group theory zipped!

world-wide quest to tame group theory

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Mar 31, 2024

Overview

I have collected the course notes here, not so much for the notes themselves –they cannot be understood on their own, without viewing the recorded live lectures– but for the hyperlinks to various source texts you might find useful later on in your research.

Course outline *(8:49 min)*

- sect. 17 *The epilogue:* detailed overview of the course

Course policy *(2:03 min)*

0.1 *Navigating the course* *(4:18 min)*

Extras If a video is marked as an 'extra', it is not a required viewing, but a supplementary discussion or a comment or (sometimes) just a rant.

My teaching philosophy : Bologna *(2:09 min)*
About teaching online *(3:15 min)*
How does one pronounce 'Euler'? 'Cvitanović'? *(4:17 min)*
If I am allowed to teach group theory... *(0:30 sec)*

The fastest way to watch any week's lecture videos is by letting YouTube run the

course playlist

and skipping 'extras' manually.

Contents

group theory - week 1

Linear algebra

Homework HW1

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 1.1 *Trace-log of a matrix*	4 points
Exercise 1.2 *Stability, diagonal case*	2 points
Exercise 1.3 *The matrix square root*	4 points

Total of 10 points = 100 % score.

Diagonalizing the matrix: that's the key to the whole thing.
— Governor Arnold Schwarzenegger

2021-05-18 Predrag Week 1 *Linear algebra*

- Light purple text in this PDF is a live hyperlink. If you encounter a ChaosBook.org web login: all copyright-protected references are on a password protected site. What password? If you are a Georgia Tech student, I can help you with that.
- The fastest way to watch any week's lecture videos is by letting YouTube run the

  course playlist

  Downside is that the playlist plays also all 'extra' videos - you can skip through those, if you are short on time. Or patience.

1.1 Week's videos, reading

- Sect. 1.4 *Matrix-valued functions*
  - *Matrices : 2 kinds*
  - *Derivative of a matrix function*
  - *Exponential, logarithm of a matrix*
  - *Determinant is a volume*
  - *log det = tr log* (updated Aug 18, 2020)
  - *Multi-matrix functions* (optional, for the QM inclined)
- Sect. 1.5 *A linear diversion*
  There are two representations of exponential of constant matrix, the Taylor series (1.7) and the compound interest (Euler product) formulas (1.8).
  - *Linear differential equations*
- Sect. 1.6 *Eigenvalues and eigenvectors*
  Hamilton-Cayley equation, projection operators (1.27), any matrix function is evaluated by spectral decomposition (1.30). Work through example 1.5.
  - *Eigenvalues and eigenvectors*
  - *What's the deal with Hamilton-Cayley?*
  - *Spectral decomposition*
  - *Spectral decomposition and completeness*
  - *Right, left eigenvectors*
  - *A projection operators workout*

1.2 Other sources (optional)

> Mopping up operations are the activities that engage most scientists throughout their careers.
> — Thomas Kuhn, *The Structure of Scientific Revolutions*

The subject of linear algebra is a vast and very alive research area, generates innumerable tomes of its own, and is way beyond what we can exhaustively cover here. Some linear operators and matrices reading (optional reading for week 1, not required for this course; whenever the text is colored, you can click on the live hyperlink in the pdf version of these notes):

- Stone and Goldbart [15], *Mathematics for Physics: A Guided Tour for Graduate Students*, Appendix A. This is an advanced summary where you will find almost everything one needs to know.
- AWH p. 113 Functions of Matrices. Anything prefixed by AWH, like "Kronecker product AWH eq. (2.55)" refers to the more pedestrian and perhaps easier to read Arfken, Weber & Harris [2] *Mathematical Methods for Physicists: A Comprehensive Guide* (Georgia Tech students can get it from GaTech Library).
- AWH Section *2.2 Matrices*
- AWH Example 2.2.6 *Exponential of a diagonal matrix*
- AWH Chapter 2 *Determinants and matrices* (click here).
- AWH Section 6.1 *Eigenvalue Equations* (click here).
- AWH Chapter 6 *Eigenvalue problems* (click here).
- In sect. 1.4 I make matrix functions appear easier than they really are. For an in-depth discussion, consult Golub and Van Loan [8] *Matrix Computations*, chap. 9 *Functions of Matrices* (click here). exercise 1.3
- Much more than you ever wanted to know about linear algebra: Axler [3] *Down with determinants!* (click here).
- Steve Trettel *Linear Algebra and the Periodic Table* is a gentle 53 min tour from vectors to function spaces to quantum mechanics. True, what they teach you as QM is 95% linear algebra, but Trettel does not mention that QM is 95% one amazing experimental fact: $\hbar$ is a nature-given *constant*. Mathematicians...
- David Austin online PreTeXt textbook *Understanding Linear Algebra*.
- Grant Sanderson *Essence of linear algebra* (3Blue1Brown). Karan Shah likes the geometrical explanations of linear algebra eigen-values / -vectors, recommends it.

Question 1.1. Henriette Roux finds course notes confusing

Q Couldn't you use one single, definitive text for methods taught in the course?

A It's a grad school, so it is research focused - I myself am (re)learning the topics that we are going through the course, using various sources. My emphasis in this course is on understanding and meaning, not on getting all signs and 2π's right, and I find reading about the topic from several perspectives helpful. But if you really find one book more comfortable, nearly all topics are covered in Arfken, Weber & Harris [2].

1.3 Special projects

Several people have been interested in taking a **special project**, instead of the final in the course. If you propose to work out in detail some group-theory needed for your own research (but you have not taken the time to master the theory), that would be ideal. Random examples of interesting topics (i.e., something that Predrag would like to learn from you:) –

1. The talk by David Weitz on melting of crystal lattices. Can you do a calculation on a Wigner lattice or a graphene, or a silicon carbide polytype used as a substrate in our graphene lab (ask Claire Berger about it), using our group theory methods as applied to space groups (2- or 3-D lattices)?

2. If you are really wild about string theory, then you can read Giles and Thorn [7] *Lattice approach to string theory*, and write up what you have learned as the project report. The Giles-Thorn (GT) discretization of the worldsheet begins with a representation of the free closed or open string propagator as a light-cone worldsheet path integral defined on a lattice. The sequel Papathanasiou and Thorn [14] *Worldsheet propagator on the lightcone worldsheet lattice* gives in Appendix B 2D lattice Neumann open string, Dirichlet open string, and closed string propagators. *Discrete Green's functions* are explained, for example, by Chung and Yau [4] who give explicitly, in their Theorem 6, a 2-dimensional lattice Green's function for a rectangular region $R^{[\ell_1 \times \ell_2]}$. The paper is cited over 100 times, maybe there is a better, more up-to-date one to read in that list.

3. 3-springs system of sect. 6.4.

I recommend that you take a final, as these are hard and time-consuming projects, and the faculty does not want to overburden you with course work. However, if a project dovetails with your research interests, it might be worth it. Fly it by me.

1.4 Matrix-valued functions

What is a matrix?

—Werner Heisenberg (1925)

What is the matrix?

—-Keanu Reeves (1999)

Why should a working physicist care about linear algebra? Physicists were blissfully ignorant of group theory until 1920's, but with Heisenberg's sojourn in Helgoland, everything changed. Quantum Mechanics was formulated as

$$\phi(t) = \hat{U}^t \phi(0) \,, \qquad \hat{U}^t = e^{-\frac{i}{\hbar} t \hat{H}} \,, \tag{1.1}$$

where $\phi(t)$ is the quantum wave function t, $\hat{U}^t$ is the unitary quantum evolution operator, and $\hat{H}$ is the Hamiltonian operator. Fine, but what does this equation *mean*? In the first lecture we deconstruct it.

The matrices that have to be evaluated are very high-dimensional, in principle infinite dimensional, and the numerical challenges can quickly get out of hand. What made it possible to solve these equations analytically in 1920's for a few iconic problems, such as the hydrogen atom, are the symmetries, or in other words group theory, which start sketching out in the second lecture (and fill in the details in the next 27 lectures).

Whenever you are confused about an "operator", think "matrix". Here we recapitulate a few matrix algebra concepts that we found essential. The punch line is (1.35): Hamilton-Cayley equation $\prod(\mathbf{M} - \lambda_i \mathbf{1}) = 0$ associates with each distinct root λ_i of a matrix $\mathbf{M}$ a projection onto ith vector subspace

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} \,.$$

What follows - for this week - is a jumble of Predrag's notes. If you understand the examples, we are on the roll. If not, ask :)

How are we to think of the quantum operator

$$\hat{H} = \hat{T} + \hat{V} \,, \qquad \hat{T} = \hat{p}^2/2m \,, \qquad \hat{V} = V(\hat{q}) \,, \tag{1.2}$$

corresponding to a classical Hamiltonian $H = T + V$, where T is kinetic energy, and V is the potential?

Expressed in terms of basis functions, the quantum evolution operator is an infinite-dimensional matrix; if we happen to know the eigenbasis of the Hamiltonian, the problem is solved already. In real life we have to guess that some complete basis set is good starting point for solving the problem, and go from there. In practice we truncate such operator representations to finite-dimensional matrices, so it pays to recapitulate a few relevant facts about matrix algebra and some of the properties of functions of finite-dimensional matrices.

Matrix derivatives. The derivative of a matrix is a matrix with elements

$$\mathbf{A}'(x) = \frac{d\mathbf{A}(x)}{dx} \,, \qquad A'_{ij}(x) = \frac{d}{dx} A_{ij}(x) \,. \tag{1.3}$$

Derivatives of products of matrices are evaluated by the chain rule

$$\frac{d}{dx}(\mathbf{AB}) = \frac{d\mathbf{A}}{dx}\mathbf{B} + \mathbf{A}\frac{d\mathbf{B}}{dx} \,. \tag{1.4}$$

A matrix and its derivative matrix in general do not commute

$$\frac{d}{dx}\mathbf{A}^2 = \frac{d\mathbf{A}}{dx}\mathbf{A} + \mathbf{A}\frac{d\mathbf{A}}{dx}\,. \tag{1.5}$$

The derivative of the inverse of a matrix, if the inverse exists, follows from $\frac{d}{dx}(\mathbf{A}\mathbf{A}^{-1}) = 0$:

$$\frac{d}{dx}\mathbf{A}^{-1} = -\frac{1}{\mathbf{A}}\frac{d\mathbf{A}}{dx}\frac{1}{\mathbf{A}}\,. \tag{1.6}$$

Matrix functions. A function of a single variable that can be expressed in terms of additions and multiplications generalizes to a matrix-valued function by replacing the variable by a matrix.

