group theory course - GaTech phys 7143

An overview

This whole course has only one message:

> If you have a symmetry, **use it!**

Here is a brief summary of the course, the ideas you want to take with you:

week 1 Linear algebra

> Projection operators : eigenvalues of a matrix split a vector space into subspaces.

week 2 Finite groups

> Groups, permutations, group multiplication tables, rearrangement theorem, subgroups, cosets, classes.

week 3 Representation theory

> Irreps, regular representation. So far, everything was intuitive: a representation of a group was bunch of 0's and 1's indicating how a group operation permutes physical objects. But now the first surprise:
>
> Any representation of any finite group can be put into unitary form, and so complex-valued vector spaces and unitary representation matrices make their entrance.

week 4 Characters

> Schur's Lemma. Unitary matrices can be diagonalized, and from that follows the Wonderful Orthogonality Theorem for Characters (coordinate independent, intrinsic numbers), and the full reducibility of any representation of any finite group.

week 5 Classes

The algebra of central or 'all-commuting' class operators, connects the reduction in terms of characters to the projection operators of week 1. The key idea:

Define a group by what objects (primitive invariant tensors) it leaves invariant.

week 6 Fundamental domain

Dynamical systems application: the Lorenz flow and its C_2 symmetry.

week 7 Lorenz to Van Gogh; Diffusion confusion

(1) Conclusion of the finite groups part of the course: Lorenz flow desymmetrization: if the system is nonlinear, its symmetry reduction is not easy.

(2) So far, everything was finite and compact. Next: two distinct ways of going infinite: (a) discrete translations, exemplified by deterministic diffusion and space groups of week 8, and (b) continuous Lie groups, exemplified by rotations of week 9.

week 8 Space groups

Translation group, Bravais lattice, wallpaper groups, reciprocal lattice, Brilluoin zone.

week 9 Continuous groups

Lie groups. Matrix representations. Invariant tensors. Lie algebra. Adjoint representation, Jacobi relation. Birdtracks.

Irreps of $SO(2)$ and $O(2)$ Clebsch-Gordan series (i.e., reduction of their products).

week 10 $SO(3)$ characters; $O(2)$ symmetry sliced

(a) Group integrals. $SO(3)$ character orthogonality.

(b) Continuous symmetry reduction for a *nonlinear* system is much harder than discrete symmetry reduction of week 7. "Slicing" is a research level topic, will not be included in the final.

week 11 $SU(2)$ and $SO(3)$

$SU(2) \simeq SO(3)$ correspondence leads to the next rude awakening; our 3-dimensional Euclidean space is not fundamental! All irreps of $SO(3)$ are built from 2-dimensional complex vectors, or 1/2 spins. Birdtrack notation for the smallest irreps of $SO(n)$.

week 12 Lorentz group; spin

(a) We now loose compactness: even though the $SO(1,3)$ Lorentz invariance group of the Minkowski space symmetries is not compact, its Lie algebra still closes, as for the compact $SO(4)$.

(b) $SO(4) \simeq SU(2) \otimes SU(2)$ correspondence leads to the Minkowski 4-dimensional space not being fundamental either - all irreps of the Lorentz group are built from combinations of 2-dimensional complex vectors, or spinors.

(c) Not included in the final: together with general relativity, this leads to replacement of the Minkowski continuum by a 4-dimensional spacetime (or quantum) foam, a candidate theory of quantum gravity.

week 13 Simple Lie algebras; SU(3)

The next profound shift:

So far all our group notions were based on tangible, spatial intuition: permutations, reflections, rotations. But now Lie groups take on a life of their own.

(a) The SO(3) theory of angular momenta generalizes to Killing-Cartan lattices, and a fully abstract enumeration of all possible semi-simple compact Lie groups.

(b) SU(2) is promoted to an *internal* isospin symmetry, decoupled from our Euclidean spatial intuition. Modern particle physics is born, with larger and larger internal symmetry groups, tacked onto higher and higher dimensional continuum spacetimes.

week 14 Flavor SU(3)

Gell-Mann–Okubo formula. The next triumph of particle physics is yet another departure; observed baryons and mesons are built up from quarks, particles by assumption unobservable in isolation.

week 15 Young tableaux

We have come full circle now: as a much simpler alternative to the Cartan-Killing construction, irreps of the *finite* symmetric group S_n classify the irreps of the *continuous* SU(n) symmetry multi-particle states.

week 16 Wigner 3- and 6-j coefficients

The goal of group theory is to predict measurable numbers, numbers independent of any particular choice of coordinate. The full reducibility says that any such number is built from 3- and 6-j coefficients: they are the total content of group theory.