl} - C_{ljm} C_{mki} = C_{iml} C_{jkm} \,. \tag{10.48}$$

Hence, the Jacobi relation is also an invariance statement, this time the statement that the structure constants are invariant tensors.

10.8 Nobel Prize in Physics 2020 (optional)

Students –really, anybody who has learned some physics– often ask me: is space continuous or discrete?

We do not know, but this week's SO(3) $\approx$ SU(2) correspondence is one of the gateway drugs to speculations about quantum underpinnings of the observed spacetime. It start's with Hamilton's quaternions - the discovery that the building blocks of our apparent 3 Euclidian dimensions are 2-dimensional complex spin 1/2 'spinors', and it leads different people to different theories of quantum spacetime - one direction is the one taken by David Ritz Finkelstein, another one leads to Roger Penrose's description of Minkowski spacetime in terms of twistors.

In what follows, Erin Wells Bonning from Emory University and Predrag Cvitanović from the Georgia Tech explain the 2020 Nobel Prize in physics in terms accessible to all.

A half of the 2020 Nobel Prize in Physics was awarded to Roger Penrose, for the discovery that black hole formation is a robust prediction of the general theory of relativity. In 1957 Penrose, then a graduate student, met Georgia Tech's late David Ritz Finkelstein in a fateful meeting that changed both men's lives forever after. It was Finkelstein's extension of the Schwarzschild metric which provided Penrose with an opening into general relativity and set him on the path to his 1965 discovery celebrated by this year's prize.

A half of the 2020 Nobel Prize in Physics was awarded jointly to Reinhard Genzel and Andrea Ghez for the discovery of –in Ghez's words- "The Monster at the heart of the Milky Way," a black hole whose existence had been hypothesized since the early 1970s. In order to visually observe an object that famously does not emit any light, precise measurements of stars moving in the black hole's gravitational field had to be carried out. The independent work of Genzel and Ghez mapping the positions of these stars over many years has led to the clearest evidence yet that the center of our Milky Way galaxy contains "The Monster", that possibly every galaxy contains a black hole, and that the environment near it looks nothing like what was expected.

- *Nobel Lecture: Roger Penrose, Nobel Prize in Physics 2020 (34 min)*
- *Nobel Prize in Physics 2020 (56 min)*
  - Abstract
  - Penrose slides for Predrag's 1/2 of the presentation
  - 2020 Nobel Prizes in Chemistry and Physics, Explained
- *Roger Penrose gets Nobel Prize. How David Ritz Finkelstein and Roger Penrose met, and exchanged their lives' paths.*
- *Negative dimensions (6 min)*
- *Andrea Ghez: "The Monster at the Heart of our Galaxy"*
- *Veritasium: "The Infinite Pattern That Never Repeats"*

10.8.1 Quaternionic speculations

Predrag: putting this here for a further re-examination - safely ignored:)

Marek Danielewski (AGH), December 29, 2020, and L. Sapa: Foundations of the Quaternion Quantum Mechanics *Foundations of the Quaternion Quantum Mechanics*, Entropy, **2020**, 22, 1424:

"We show that quaternion quantum mechanics has well-founded mathematical roots and can be derived from the model of the elastic continuum by Cauchy, i.e., it can be regarded as representing the physical reality of elastic continuum. Starting from the Cauchy theory (classical balance equations for isotropic Cauchy-elastic material) and using the Hamilton quaternion algebra, we present a rigorous derivation of the quaternion form of the non- and relativistic wave equations. The family of the wave equations and the Poisson equation are a straightforward consequence of the quaternion representation of the Cauchy model of the elastic continuum. This is the most general kind of quantum mechanics possessing the same kind of calculus of assertions as conventional quantum mechanics. The problem of the Schrödinger equation, where imaginary 'i' should emerge, is solved. This interpretation is an attempt to describe the ontology of quantum mechanics, and demonstrates that, besides Bohmian mechanics, the complete ontological interpretations of quantum theory exists."

It has a quack feel to it, but should be easy to work through...