In particular, the exponential of a constant matrix can be defined either by its series expansion, or as a limit of an infinite product:

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{A}^k\,, \qquad \mathbf{A}^0 = \mathbf{1} \tag{1.7}$$

$$= \lim_{N\to\infty}\left(\mathbf{1} + \frac{1}{N}\mathbf{A}\right)^N \tag{1.8}$$

The first equation follows from the second one by the binomial theorem, so these indeed are equivalent definitions. That the terms of order $O(N^{-2})$ or smaller do not matter for a function of a single variable follows from the bound

$$\left(1+\frac{x-\epsilon}{N}\right)^N < \left(1+\frac{x+\delta x_N}{N}\right)^N < \left(1+\frac{x+\epsilon}{N}\right)^N\,,$$

where $|\delta x_N| < \epsilon$. If $\lim \delta x_N \to 0$ as $N \to \infty$, the extra terms do not contribute. A proof for matrices would probably require defining the norm of a matrix (and, more generally, a norm of an operator acting on a Banach space) first. If you know an easy proof, let us know.

Logarithm of a matrix. The logarithm of a matrix is defined by the power series

$$\ln(\mathbf{1}-\mathbf{A}) = -\sum_{k=1}^{\infty}\frac{\mathbf{A}^k}{k}\,. \tag{1.9}$$

log det = tr log matrix identity. Consider now the determinant

$$\det\left(e^{\mathbf{A}}\right) = \lim_{N\to\infty}\left(\det\,(\mathbf{1}+\mathbf{A}/N)\right)^N\,.$$

To the leading order in $1/N$

$$\det\,(\mathbf{1}+\mathbf{A}/N) = 1 + \frac{1}{N}\mathrm{tr}\,\mathbf{A} + O(N^{-2})\,.$$

hence

$$\det e^{\mathbf{A}} = \lim_{N\to\infty}\left(1+\frac{1}{N}\mathrm{tr}\,\mathbf{A}+O(N^{-2})\right)^N = \lim_{N\to\infty}\left(1+\frac{\mathrm{tr}\,\mathbf{A}}{N}\right)^N = e^{\mathrm{tr}\,\mathbf{A}} \quad (1.10)$$

Defining $\mathbf{M} = e^{\mathbf{A}}$ we can write this as

$$\ln\det\mathbf{M} = \mathrm{tr}\,\ln\mathbf{M}\,, \quad (1.11)$$

a crazy useful identity that you will run into over and over again.

Functions of several matrices. Due to non-commutativity of matrices, generalization of a function of several variables to a function of several matrices is not as straightforward. Expression involving several matrices depend on their commutation relations. For example, the Baker-Campbell-Hausdorff commutator expansion

$$e^{t\mathbf{A}}\mathbf{B}e^{-t\mathbf{A}} = \mathbf{B}+t[\mathbf{A},\mathbf{B}]+\frac{t^2}{2}[\mathbf{A},[\mathbf{A},\mathbf{B}]]+\frac{t^3}{3!}[\mathbf{A},[\mathbf{A},[\mathbf{A},\mathbf{B}]]]+\cdots \quad (1.12)$$

sometimes used to establish the equivalence of the Heisenberg and Schrödinger pictures of quantum mechanics, follows by recursive evaluation of t derivatives

$$\frac{d}{dt}\left(e^{t\mathbf{A}}\mathbf{B}e^{-t\mathbf{A}}\right) = e^{t\mathbf{A}}[\mathbf{A},\mathbf{B}]e^{-t\mathbf{A}}\,.$$

Expanding $\exp(\mathbf{A}+\mathbf{B})$, $\exp\mathbf{A}$, $\exp\mathbf{B}$ to first few orders using (1.7) yields

$$e^{(\mathbf{A}+\mathbf{B})/N} = e^{\mathbf{A}/N}e^{\mathbf{B}/N} - \frac{1}{2N^2}[\mathbf{A},\mathbf{B}]+O(N^{-3})\,, \quad (1.13)$$

and the Trotter product formula: if $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{A}=\mathbf{B}+\mathbf{C}$ are matrices, then

$$e^{\mathbf{A}} = \lim_{N\to\infty}\left(e^{\mathbf{B}/N}e^{\mathbf{C}/N}\right)^N \quad (1.14)$$

In particular, we can now make sense of the quantum evolution operator (1.1) as a succession of short free flights (kinetic term) interspersed by small acceleration kicks (potential term),

$$e^{-it\hat{H}} = \lim_{N\to\infty}\left(e^{-i\Delta t\,\hat{T}}e^{-i\Delta t\,\hat{V}}\right)^N\,, \qquad \Delta t = t/N\,, \quad (1.15)$$

where we have set $\hbar = 1$.

1.5 A linear diversion

Linear is good, nonlinear is bad.
—Jean Bellissard

(Notes based of ChaosBook.org/chapters/stability.pdf)

Linear fields are the simplest vector fields, described by linear differential equations which can be solved explicitly, with solutions that are good for all times. The state space for linear differential equations is $\mathcal{M} = \mathbb{R}^d$, and the equations of motion are written in terms of a vector x and a constant stability matrix A as

$$\dot{x} = v(x) = Ax\,. \tag{1.16}$$

Solving this equation means finding the state space trajectory

$$x(t) = (x_1(t), x_2(t), \ldots, x_d(t))$$

passing through a given initial point x_0. If $x(t)$ is a solution with $x(0) = x_0$ and $y(t)$ another solution with $y(0) = y_0$, then the linear combination $ax(t) + by(t)$ with $a, b \in \mathbb{R}$ is also a solution, but now starting at the point $ax_0 + by_0$. At any instant in time, the space of solutions is a d-dimensional vector space, spanned by a basis of d linearly independent solutions.

How do we solve the linear differential equation (1.16)? If instead of a matrix equation we have a scalar one, $\dot{x} = \lambda x$, the solution is $x(t) = e^{t\lambda} x_0$. In order to solve the d-dimensional matrix case, it is helpful to rederive this solution by studying what happens for a short time step Δt. If time $t = 0$ coincides with position $x(0)$, then

$$\frac{x(\Delta t) - x(0)}{\Delta t} = \lambda x(0)\,, \tag{1.17}$$

which we iterate m times to obtain Euler's formula for compounding interest

$$x(t) \approx \left(1 + \frac{t}{m}\lambda\right)^m x(0) \approx e^{t\lambda} x(0)\,. \tag{1.18}$$

The term in parentheses acts on the initial condition $x(0)$ and evolves it to $x(t)$ by taking m small time steps $\Delta t = t/m$. As $m \to \infty$, the term in parentheses converges to $e^{t\lambda}$. Consider now the matrix version of equation (1.17):

$$\frac{x(\Delta t) - x(0)}{\Delta t} = Ax(0)\,. \tag{1.19}$$

A representative point x is now a vector in $\mathbb{R}^d$ acted on by the matrix A, as in (1.16). Denoting by $\mathbf{1}$ the identity matrix, and repeating the steps (1.17) and (1.18) we obtain Euler's formula for the exponential of a matrix:

$$x(t) = J^t x(0)\,, \qquad J^t = e^{tA} = \lim_{m\to\infty} \left(\mathbf{1} + \frac{t}{m}A\right)^m\,, \tag{1.20}$$

where $J^t = J(t)$ is a short hand for $\exp(tA)$.

1.6 Eigenvalues and eigenvectors

> 10. Try to leave out the part that readers tend to skip.
> — Elmore Leonard's Ten Rules of Writing.

Eigenvalues of a $[d\times d]$ matrix $\mathbf{M}$ are the roots of its characteristic polynomial

$$\det(\mathbf{M}-\lambda\mathbf{1}) = \prod(\lambda_i - \lambda) = 0\,. \tag{1.21}$$

Given a nonsingular matrix $\mathbf{M}$, with all $\lambda_i \neq 0$, acting on d-dimensional vectors $\mathbf{x}$, we would like to determine *eigenvectors* $\mathbf{e}^{(i)}$ of $\mathbf{M}$ on which $\mathbf{M}$ acts by scalar multiplication by eigenvalue λ_i

$$\mathbf{M}\,\mathbf{e}^{(i)} = \lambda_i \mathbf{e}^{(i)}\,. \tag{1.22}$$

If $\lambda_i \neq \lambda_j$, $\mathbf{e}^{(i)}$ and $\mathbf{e}^{(j)}$ are linearly independent. There are at most d distinct eigenvalues which we order by their real parts, $\mathrm{Re}\,\lambda_i \geq \mathrm{Re}\,\lambda_{i+1}$.

If all eigenvalues are distinct, $\mathbf{e}^{(j)}$ are d linearly independent vectors which can be used as a (non-orthogonal) basis for any d-dimensional vector $\mathbf{x} \in \mathbb{R}^d$

$$\mathbf{x} = x_1\,\mathbf{e}^{(1)} + x_2\,\mathbf{e}^{(2)} + \cdots + x_d\,\mathbf{e}^{(d)}\,. \tag{1.23}$$

However, r, the number of distinct eigenvalues, is in general smaller than the dimension of the matrix, $r \leq d$ (see example 1.3).

From (1.22) it follows that

$$(\mathbf{M} - \lambda_i\mathbf{1})\,\mathbf{e}^{(j)} = (\lambda_j - \lambda_i)\,\mathbf{e}^{(j)}\,,$$

matrix $(\mathbf{M} - \lambda_i\mathbf{1})$ annihilates $\mathbf{e}^{(i)}$, thus the product of all such factors annihilates any vector, and the matrix $\mathbf{M}$ satisfies its characteristic equation

$$\prod_{i=1}^{d}(\mathbf{M} - \lambda_i\mathbf{1}) = 0\,. \tag{1.24}$$

This humble fact has a name: the Hamilton-Cayley theorem. If we delete one term from this product, we find that the remainder projects $\mathbf{x}$ from (1.23) onto the corresponding eigenspace:

$$\prod_{j\neq i}(\mathbf{M} - \lambda_j\mathbf{1})\mathbf{x} = \prod_{j\neq i}(\lambda_i - \lambda_j)\,x_i\mathbf{e}^{(i)}\,.$$

Dividing through by the $(\lambda_i - \lambda_j)$ factors yields the *projection operators*

$$\mathbf{P}_i = \prod_{j\neq i}\frac{\mathbf{M} - \lambda_j\mathbf{1}}{\lambda_i - \lambda_j}\,, \tag{1.25}$$

which are *orthogonal* and *complete*:

$$\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_j\,, \quad (\text{no sum on } j)\,, \qquad \sum_{i=1}^{r}\mathbf{P}_i = \mathbf{1}\,, \tag{1.26}$$

with the dimension of the ith subspace given by $d_i = \mathrm{tr}\,\mathbf{P}_i$. For each distinct eigenvalue λ_i of $\mathbf{M}$,

$$(\mathbf{M} - \lambda_j\mathbf{1})\mathbf{P}_j = \mathbf{P}_j(\mathbf{M} - \lambda_j\mathbf{1}) = 0\,, \tag{1.27}$$

the colums/rows of $\mathbf{P}_j$ are the right/left eigenvectors $\mathbf{e}^{(j)}$, $\mathbf{e}_{(j)}$ of $\mathbf{M}$ which (provided $\mathbf{M}$ is not of Jordan type, see example 1.3) span the corresponding linearized subspace.

The main take-home is that once the distinct non-zero eigenvalues $\{\lambda_i\}$ are computed, projection operators are polynomials in $\mathbf{M}$ which need no further diagonalizations or orthogonalizations.

It follows from the characteristic equation (1.27) that λ_i is the eigenvalue of $\mathbf{M}$ on $\mathbf{P}_i$ subspace:

$$\mathbf{M}\,\mathbf{P}_i = \lambda_i \mathbf{P}_i \qquad (\text{no sum on } i)\,. \tag{1.28}$$

Using $\mathbf{M} = \mathbf{M}\,\mathbf{1}$ and completeness relation (1.26) we can rewrite $\mathbf{M}$ as

$$\mathbf{M} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \cdots + \lambda_d \mathbf{P}_d\,. \tag{1.29}$$

Any matrix function $f(\mathbf{M})$ takes the scalar value $f(\lambda_i)$ on the $\mathbf{P}_i$ subspace, $f(\mathbf{M})\,\mathbf{P}_i = f(\lambda_i)\,\mathbf{P}_i$, and is thus easily evaluated through its *spectral decomposition*

$$f(\mathbf{M}) = \sum_i f(\lambda_i)\mathbf{P}_i\,. \tag{1.30}$$

This, of course, is the reason why anyone but a fool works with irreducible reps: they reduce matrix (AKA "operator") evaluations to manipulations with numbers.

By (1.27) every column of $\mathbf{P}_i$ is proportional to a right eigenvector $\mathbf{e}^{(i)}$, and its every row to a left eigenvector $\mathbf{e}_{(i)}$. In general, neither set is orthogonal, but by the idempotence condition (1.26), they are mutually orthogonal,

$$\mathbf{e}_{(i)} \cdot \mathbf{e}^{(j)} = c_j\,\delta_i^j\,. \tag{1.31}$$

The non-zero constant c_j is convention dependent and not worth fixing, unless you feel nostalgic about Clebsch-Gordan coefficients. We shall set $c_j = 1$. Then it is convenient to collect all left and right eigenvectors into a single matrix.

Example 1.1. ***Linear stability of 2-dimensional flows:*** *For a 2-dimensional flow the eigenvalues λ_1, λ_2 of A are either real, leading to a linear motion along their eigenvectors, $x_j(t) = x_j(0)\exp(t\lambda_j)$, or form a complex conjugate pair $\lambda_1 = \mu + i\omega\,, \lambda_2 = \mu - i\omega\,$, leading to a circular or spiral motion in the $[x_1, x_2]$ plane, see example 1.2.*

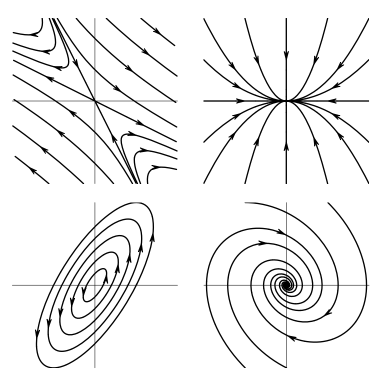

Figure 1.1: Streamlines for several typical 2-dimensional flows: saddle (hyperbolic), in node (attracting), center (elliptic), in spiral.

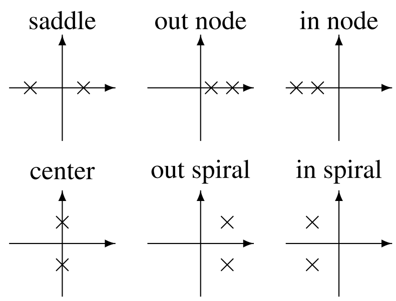

Figure 1.2: Qualitatively distinct types of exponents $\{\lambda^{(1)}, \lambda^{(2)}\}$ of a $[2\times 2]$ Jacobian matrix. Here the eigenvalues of the Jacobian matrix are *multipliers* $\Lambda^{(j)}$, and the *exponents* are defined as the deformation rates $\lambda^{(j)} = \log(\Lambda^{(j)})/t$.

These two possibilities are refined further into sub-cases depending on the signs of the real part. In the case of real $\lambda_1 > 0$, $\lambda_2 < 0$, x_1 grows exponentially with time, and x_2 contracts exponentially. This behavior, called a saddle*, is sketched in figure 1.1, as are the remaining possibilities: in/out nodes, inward/outward spirals, and the center. The magnitude of out-spiral $|x(t)|$ diverges exponentially when $\mu > 0$, and in-spiral contracts into $(0, 0)$ when $\mu < 0$; whereas, the phase velocity ω controls its oscillations.*

If eigenvalues $\lambda_1 = \lambda_2 = \lambda$ are degenerate, the matrix might have two linearly independent eigenvectors, or only one eigenvector, see example 1.3. We distinguish two cases: (a) A can be brought to diagonal form and (b) A can be brought to Jordan form*, which (in dimension 2 or higher) has zeros everywhere except for the repeating eigenvalues on the diagonal and some 1's directly above it. For every such Jordan $[d_\alpha\times d_\alpha]$ block there is only one eigenvector per block.*

We sketch the full set of possibilities in figures 1.1 and 1.2.

Example 1.2. *Complex eigenvalues: in-out spirals.* *As $\mathbf{M}$ has only real entries, it will in general have either real eigenvalues, or complex conjugate pairs of eigenvalues. Also the corresponding eigenvectors can be either real or complex. All coordinates used in defining a dynamical flow are real numbers, so what is the meaning of a* complex *eigenvector?*

If λ_k, λ_{k+1} eigenvalues that lie within a diagonal $[2\times 2]$ sub-block $\mathbf{M}' \subset \mathbf{M}$ form a complex conjugate pair, $\{\lambda_k, \lambda_{k+1}\} = \{\mu + i\omega, \mu - i\omega\}$, the corresponding complex eigenvectors can be replaced by their real and imaginary parts, $\{\mathbf{e}^{(k)}, \mathbf{e}^{(k+1)}\} \to \{\mathrm{Re}\,\mathbf{e}^{(k)}, \mathrm{Im}\,\mathbf{e}^{(k)}\}$. In this 2-dimensional real representation, $\mathbf{M}' \to A$, the block A is a sum of the rescaling$\times$identity and the generator of $SO(2)$ rotations in the $\{\mathrm{Re}\,\mathbf{e}^{(1)}, \mathrm{Im}\,\mathbf{e}^{(1)}\}$ plane.

$$A = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix} = \mu \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

Trajectories of $\dot{\mathbf{x}} = A\,\mathbf{x}$, given by $\mathbf{x}(t) = J^t\,\mathbf{x}(0)$, where (omitting $\mathbf{e}^{(3)}, \mathbf{e}^{(4)}, \cdots$ eigen-directions)

$$J^t = e^{tA} = e^{t\mu} \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix} , \tag{1.32}$$

spiral in/out around $(x, y) = (0, 0)$, see figure 1.1, with the rotation period T and the radial expansion /contraction multiplier along the $\mathbf{e}^{(j)}$ eigen-direction per a turn of the exercise 1.4 *spiral:*

$$T = 2\pi/\omega\,, \qquad \Lambda_{radial} = e^{T\mu}\,. \tag{1.33}$$

We learn that the typical turnover time scale in the neighborhood of the equilibrium $(x, y) = (0, 0)$ *is of order* $\approx T$ *(and not, let us say,* $1000\,T$*, or* $10^{-2}T$*).*

Example 1.3. ***Degenerate eigenvalues.*** *While for a matrix with generic real elements all eigenvalues are distinct with probability 1, that is not true in presence of symmetries, or spacial parameter values (bifurcation points). What can one say about situation where* d_α *eigenvalues are degenerate,* $\lambda_\alpha = \lambda_i = \lambda_{i+1} = \cdots = \lambda_{i+d_\alpha-1}$*? Hamilton-Cayley (1.24) now takes form*

$$\prod_{\alpha=1}^{r} (\mathbf{M} - \lambda_\alpha \mathbf{1})^{d_\alpha} = 0\,, \qquad \sum_\alpha d_\alpha = d\,. \tag{1.34}$$

We distinguish two cases:

$\mathbf{M}$ can be brought to diagonal form. *The characteristic equation (1.34) can be replaced by the minimal polynomial,*

$$\prod_{\alpha=1}^{r} (\mathbf{M} - \lambda_\alpha \mathbf{1}) = 0\,, \tag{1.35}$$

where the product includes each distinct eigenvalue only once. Matrix $\mathbf{M}$ *acts multiplicatively*

$$\mathbf{M}\,\mathbf{e}^{(\alpha,k)} = \lambda_i \mathbf{e}^{(\alpha,k)}\,, \tag{1.36}$$

on a d_α*-dimensional subspace spanned by a linearly independent set of basis eigenvectors* $\{\mathbf{e}^{(\alpha,1)}, \mathbf{e}^{(\alpha,2)}, \cdots, \mathbf{e}^{(\alpha,d_\alpha)}\}$*. This is the easy case. Luckily, if the degeneracy is due to a finite or compact symmetry group, relevant* $\mathbf{M}$ *matrices can always be brought to such diagonalizable form.*