For a different approach, straightforward, no quackery, see Pavel A. Bolokhov *Quaternionic wave function* arXiv:1712.04795: " quaternions form a natural language for the description of quantum-mechanical wave functions with spin. We use the quaternionic spinor formalism which is in one-to-one correspondence with the usual spinor language. No unphysical degrees of freedom are admitted, in contrast to the majority of literature on quaternions. We build a Dirac Lagrangian in the quaternionic form, derive the Dirac equation and take the nonrelativistic limit to find the Schrödinger's equation. We show that the quaternionic formalism is a natural choice to start with, while in the transition to the noninteracting nonrelativistic limit, the quaternionic description effectively reduces to the regular complex wave function language. We provide an easy-to-use grammar for switching between the ordinary spinor language and the description in terms of quaternions. As an illustration of the broader range of the formalism, we also derive the Maxwell's equation from the quaternionic Lagrangian of Quantum Electrodynamics. In order to derive the equations of motion, we develop the variational calculus appropriate for this formalism. "

Commentary:

Manfried Faber, Richard Gill Quaternions were invented by Benjamin Olinde Rodrigues, before Hamilton. (He is also known for Rodrigues formula for Legendre polynomials.) In 1840 he published a result on transformation groups which applied Leonhard Euler's four squares formula, a precursor to the quaternions of William Rowan Hamilton, to the problem of representing rotations in space. In 1846 Arthur Cayley acknowledged Euler's and Rodrigues' priority describing orthogonal transformations.

Manfried Faber MathsHistory.st-andrews: In 1840 Rodrigues published a mathematical paper which contains the second result for which he is known today, namely his work on transformation groups where he derived the formula for the composition of successive finite rotations by an entirely geometric method. Rodrigues' composition of rotations is basically the composition of unit quaternions.

Predrag I teach my students that SU(2) is double cover of SO(3) and do not do more with quaternions. Octonions is another story...

Richard Gill According to Stigler's law of eponomy, everything worth remembering is associated with the name of someone we want to remember, who did something else.

10.9 What *really* happened (optional)

They do not make Norwegians as they used to. In his brief biographical sketch of Sophus Lie, Burkman writes: "I feel that I would be remiss in my duties if I failed to mention avery interesting event that took place in Lie's life. Klein (a German) and Lie had moved to Paris in the spring of 1870 (they had earlier been working in Berlin). However, in July 1870, the Franco-Prussian war broke out. Being a German alien in

France, Klein decided that it would be safer to return to Germany; Lie also decided to go home to Norway. However (in a move that I think questions his geometric abilities), Lie decided that to go from Paris to Norway, he would walk to Italy (and then presumably take a ship to Norway). The trip did not go as Lie had planned. On the way, Lie ran into some trouble–first some rain, and he had a habit of taking off his clothes and putting them in his backpack when he walked in the rain (so he was walking to Italy in the nude). Second, he ran into the French military (quite possibly while walking in the nude) and they discovered in his sack (in addition to his hopefully dry clothing) letters written to Klein in German containing the words 'lines' and 'spheres' (which the French interpreted as meaning 'infantry' and 'artillery'). Lie was arrested as a (insane) German spy. However, due to intervention by Gaston Darboux, he was released four weeks later and returned to Norway to finish his doctoral dissertation."

10.10 Birdtracks - updated history

Predrag Cvitanović November 7, 2018

Young tableaux and (non-Hermitian) Young projection operators were introduced by Young [26] in 1933 (Tung monograph [25] is a standard exposition). In 1937 R. Brauer [5] introduced diagrammatic notation for δ_{ij} in order to represent "Brauer algebra" permutations, index contractions, and matrix multiplication diagrammatically. R. Penrose's papers were the first to cast the Young projection operators into a diagrammatic form. In 1971 monograph [18] Penrose introduced diagrammatic notation for symmetrization operators, Levi-Civita tensors [20], and "strand networks" [17]. Penrose credits Aitken [2] with introducing this notation in 1939, but inspection of Aitken's book reveals a few Brauer diagrams for permutations, and no (anti)symmetrizers. Penrose's [19] 1952 initial ways of drawing symmetrizers and antisymmetrizers are very aesthetical, but the subsequent developments gave them a distinctly ostrich flavor [19]. In 1974 G. 't Hooft introduced a double-line notation for U(n) gluon group-theory weights [1]. In 1976 Cvitanović [9] introduced analogous notation for SU(N), SO(n) and Sp(n). For several specific, few-index tensor examples, diagrammatic Young projection operators were constructed by Canning [6], Mandula [15], and Stedman [23].

The 1975–2008 Cvitanović diagrammatic formulation of the theory of all semi-simple Lie groups [11] as a way to compute group theoretic wights without any recourse to symbols goes conceptually and profoundly beyond the Penrose notation (indeed, Cvitanović "birdtracks" bear no resemblance to Penrose's "fornicating ostriches" [19]).

A chapter in Cvitanović 2008 monograph [11] sketches how birdtrack (diagrammatic) Young projection operators for arbitrary irreducible representation of SU(N) could be constructed (this text is augmented by a 2005 appendix by Elvang, Cvitanović and Kennedy [13] which, however, contains a significant error). Keppeler and Sjödahl [14] systematized the construction by offering a simple method to construct Hermitian Young projection operators in the birdtrack formalism. Their iteration is easy to understand, and the proofs of Hermiticity are simple. However, in practice, the algorithm is inefficient - the expression balloon quickly, the Young projection operators soon become unwieldy and impractical, if not impossible to implement.