$\mathbf{M}$ can only be brought to upper-triangular, Jordan form. *This is the messy case, so we only illustrate the key idea in example 1.4.*

Example 1.4. ***Decomposition of 2-dimensional vector spaces:*** *Enumeration of every possible kind of linear algebra eigenvalue / eigenvector combination is beyond what we can reasonably undertake here. However, enumerating solutions for the simplest case, a general* $[2\times2]$ *non-singular matrix*

$$\mathbf{M} = \left[\begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array}\right].$$

takes us a long way toward developing intuition about arbitrary finite-dimensional matrices. The eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\mathrm{tr}\,\mathbf{M} \pm \frac{1}{2}\sqrt{(\mathrm{tr}\,\mathbf{M})^2 - 4\det\mathbf{M}} \tag{1.37}$$

are the roots of the characteristic (secular) equation (1.21):

$$\begin{aligned} \det(\mathbf{M} - \lambda\,\mathbf{1}) &= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \\ &= \lambda^2 - \mathrm{tr}\,\mathbf{M}\,\lambda + \det\mathbf{M} \;=\; 0\,. \end{aligned}$$

Distinct eigenvalues case *has already been described in full generality. The left/right eigenvectors are the rows/columns of projection operators (see example 1.5)*

$$P_1 = \frac{\mathbf{M} - \lambda_2 \mathbf{1}}{\lambda_1 - \lambda_2}\,, \qquad P_2 = \frac{\mathbf{M} - \lambda_1 \mathbf{1}}{\lambda_2 - \lambda_1}\,, \qquad \lambda_1 \neq \lambda_2\,. \tag{1.38}$$

Degenerate eigenvalues. *If $\lambda_1 = \lambda_2 = \lambda$, we distinguish two cases: (a) $\mathbf{M}$ can be brought to diagonal form. This is the easy case. (b) $\mathbf{M}$ can be brought to* Jordan form, *with zeros everywhere except for the diagonal, and some 1's directly above it; for a $[2\times2]$ matrix the Jordan form is*

$$\mathbf{M} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad \mathbf{e}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$\mathbf{v}^{(2)}$ helps span the 2-dimensional space, $(\mathbf{M}-\lambda)^2\mathbf{v}^{(2)} = 0$, but is not an eigenvector, as $\mathbf{M}\mathbf{v}^{(2)} = \lambda\mathbf{v}^{(2)} + \mathbf{e}^{(1)}$. For every such Jordan $[d_\alpha \times d_\alpha]$ block there is only one eigenvector per block. Noting that

$$\mathbf{M}^m = \begin{bmatrix} \lambda^m & m\lambda^{m-1} \\ 0 & \lambda^m \end{bmatrix},$$

we see that instead of acting multiplicatively on $\mathbb{R}^2$, Jacobian matrix $J^t = \exp(t\mathbf{M})$

$$e^{t\mathbf{M}} \begin{pmatrix} u \\ v \end{pmatrix} = e^{t\lambda} \begin{pmatrix} u + tv \\ v \end{pmatrix} \tag{1.39}$$

picks up a power-low correction. That spells trouble (logarithmic term $\ln t$ if we bring the extra term into the exponent).

Example 1.5. *Projection operator decomposition in 2 dimensions:* *Let's illustrate how the distinct eigenvalues case works with the $[2\times2]$ matrix [11]*

$$\mathbf{M} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}.$$

Its eigenvalues $\{\lambda_1, \lambda_2\} = \{5, 1\}$ are the roots (1.37):

$$\det(\mathbf{M} - \lambda\mathbf{1}) = \lambda^2 - 6\,\lambda + 5 = (5-\lambda)(1-\lambda) = 0\,.$$

That $\mathbf{M}$ satisfies its secular equation (Hamilton-Cayley theorem) can be verified by explicit calculation:

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}^2 - 6\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Associated with each root λ_i is the projection operator *(1.38)*

$$\mathbf{P}_1 = \frac{1}{4}(\mathbf{M} - \mathbf{1}) = \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \tag{1.40}$$

$$\mathbf{P}_2 = -\frac{1}{4}(\mathbf{M} - 5\cdot\mathbf{1}) = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix}. \tag{1.41}$$

Matrices $\mathbf{P}_i$ are orthonormal and complete. The dimension of the ith subspace is given by $d_i = \mathrm{tr}\,\mathbf{P}_i$; in case at hand both subspaces are 1-dimensional. From the characteristic equation it follows that $\mathbf{P}_i$ satisfies the eigenvalue equation $\mathbf{M}\,\mathbf{P}_i = \lambda_i \mathbf{P}_i$. Two consequences are immediate. First, we can easily evaluate any function of $\mathbf{M}$ by spectral decomposition, for example

$$\mathbf{M}^7 - 3\cdot\mathbf{1} = (5^7-3)\mathbf{P}_1 + (1-3)\mathbf{P}_2 = \begin{bmatrix} 58591 & 19531 \\ 58593 & 19529 \end{bmatrix}.$$

Second, as $\mathbf{P}_i$ satisfies the eigenvalue equation, its every column is a right eigenvector, and every row a left eigenvector. Picking first row/column we get the eigenvectors:

$$\begin{aligned} \{\mathbf{e}^{(1)}, \mathbf{e}^{(2)}\} &= \{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-3\end{bmatrix}\} \\ \{\mathbf{e}_{(1)}, \mathbf{e}_{(2)}\} &= \{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\}, \end{aligned}$$

*with overall scale arbitrary. The matrix is not symmetric, so $\{\mathbf{e}^{(j)}\}$ do not form an orthogonal basis. The left-right eigenvector dot products $\mathbf{e}_{(j)} \cdot \mathbf{e}^{(k)}$, however, are orthogonal as in (1.31), by inspection. (Continued in example **??**.)*

Example 1.6. ***Computing matrix exponentials.*** *If A is diagonal (the system is uncoupled), then e^{tA} is given by*

$$\exp\begin{pmatrix} \lambda_1 t & & & \\ & \lambda_2 t & & \\ & & \ddots & \\ & & & \lambda_d t \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_d t} \end{pmatrix}.$$

If A is diagonalizable, $A = FDF^{-1}$, where D is the diagonal matrix of the eigenvalues of A and F is the matrix of corresponding eigenvectors, the result is simple: $A^n = (FDF^{-1})(FDF^{-1})\dots(FDF^{-1}) = FD^nF^{-1}$. Inserting this into the Taylor series for e^x gives $e^{At} = Fe^{Dt}F^{-1}$.

But A may not have d linearly independant eigenvectors, making F singular and forcing us to take a different route. To illustrate this, consider $[2\times 2]$ matrices. For any linear system in $\mathbb{R}^2$, there is a similarity transformation

$$B = U^{-1}AU,$$

where the columns of U consist of the generalized eigenvectors of A such that B has one of the following forms:

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \qquad B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad B = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}.$$

These three cases, called normal forms, correspond to A having (1) distinct real eigenvalues, (2) degenerate real eigenvalues, or (3) a complex pair of eigenvalues. It follows that

$$e^{Bt} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix}, \qquad e^{Bt} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \qquad e^{Bt} = e^{at}\begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix},$$

and $e^{At} = Ue^{Bt}U^{-1}$. What we have done is classify all $[2\times 2]$ matrices as belonging to one of three classes of geometrical transformations. The first case is scaling, the second is a shear, and the third is a combination of rotation and scaling. The generalization of these normal forms to $\mathbb{R}^d$ is called the Jordan normal form. (J. Halcrow)

1.6.1 Yes, but how do you really do it?

As $\mathbf{M}$ has only real entries, it will in general have either real eigenvalues (over-damped oscillator, for example), or complex conjugate pairs of eigenvalues (under-damped oscillator, for example). That is not surprising, but also the corresponding eigenvectors can be either real or complex. All coordinates used in defining the flow are real numbers, so what is the meaning of a *complex* eigenvector?

Due to the reality of $\mathbf{M}$, complex eigenvalues form complex conjugate pairs,

$$\{\lambda_k, \lambda_{k+1}\} = \{\mu + i\omega, \mu - i\omega\}\,,$$

and the sum of terms in the spectral decomposition (1.29) of $\mathbf{M}$ is real,

$$\begin{aligned}\mathbf{M} &= \cdots + (\mu_k + i\omega_k)\mathbf{P}_k + (\mu_k - i\omega_k)\mathbf{P}_{k+1} + \cdots \\ &= \cdots + \mu_k \mathbf{R}_k + \omega_k \mathbf{Q}_k + \cdots\,,\end{aligned} \tag{1.42}$$

where $\mathbf{R}_k = \mathbf{P}_k + \mathbf{P}_{k+1}$ and $\mathbf{Q}_k = i(\mathbf{P}_k - \mathbf{P}_{k+1})$ are matrices with real elements.

$$\mathbf{P}_k = \frac{1}{2}(\mathbf{R} + i\mathbf{Q})\,, \qquad \mathbf{P}_{k+1} = \mathbf{P}_k^*\,,$$

Any matrix function $f(\mathbf{M})$ takes the scalar value $f(\lambda_i)$ on the $\mathbf{P}_i$ subspace, $f(\mathbf{M})\,\mathbf{P}_i = f(\lambda_i)\,\mathbf{P}_i$, and is thus easily evaluated through its

they are in a sense degenerate:

is also a projection operator pair, but this time

If two eigenvalues form a complex conjugate pair, $\{\lambda_k, \lambda_{k+1}\} = \{\mu + i\omega, \mu - i\omega\}$, they are in a sense degenerate: while a real λ_k characterizes a motion along a line, a complex λ_k characterizes a spiralling motion in a plane. We determine this plane by replacing the corresponding complex eigenvectors by their real and imaginary parts, $\{\mathbf{e}^{(k)}, \mathbf{e}^{(k+1)}\} \to \{\mathrm{Re}\,\mathbf{e}^{(k)}, \mathrm{Im}\,\mathbf{e}^{(k)}\}$, or, in terms of projection operators: Substitution

$$\begin{bmatrix}\mathbf{P}_k \\ \mathbf{P}_{k+1}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & i \\ 1 & -i\end{bmatrix}\begin{bmatrix}\mathbf{R} \\ \mathbf{Q}\end{bmatrix}\,,$$

suggest introduction of a $\det U = 1$, special unitary matrix

$$U = \frac{e^{i\pi/2}}{\sqrt{2}}\begin{bmatrix}1 & i \\ 1 & -i\end{bmatrix} \tag{1.43}$$

which brings the $\lambda_k \mathbf{P}_k + \lambda_{k+1}\mathbf{P}_{k+1}$ complex eigenvalue pair in the spectral decomposition into the real form,

$$U^\top \begin{bmatrix}\mu + i\omega & 0 \\ 0 & \mu - i\omega\end{bmatrix} U = \begin{bmatrix}\mu & -\omega \\ \omega & \mu\end{bmatrix}\,.$$

$$[\mathbf{P}_k, \mathbf{P}_{k+1}]\begin{bmatrix}\lambda & 0 \\ 0 & \lambda^*\end{bmatrix}\begin{bmatrix}\mathbf{P}_k \\ \mathbf{P}_{k+1}\end{bmatrix} = [\mathbf{R}, \mathbf{Q}]\begin{bmatrix}\mu & -\omega \\ \omega & \mu\end{bmatrix}\begin{bmatrix}\mathbf{R} \\ \mathbf{Q}\end{bmatrix}\,, \tag{1.44}$$

where we have dropped the superscript $^{(k)}$ for notational brevity.

To summarize, spectrally decomposed matrix $\mathbf{M}$ acts along lines on subspaces corresponding to real eigenvalues, and as a $[2\times 2]$ rotation in a plane on subspaces corresponding to complex eigenvalue pairs.

Commentary

Remark 1.1. Projection operators. The construction of projection operators given in sect. 1.6.1 is taken from refs. [5, 6]. Who wrote this down first we do not know, lineage certainly goes all the way back to Lagrange polynomials [13], but projection operators tend to get drowned in sea of algebraic details. Arfken and Weber [1] ascribe spectral decomposition (1.30) to Sylvester. Halmos [9] is a good early reference - but we like Harter's exposition [10–12] best, for its multitude of specific examples and physical illustrations. In particular, by the time we get to (1.27) we have tacitly assumed full diagonalizability of matrix $\mathbf{M}$. That is the case for the compact groups we will study here (they are all subgroups of $\mathrm{U}(n)$) but not necessarily in other applications. A bit of what happens then (nilpotent blocks) is touched upon in example 1.4. Harter in his lecture Harter's lecture 5 (starts about min. 31 into the lecture) explains this in great detail - its well worth your time.

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Exercises

1.1. **Trace-log of a matrix.** Prove that

$$\det M = e^{\mathrm{tr} \ln M} .$$

for an arbitrary nonsingular finite dimensional matrix M, $\det M \neq 0$.

1.2. **Stability, diagonal case.** Verify that for a diagonalizable matrix A the exponential is also diagonalizable

$$J^t = e^{tA} = \mathbf{U}^{-1} e^{tA_D} \mathbf{U} , \quad A_D = \mathbf{U} \mathbf{A} \mathbf{U}^{-1} . \tag{1.45}$$

1.3. **The matrix square root.** Consider matrix

$$A = \left[\begin{array}{cc} 4 & 10 \\ 0 & 9 \end{array} \right] .$$

Generalize the square root function $f(x) = x^{1/2}$ to a square root $f(A) = A^{1/2}$ of a matrix A.
a) Which one(s) of these is/are the square root of A

$$\left[\begin{array}{cc} 2 & 2 \\ 0 & 3 \end{array} \right] , \left[\begin{array}{cc} -2 & 10 \\ 0 & 3 \end{array} \right] , \left[\begin{array}{cc} -2 & -2 \\ 0 & -3 \end{array} \right] , \left[\begin{array}{cc} 2 & -10 \\ 0 & -3 \end{array} \right] ?$$

b) Assume that the eigenvalues of a $[d \times d]$ matrix are all distinct. How many square root matrices does such matrix have?
c) Given a $[2{\times}2]$ matrix A with a distinct pair of eigenvalues $\{\lambda_1, \lambda_2\}$, write down a formula that generates all square root matrices $A^{1/2}$. Hint: one can do this using the 2 projection operators associates with the matrix A. 2 points

1.4. **Real representation of complex eigenvalues.** (Verification of example 1.2.) λ_k, λ_{k+1} eigenvalues form a complex conjugate pair, $\{\lambda_k, \lambda_{k+1}\} = \{\mu + i\omega, \mu - i\omega\}$. Show that

(a) corresponding projection operators are complex conjugates of each other,

$$\mathbf{P} = \mathbf{P}_k , \qquad \mathbf{P}^* = \mathbf{P}_{k+1} ,$$

where we denote $\mathbf{P}_k$ by $\mathbf{P}$ for notational brevity.

(b) $\mathbf{P}$ can be written as

$$\mathbf{P} = \frac{1}{2}(\mathbf{R} + i\mathbf{Q}) ,$$

where $\mathbf{R} = \mathbf{P}_k + \mathbf{P}_{k+1}$ and $\mathbf{Q}$ are matrices with real elements.

(c)

$$\begin{bmatrix} \mathbf{P}_k \\ \mathbf{P}_{k+1} \end{bmatrix} = \frac{1}{2} \left[\begin{array}{cc} 1 & i \\ 1 & -i \end{array} \right] \begin{bmatrix} \mathbf{R} \\ \mathbf{Q} \end{bmatrix} .$$

(d) The $\cdots + \lambda_k \mathbf{P}_k + \lambda_k^* \mathbf{P}_{k+1} + \cdots$ complex eigenvalue pair in the spectral decomposition (1.29) is now replaced by a real $[2\times 2]$ matrix

$$\cdots + \left[\begin{array}{cc} \mu & -\omega \\ \omega & \mu \end{array} \right] \begin{bmatrix} \mathbf{R} \\ \mathbf{Q} \end{bmatrix} + \cdots$$

or whatever you find the clearest way to write this real representation.

group theory - week 2

Finite groups - definitions

Homework HW2

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 2.1 $G_x \subset G$	1 point
Exercise 2.2 *Transitivity of conjugation*	1 point
Exercise 2.3 *Isotropy subgroup of* gx	1 points
Exercise 2.4 D_3*: symmetries of an equilateral triangle*	5 points
Exercise 2.5 C_4*-invariant potential*	7 (+2) points

Bonus points

Exercise 2.6 (a), (b) and (c) *Permutation of three objects*	2 points
Exercise 2.7 *Three masses on a loop*	6 points

Total of 15 points = 100 % score.

Extra points accumulate, can help you later if you miss a few problems.

2.1 Week's videos, reading

If I had had more time, I would have written less
— Blaise Pascal, a remark made to a correspondent

Please do not get intimated by the length of this week's notes - they are here more for me than for you, as notes on these topics for future reference. If you understand the main sequence of video clips, and recommended reading, that should suffice to do the problems. The rest is optional, you can quickly skim over...

2.1.1 Don't wonna know group theory

The fastest way to watch any week's lecture videos is by letting YouTube run the

- *Section playlist*
  - Sect. 2.4 *Using symmetries*
  - Sect. 2.5 *Normal modes*: The free vibrations of systems, for undamped systems with total energy conserved for which the frequencies of oscillation are real.
- *Normal modes* (9:06 min)
  - example 2.5 *Vibrational spectra of molecules* is taken from Gutkin lecture notes example 5.1 C_n *symmetry*. The corresponding projection operators (1.25) are worked out in example 2.6.
  - Example 2.4 *Vibrations of a classical CO_2 molecule*
- *A Hamiltonian with a symmetry* (4:46 min)
- *CO_2 molecule* (4:07 min)
- *Projection operators (5:33 min)*
- *(Anti)symmetric subspaces (3:04 min)*
- *Zero mode (5:19 min)*

2.1.2 Finite groups

Groups, permutations, rearrangement theorem, subgroups, cosets, classes, all exemplified by the $D_3 \cong C_{3v} \cong S_3$ symmetries of an equilateral triangle.

- *Section playlist*
- Dresselhaus *et al.* [4] Chapter 1 *Basic Mathematical Background: Introduction*. The MIT course 6.734 online version contains much of the same material.
- ChaosBook Chapter 10. Flips, slides and turns

- *Discrete symmetry, an example: 3-disk pinball (4:03 min)*
- *What is a group?* (10:56 min)
  - (extra) *Discussion: There might be many examples of it, but a 'group' itself is an abstract notion.* (3 min)
  - (extra) *Discussion: permutations, symmetric group, simple groups, Italian renaissance, French revolution, Galois* (5:23 min)
- *by Socratica: (cannot add it to the section YouTube playlist) a delightful introduction to group multiplication (or Cayley) tables. (7:32)*
- *Active, passive coordinate transformations (3:08 min)*
- *Following Mefisto: symmetry defined three (3) times (7:01 min)*
- *Subgroups, classes, group orbits, reduced state space* (7:57 min)

2.2 Other sources (optional)

- *Group theory and why I love 808,017,· · · ,000* is a great video on group theory from 3Blue1Brown, writes Andrew Wu. I agree: Well worth of your time, more motivational than my lectures. What it actually focuses on - the monster group - is totally useless to us.
- AWH Example 6.2.3 *Degenerate eigenproblem*.
- AWH Example 6.5.2 *Normal modes*.
- For a deep dive into this material, here is your rabbit hole.
- For deeper insights, read Roger Penrose [7] (click here).
- Nathan Carter *Visual Group Theory* (read it online through your university library) seems very good. The next two online courses are based on it:
  - Dana Ernst *An inquiry-based approach to abstract algebra*.
  - Matt Macauley Twitter *Group actions* course and *Modern Algebra* course.
- Tom Judson's online *Abstract Algebra: Theory and Applications*.
- AWH Chapter 17 *Group Theory* (click here).
- For a typical (but for this course advanced) application see, for example, Stone and Goldbart [10], *Mathematics for Physics: A Guided Tour for Graduate Students*, Section 14.3.2 *Vibrational spectrum of* H_2O (click here).
- ◦ Glance through sect. 2.6 *Group presentations* and sect. 2.8 *Literature*, but I do not expect you to understand this material.

Discussion 4 - Homework (3 min)

There is no need to learn all these "Greek" words.