The Alcock-Zeilinger algorithm, based on the simplification rules of ref. [3], leads to explicitly Hermitian and drastically more compact expressions for the projection operators than the Keppeler-Sjödahl algorithm [14]. Alcock-Zeilinger fully supersedes Cvitanović's formulation, and any future full exposition of reduction of SU(N) tensor products into irreducible representations should be based on the Alcock-Zeilinger algorithm.

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Exercises

10.1. **Conjugacy classes of SO(3):** Show that all SO(3) rotations (10.7) by the same angle θ around any rotation axis $\boldsymbol{n}$ are conjugate to each other:

$$e^{i\phi \boldsymbol{n}_2 \cdot \boldsymbol{L}} e^{i\theta \boldsymbol{n}_1 \cdot \boldsymbol{L}} e^{-i\phi \boldsymbol{n}_2 \cdot \boldsymbol{L}} = e^{i\theta \boldsymbol{n}_3 \cdot \boldsymbol{L}} \tag{10.49}$$

Check this for infinitesimal ϕ, and argue that from that it follows that it is also true for finite ϕ. Hint: use the Lie algebra commutators (10.9).

10.2. **The character of SO(3) 3-dimensional representation:** Show that for the 3-dimensional special orthogonal representation (10.7), the character is

$$\chi = 2\cos(\theta) + 1\,. \tag{10.50}$$

Hint: evaluate the character explicitly for $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$, then explain what is the intuitive meaning of 'class' for rotations.

10.3. **The orthonormality of SO(3) characters:** Verify that given the Haar measure (10.13), the characters (10.12) are orthogonal:

$$\langle \chi(j) | \chi(j') \rangle = \int_G dg\, \chi^{(j)}(g^{-1})\, \chi^{(j')}(g) = \delta_{jj'}\,. \tag{10.51}$$

10.4. **U(1) equivariance of two-modes system for finite angles:** Show that the vector field in two-modes system (10.2) is equivariant under (10.22), the unitary group U(1) acting on $\mathbb{R}^4 \cong \mathbb{C}^2$ as the $k = 1$ and 2 modes:

$$g(\theta)(z_1, z_2) = (e^{i\theta} z_1, e^{i2\theta} z_2)\,, \quad \theta \in [0, 2\pi)\,. \tag{10.52}$$

10.5. **Integrate the two-modes system:** Integrate (10.5) and plot a long trajectory of two-modes in the 4d state space, (x_1, y_1, y_2) projection, as in figure 10.4 (a). To save you time (typing in (10.5) is tedious), we have prepared for you python code, and online graded problem set here. If you do this exercise, please get started early, in order to make sure that the autograder is working, and forward to us the grades that you receive from the autograder.

10.6. **SO(2) or harmonic oscillator slice:** Construct a moving frame slice for action of SO(2) on $\mathbb{R}^2$

$$(x, y) \mapsto (x\cos\theta - y\sin\theta,\; x\sin\theta + y\cos\theta)$$

by, for instance, the positive y axis: $x = 0,\, y > 0$. Write out explicitly the group transformation that brings any point back to the slice. What invariant is preserved by this construction?

10.7. **Invariant subspace of the two-modes system:** Show that $(0, 0, x_2, y_2)$ is a flow invariant subspace of the two-modes system (10.5), i.e., show that a trajectory with the initial point within this subspace remains within it forever.

10.8. **Slicing the two-modes system:** Choose the simplest slice template point that fixes the 1. Fourier mode,

$$\hat{x}' = (1, 0, 0, 0)\,. \tag{10.53}$$

(a) Show for the two-modes system (10.5), that the velocity within the slice, and the phase velocity along the group orbit are

$$\hat{v}(\hat{x}) = v(\hat{x}) - \dot{\phi}(\hat{x})t(\hat{x}) \tag{10.54}$$

$$\dot{\phi}(\hat{x}) = -v_2(\hat{x})/\hat{x}_1 \tag{10.55}$$

(b) Determine the chart border (the locus of point where the group tangent is either not transverse to the slice or vanishes).

(c) What is its dimension?

(d) What is its relation to the invariant subspace of exercise 10.7?

(e) Can a symmetry-reduced trajectory cross the chart border?

10.9. **The symmetry reduced two-modes flow:** Pick an initial point $\hat{x}(0)$ that satisfies the slice condition for the template choice (10.53) and integrate (10.54) & (10.55). Plot the three dimensional slice hyperplane spanned by (x_1, x_2, y_2) to visualize the symmetry reduced dynamics. Does it look like figure 10.4 (b)?