- If instead, bedside crocheting is your thing, click here.

2.3 Using group theory without knowing any

It's a matter of no small pride for a card-carrying dirt physics theorist to claim full and total ignorance of group theory (read ChaosBook Appendix A.6 Gruppenpest). So what we will do first is work out a few examples of physical applications of group theory that you already know without knowing that you have been using "Group Theory."

2.4 Using symmetries

Tyger Tyger burning bright,
In the forests of the night:
What immortal hand or eye,
Dare frame thy fearful symmetry?
—William Blake, *The Tyger*

The big idea #1 of this is week is *symmetry*.

If our physical problem is defined by a (perhaps complicated) Hamiltonian $\mathbf{H}$, another matrix $\mathbf{M}$ (hopefully a very simple matrix) is a symmetry if it commutes with the Hamiltonian

$$[\mathbf{M}, \mathbf{H}] = 0\,. \tag{2.1}$$

Than we can use the spectral decomposition (1.30) of $\mathbf{M}$ to block-diagonalize $\mathbf{H}$ into a sum of lower-dimensional sub-matrices,

$$\mathbf{H} = \sum_i \mathbf{H}_i\,, \qquad \mathbf{H}_i = \mathbf{P}_i \mathbf{H} \mathbf{P}_i\,, \tag{2.2}$$

and thus significantly simplify the computation of eigenvalues and eigenvectors of $\mathbf{H}$, the matrix of physical interest.

2.5 Normal modes

The big idea #2 of this is week is: *many body systems* (molecules, neuronal networks, ...) are ruled by *collective modes*, not individual particles (atoms, neurons, ...).

D_3	e	C	C^2	$\sigma^{(1)}$	$\sigma^{(2)}$	$\sigma^{(3)}$
e	e	C	C^2	$\sigma^{(1)}$	$\sigma^{(2)}$	$\sigma^{(3)}$
C	C	C^2	e	$\sigma^{(3)}$	$\sigma^{(1)}$	$\sigma^{(2)}$
C^2	C^2	e	C	$\sigma^{(2)}$	$\sigma^{(3)}$	$\sigma^{(1)}$
$\sigma^{(1)}$	$\sigma^{(1)}$	$\sigma^{(2)}$	$\sigma^{(3)}$	e	C	C^2
$\sigma^{(2)}$	$\sigma^{(2)}$	$\sigma^{(3)}$	$\sigma^{(1)}$	C^2	e	C
$\sigma^{(3)}$	$\sigma^{(3)}$	$\sigma^{(1)}$	$\sigma^{(2)}$	C	C^2	e

Table 2.1: The dihedral group D_3 group multiplication table. Actually, we prefer cyclic and dihedral groups notation ‘rotations’ r^ℓ and ‘flips’ σ_m, as in table 4.1.

In the linear, harmonic oscillator approximation, the classical dynamics of a molecule is governed by the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{m_i}{2} \dot{x}_i^2 + \frac{1}{2} \sum_{i,j=1}^{N} x_i^\top V_{ij} x_j \,,$$

where $\{x_i\}$ are small deviations from the equilibrium, resting points of the molecules labelled i. V_{ij} is a symmetric matrix, so it can be brought to a diagonal form by an orthogonal transformation, to a set of N uncoupled harmonic oscillators or *normal modes* of frequencies $\{\omega_i\}$.

$$x \to y = Ux, \qquad H = \sum_{i=1}^{N} \frac{m_i}{2} \left(\dot{y}_i^2 + \omega_i^2 y_i^2 \right) \,. \tag{2.3}$$

2.6 Group presentations

Group theory? It is all about class & character.
— Predrag Cvitanović, *One minute elevator pitch*

Group multiplication (or Cayley) tables, such as Table 2.1, *define* each distinct discrete group, but they can be hard to digest. A Cayley graph, with links labeled by generators, and the vertices corresponding to the group elements, has the same information as the group multiplication table, but is often a more insightful presentation of the group.

For example, the Cayley graph figure 2.1 is a clear presentation of the dihedral group D_4 of order 8,

$$D_4 = (e, a, a^2, a^3, b, ba, ba^2, ba^3) \,, \quad \text{generators } a^4 = e \,,\; b^2 = e \,. \tag{2.4}$$

Quaternion group is also of order 8, but with a distinct multiplication table / Cayley graph, see figure 2.2. For more of such, see, for example, mathoverflow Cayley graph discussion.

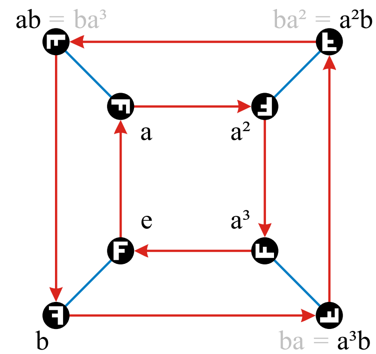


Figure 2.1: A Cayley graph presentation of the dihedral group D_4. The 'root vertex' of the graph, marked e, is here indicated by the letter $\mathbb{F}$, the links are multiplications by two generators: a cyclic rotation by left-multiplication by element a (directed red link), and the flip by b (undirected blue link). The vertices are the 8 possible orientations of the transformed letter $\mathbb{F}$.

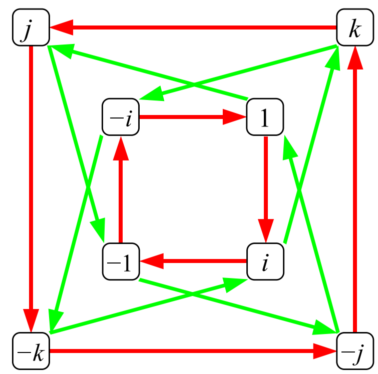
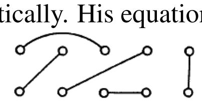


Figure 2.2: A Cayley graph presentation of the quaternion group Q_8. It is also of order 8, but distinct from D_4.

2.7 Permutations in birdtracks

The text that follows is a very condensed extract of birdtracks.eu chapter 6 *Permutations*, from *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [3]. I am usually reluctant to use birdtrack notations in front of graduate students indoctrinated by their professors in the 1890's tensor notation, but I'm emboldened by the very enjoyable article on *The new language of mathematics* by Dan Silver [9]. Your professor's notation is as convenient for actual calculations as -let's say- long division using roman numerals. So leave them wallowing in their early progressive rock of 1968, King Crimsons of their youth. You chill to beats younger than Windows 98, to grime, to trap, to hardvapour, to birdtracks.

In 1937 R. Brauer [2] introduced diagrammatic notation for the Kronecker δ_{ij} operation, in order to represent "Brauer algebra" permutations, index contractions, and matrix multiplication diagrammatically. His equation (39)



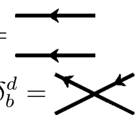
(send index 1 to 2, 2 to 4, contract ingoing (3·4), outgoing (1·3)) is the earliest published diagrammatic notation I know about. While in kindergarten (disclosure: we were too poor to afford kindergarten) I sat out to revolutionize modern group theory [3]. But I suffered a terrible setback; in early 1970's Roger Penrose pre-invented my "birdtracks," or diagrammatic notation, for symmetrization operators [6], Levi-Civita tensors [8], and "strand networks" [5]. Here is a little flavor of how one birdtracks:

We can represent the operation of permuting indices (d "billiard ball labels," tensors with d indices) by a matrix with indices bunched together:

$$\sigma_\alpha^\beta = \sigma_{b_1 \dots b_p}^{a_1 a_2 \dots a_q}{}_{,c_q \dots c_2 c_1}^{d_p \dots d_1} \,. \tag{2.5}$$

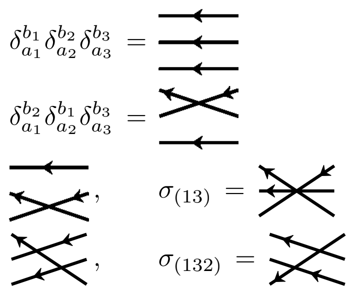
To draw this, Brauer style, it is convenient to turn his drawing on a side. For 2-index tensors, there are two permutations:

$$\text{identity:} \quad \mathbf{1}_{ab,}{}^{cd} = \delta_a^d \delta_b^c =$$
$$\text{flip:} \quad \sigma_{(12)ab,}{}^{cd} = \delta_a^c \delta_b^d = \,. \tag{2.6}$$



For 3-index tensors, there are six permutations:

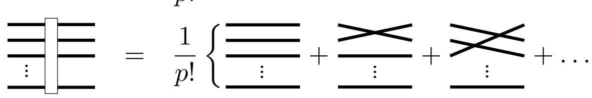
$$\mathbf{1}_{a_1 a_2 a_3,}{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} =$$
$$\sigma_{(12)a_1 a_2 a_3,}{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} \delta_{a_3}^{b_3} =$$
$$\sigma_{(23)} = \,, \qquad \sigma_{(13)} =$$
$$\sigma_{(123)} = \,, \qquad \sigma_{(132)} = \,. \tag{2.7}$$



Here group element labels refer to the standard permutation cycles notation. There is really no need to indicate the "time direction" by arrows, so we omit them from now on.

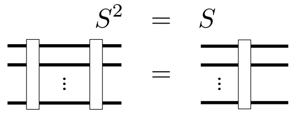
The symmetric sum of all permutations,

$$S_{a_1 a_2 \dots a_p,}{}^{b_p \dots b_2 b_1} = \frac{1}{p!}\left\{ \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_p}^{b_p} + \delta_{a_2}^{b_1} \delta_{a_1}^{b_2} \dots \delta_{a_p}^{b_p} + \dots \right\}$$
$$S = \quad = \frac{1}{p!}\left\{ \quad + \quad + \quad + \dots \right\} \,, \tag{2.8}$$



yields the symmetrization operator S. In birdtrack notation, a white bar drawn across p lines [6] will always denote symmetrization of the lines crossed. A factor of $1/p!$ has been introduced in order for S to satisfy the projection operator normalization

$$S^2 = S$$
$$\quad = \quad . \tag{2.9}$$

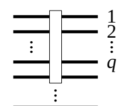


You have already seen such "fully-symmetric representation," in the discussion of discrete Fourier transforms, ChaosBook Example A24.3 *'Configuration-momentum'*

Fourier space duality, but you are not likely to recognize it. There the average was not over all permutations, but the zero-th Fourier mode $\tilde{\phi}_0$ was the average over only cyclic permutations. Every finite discrete group has such fully-symmetric representation, and in statistical mechanics and quantum mechanics this is often the most important state (the 'ground' state).

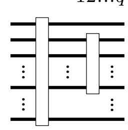
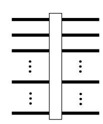
A subset of indices $a_1, a_2, \ldots a_q$, $q < p$ can be symmetrized by symmetrization matrix $S_{12\ldots q}$

$$(S_{12\ldots q})_{a_1 a_2 \ldots a_q \ldots a_p},{}^{b_p \ldots b_q \ldots b_2 b_1} = \frac{1}{q!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_q}^{b_q} + \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_q}^{b_q} + \ldots\right\}\delta_{a_q+1}^{b_q+1}\ldots\delta_{a_p}^{b_p}$$

$$S_{12\ldots q} =$$  . (2.10)

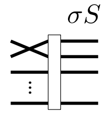
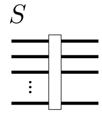
Overall symmetrization also symmetrizes any subset of indices:

$$SS_{12\ldots q} = S$$

 =  . (2.11)

Any permutation has eigenvalue 1 on the symmetric tensor space:

$$\sigma S = S$$

 =  . (2.12)

Diagrammatically this means that legs can be crossed and uncrossed at will.

One can construct a projection operator onto the fully antisymmetric space in a similar manner [3]. Other representations are trickier - that's precisely what the theory of finite groups is about.

2.8 Other sources (optional)

The exposition (or the corresponding chapter in Tinkham [11]) that we follow here largely comes from Wigner's classic *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* [12], which is a harder going, but the more group theory you learn the more you'll appreciate it.

The structure of finite groups was understood by late 19th century. A full list of finite groups was another matter. The complete proof of the classification of all finite groups takes about 3 000 pages, a collective 40-years undertaking by over 100 mathematicians, read the wiki. Not all finite groups are as simple or easy to figure out as D_3. For example, the order of the Ree group ${}^2F_4(2)'$ is $212(26+1)(24-1)(23+1)(2-1)/2 = 17\,971\,200$.

From Emory Math Department: A pariah is real! The simple finite groups fit into 18 families, except for the 26 sporadic groups. 20 sporadic groups AKA the Happy Family are parts of the Monster group. The remaining six loners are known as the pariahs.

Hang in there! And relax. None of this will be on the test. As a matter of fact, there will be no test.

Question 2.1. Henriette Roux asks

Q What did you do this weekend?

A The same as every other weekend - prepared week's lecture, with my helpers Avi the Little, Edvard the Nordman, and Malbec el Argentino, under Master Roger's watchful eye, see here.

2.9 Examples

Example 2.1. ***Discrete symmetries in physics:***

- *Point groups i.e., subgroups of $O(3)$.*
- *Point groups + discrete translations e.g., symmetry groups of crystals.*
- *Permutation groups*

$$S\Psi(x_1, x_2, \dots x_n) = \Psi(x_2, x_1, \dots x_n).$$

- *Boson wave functions are symmetric while fermion wave functions are anti-symmetric under exchange of variables.*

(B. Gutkin)

Example 2.2. ***The group multiplication table for D_3:*** *See table 4.1.*

Example 2.3. ***Reflection and discrete rotation symmetries:***

(a) *Reflection symmetry $V(x) = PV(x) = V(-x)$:*

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = E_n\psi(x) \tag{2.13}$$

(see figure 2.3). If $\psi(x)$ is solution then $P\psi(x)$ is also solution. From this and non-degeneracy of the spectrum follows that either $P\psi(x) = \psi(x)$ or $P\psi(x) = -\psi(x)$. The first case corresponds to symmetric functions while the second one to anti-symmetric one. Thus the whole spectrum can be decomposed in accordance to a symmetry of the Hamiltonian (equations of motion).

(b) *Rotation symmetry $V(x) = gV(x)$, $G = \{e, g, g^2\}$: By the same argument we have three possibilities:*

$$g\psi(x) = \psi(x); \qquad g\psi(x) = e^{i2\pi/3}\psi(x); \qquad g^{-1}\psi(x) = e^{-i2\pi/3}\psi(x).$$

In addition, by the time reversal symmetry if $\psi(x)$ is solution then $\psi^*(x)$ is solution with the same eigenvalue as well. From this follows that the spectrum must be degenerate. The spectrum is split into a real eigenfunction $\{\psi_1(x)\}$, and a degenerate pair of real eigenfunctions

$$\psi_2(x) = \psi(x) + \psi^*(x); \psi_3(x) = i(\psi(x) - \psi^*(x))\,, \quad \text{where } g\psi(x) = e^{i2\pi/3}\psi(x)$$

invariant under rotations by 1/3-rd of a circle.

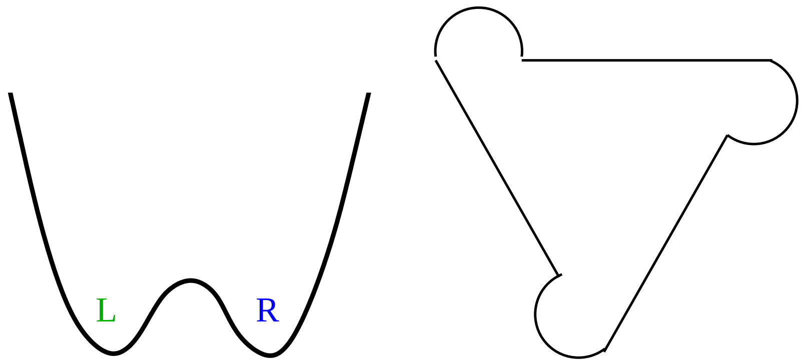


Figure 2.3: (left) A reflection-symmetric double-well potential. (right) A $1/3$rd-circle rotation-symmetric plane billiard (infinite wall potential in $2D$). (B. Gutkin)

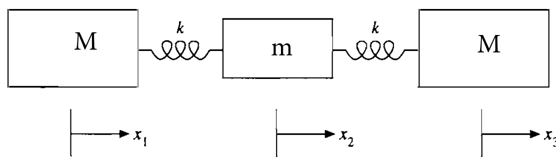


Figure 2.4: A classical colinear CO_2 molecule [1].

(B. Gutkin)

Example 2.4. ***Vibrations of a classical CO$_2$ molecule:*** *Consider one carbon and two oxygens constrained to the x-axis [1] and joined by springs of stiffness k, as shown in figure 2.4. Newton's second law says*

$$\begin{aligned}
\ddot{x}_1 &= -\frac{k}{M}(x_1 - x_2) \\
\ddot{x}_2 &= -\frac{k}{m}(x_2 - x_3) - \frac{k}{m}(x_2 - x_1) \\
\ddot{x}_3 &= -\frac{k}{M}(x_3 - x_2) \,.
\end{aligned} \tag{2.14}$$

The normal modes, with time dependence $x_j(t) = x_j \exp(it\omega)$, are the common frequency ω vibrations that satisfy (2.14),

$$\mathbf{H}\mathbf{x} = \begin{pmatrix} A & -A & 0 \\ -a & 2\,a & -a \\ 0 & -A & A \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \omega^2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} , \tag{2.15}$$

where $a = k/m$, $A = k/M$. Secular determinant $\det(\mathbf{H} - \omega^2 \mathbf{1}) = 0$ now yields a cubic equation for ω^2.

You might be tempted to stick this $[3\times 3]$ matrix into `Mathematica` or whatever, but please do that in some other course. What would understood by staring at the output? In this course we think.

First thing to always *ask yourself is: does the system have a* symmetry*? Yes! Note that the CO$_2$ molecule (2.14) of figure 2.4 is invariant under $x_1 \leftrightarrow x_3$ interchange, i.e., coordinate relabeling by matrix σ that commutes with our law of motion $\mathbf{H}$,*

$$\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \sigma\mathbf{H} = \mathbf{H}\sigma = \begin{pmatrix} 0 & -A & A \\ -a & 2a & -a \\ A & -A & 0 \end{pmatrix}. \tag{2.16}$$

We can now use the symmetry operator σ to simplify the calculation. As $\sigma^2 = \mathbf{1}$, its eigenvalues are ± 1, and the corresponding symmetrization, anti-symmetrization projection operators *(1.38) are*

$$\mathbf{P}_+ = \frac{1}{2}(\mathbf{1} + \sigma), \qquad \mathbf{P}_- = \frac{1}{2}(\mathbf{1} - \sigma). \tag{2.17}$$

The dimensions $d_i = \mathrm{tr}\,\mathbf{P}_i$ of the two subspaces are

$$d_+ = 2, \qquad d_- = 1. \tag{2.18}$$

As σ and $\mathbf{H}$ commute, we can now use spectral decomposition (1.30) to block-diagonalize $\mathbf{H}$ to a 1-dimensional and a 2-dimensional matrix.

On the 1-dimensional antisymmetric subspace, the trace of a $[1\times 1]$ matrix equals its sole matrix element equals it eigenvalue

$$\lambda_- = \mathbf{H}\mathbf{P}_- = \frac{1}{2}(\mathrm{tr}\,\mathbf{H} - \mathrm{tr}\,\mathbf{H}\sigma) = (a + A) - a = \frac{k}{M},$$

so the corresponding eigenfrequency is $\omega_-^2 = k/M$. To understand its physical meaning, write out the antisymmetric subspace projection operator (2.18) explicitly. Its non-vanishing columns are proportional to the sole eigenvector

$$\mathbf{P}_- = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{e}^{(-)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}. \tag{2.19}$$

In this subspace the outer oxygens are moving in opposite directions, with the carbon stationary.

On the 2-dimensional symmetric subspace, the trace yields the sum of the remaining two eigenvalues

$$\lambda_+ + \lambda_0 = \mathrm{tr}\,\mathbf{H}\mathbf{P}_+ = \frac{1}{2}(\mathrm{tr}\,\mathbf{H} + \mathrm{tr}\,\mathbf{H}\sigma) = (a + A) + a = \frac{k}{M} + 2\,\frac{k}{m}.$$

We could disentangle the two eigenfrequencies by evaluating $\mathrm{tr}\,\mathbf{H}^2\mathbf{P}_+$, for example, but thinking helps again.

There is still another, translational symmetry, so obvious that we forgot it; if we change the origin of the x-axis, the three coordinates $x_j \to x_j - \delta x$ change, for any continuous translation δx, but the equations of motion (2.14) do not change their form,

$$\mathbf{H}\,\mathbf{x} = \mathbf{H}\,\mathbf{x} + \mathbf{H}\,\delta\mathbf{x} = \omega^2\mathbf{x} \quad \Rightarrow \quad \mathbf{H}\,\delta\mathbf{x} = 0. \tag{2.20}$$

So any translation $\mathbf{e}^{(0)} = \delta\mathbf{x} = (\delta x, \delta x, \delta x)$ is a nul, 'zero mode' eigenvector of $\mathbf{H}$ in (2.16), with eigenvalue $\lambda_0 = \omega_0^2 = 0$, and thus the remaining eigenfrequency is $\omega_+^2 = k/M + 2\,k/m$. As we can add any nul eigenvector $\mathbf{e}^{(0)}$ to the corresponding $\mathbf{e}^{(+)}$ eigenvector, there is some freedom in choosing $\mathbf{e}^{(+)}$. One visualization of the *corresponding eigenvector is the carbon moving opposite to the two oxygens, with total momentum set to zero.*

(Taken from AWH Example 6.2.3 Degenerate eigenproblem*, but done here using symmetries.)*

Example 2.5. *Vibrational spectra of molecules:* *In the linear, harmonic oscillator approximation the classical dynamics of the molecule is governed by the Hamiltonian*

$$H = \sum_{i=1}^{N} \frac{m_i}{2} \dot{x}_i^2 + \frac{1}{2} \sum_{i,j=1}^{N} x_i^\top V_{ij} x_j \,,$$

where $\{x_i\}$ *are small deviations from the resting the equilibrium, resting points of the molecules labelled* i. V_{ij} *is a symmetric matrix, so it can be brought to a diagonal form by an orthogonal transformation, a set of* N *uncoupled harmonic oscillators or* normal modes *of frequencies* $\{\omega_i\}$.

$$x \to y = Ux, \qquad H = \sum_{i=1}^{N} \frac{m_i}{2} \left(\dot{y}_i^2 + \omega_i^2 y_i^2 \right) \,. \tag{2.21}$$

Consider now the ring of pair-wise interactions of two kinds of molecules sketched in figure 2.5 (a), given by the potential

$$V(z) = \frac{1}{2} \sum_{i=1}^{N} \left(k_1 (x_i - y_i)^2 + k_2 (x_{i+1} - y_i)^2 \right) \,, \qquad z_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \,, \tag{2.22}$$

whose $[2N \times 2N]$ *matrix form is (aside to the cognoscenti: this is a Toeplitz matrix):*

$$V_{ij} = \frac{1}{2} \begin{pmatrix} k_1+k_2 & -k_1 & 0 & 0 & 0 & \dots & 0 & 0 & -k_2 \\ -k_1 & k_1+k_2 & -k_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -k_2 & k_1+k_2 & -k_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -k_1 & k_1+k_2 & -k_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -k_2 & k_1+k_2 & -k_1 \\ -k_2 & 0 & 0 & 0 & 0 & \dots & 0 & -k_1 & k_1+k_2 \end{pmatrix}$$

This potential matrix is a holy mess. How do we find an orthogonal transformation (2.21) that diagonalizes it? Look at figure 2.5 (a). Molecules lie on a circle, so that suggests we should use a Fourier representation. As the $i = 1$ *labelling of the starting molecule on a ring is arbitrary, we are free to relabel them, for example use the next molecule pair as the starting one. This relabelling is accomplished by the* $[2N \times 2N]$ *permutation matrix (or 'one-step shift', 'stepping' or 'translation' matrix)* M *of form*

$$\underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 & I \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{pmatrix}}_{M} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z_n \\ z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{pmatrix} , \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad z_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \tag{2.23}$$

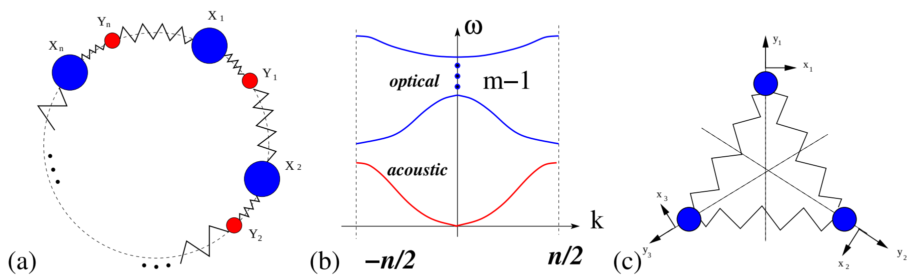


Figure 2.5: (a) Chain with circular symmetry. (b) Dependance of frequency on the representation wavenumber k. (c) Molecule with D_3 symmetry. (B. Gutkin)

Projection operators corresponding to M are worked out in example 2.6. They are N distinct $[2N\times 2N]$ matrices,

$$\mathbf{P}_k = \begin{pmatrix} I & \bar\lambda I & \bar\lambda^2 I & \dots & \bar\lambda^{N-2} I & \bar\lambda^{N-1} I \\ \lambda I & I & \bar\lambda I & \dots & \bar\lambda^{N-3} I & \bar\lambda^{N-2} I \\ \lambda^2 I & \lambda I & I & \dots & \bar\lambda^{N-4} I & \bar\lambda^{N-3} I \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda^{N-2} I & \lambda^{N-3} I & \lambda^{N-4} I & \dots & I & \bar\lambda I \\ \lambda^{N-1} I & \lambda^{N-2} I & \lambda^{N-2} I & \dots & \lambda I & I \end{pmatrix}, \qquad \lambda = \exp\left(\frac{2\pi i}{N}k\right) \tag{2.24}$$

which decompose the $2N$-dimensional configuration space of the molecule ring into a direct sum of N 2-dimensional spaces, one for each discrete Fourier mode $k = 0, 1, 2, \cdots, N-1$.

The system (2.22) is clearly invariant under the cyclic permutation relabelling M, $[V, M] = 0$ (though checking this by explicit matrix multiplications might be a bit tedious), so the $\mathbf{P}_k$ decompose the interaction potential V as well, and reduce its action to the kth 2-dimensional subspace. Thus the $[2N\times 2N]$ diagonalization (2.21) is now reduced to a $[2\times 2]$ diagonalization which one can do by hand. The resulting kth space is spanned by two $2N$-dimensional vectors, which we guess to be of form:

$$\eta_1 = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 \\ 0 \\ \lambda \\ 0 \\ \vdots \\ \lambda^{n-1} \\ 0 \end{pmatrix}, \qquad \eta_2 = \frac{1}{\sqrt{n}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \lambda \\ \vdots \\ 0 \\ \lambda^{n-1} \end{pmatrix}.$$

In order to find eigenfrequences we have to consider action of V on these two vectors:

$$V\eta_1 = (k_1 + k_2)\eta_1 - (k_1 + k_2\lambda)\eta_2\,, \qquad V\eta_2 = (k_1 + k_2)\eta_2 - (k_1 + k_2\bar\lambda)\eta_1\,.$$

The corresponding eigenfrequencies are determined by the equation:

$$
\begin{aligned}
0 &= \det\left(\begin{pmatrix} k_1+k_2 & -(k_1+k_2\lambda) \\ -(k_1+k_2\bar{\lambda}) & k_1+k_2 \end{pmatrix} - \frac{\omega^2}{2} I\right) \qquad \Longrightarrow \\
\frac{1}{2}\omega_\pm^2(k) &= k_1+k_2 \pm |k_1+k_2\lambda^k|\,,
\end{aligned}
\tag{2.25}
$$

one acoustic ($\omega(0)=0$), one optical, see figure 2.5 (b) and the acoustic and optical phonons wiki. *(B. Gutkin)*

Example 2.6. ***Projection operators for cyclic group C_N.***
Consider a cyclic group $C_N = \{e, g, g^2, \cdots g^{N-1}\}$, and let $M = D(g)$ be a $[2N\times 2N]$ representation of the one-step shift g. In the projection operator formulation (1.25), the N distinct eigenvalues of M, the Nth roots of unity $\lambda_n = \lambda^n$, $\lambda = \exp(i\,2\pi/N)$, $n = 0, \ldots N-1$, split the $2N$-dimensional space into N 2-dimensional subspaces by means of projection operators

$$
\mathbf{P}_n = \prod_{m\neq n} \frac{M - \lambda_m I}{\lambda_n - \lambda_m} = \prod_{m=1}^{N-1} \frac{\lambda^{-n} M - \lambda^m I}{1-\lambda^m}\,,
\tag{2.26}
$$

where we have multiplied all denominators and numerators by λ^{-n}. The numerator is now a matrix polynomial of form $(x-\lambda)(x-\lambda^2)\cdots(x-\lambda^{N-1})$, with the zeroth root $(x-\lambda^0) = (x-1)$ quotiented out from the defining matrix equation $M^N - 1 = 0$. Using

$$
\frac{1-x^N}{1-x} = 1 + x + \cdots + x^{N-1} = (x-\lambda)(x-\lambda^2)\cdots(x-\lambda^{N-1})
$$

we obtain the projection operator in form of a discrete Fourier sum (rather than the product (1.25)),

$$
\mathbf{P}_n = \frac{1}{N}\sum_{m=0}^{N-1} e^{i\,\frac{2\pi}{N}nm}\, M^m\,.
$$

This form of the projection operator is the simplest example of the key group theory tool, projection operator expressed as a sum over characters,

$$
\mathbf{P}_n = \frac{1}{|G|}\sum_{g\in G} \bar{\chi}(g) D(g)\,,
$$

upon which stands all that follows in this course. *(B. Gutkin and P. Cvitanović)*

2.10 What are cosets good for? (a discussion)

Question 2.2. Henriette Roux asks
Q What are cosets good for?
A Apologies for glossing over their meaning in the lecture. I try to minimize group-theory jargon, but cosets cannot be ignored.

Dresselhaus *et al.* [4] (click here) Chapter 1 *Basic Mathematical Background: Introduction* needs them to show that the dimension of a subgroup is a divisor of the dimension of the group. For example, C_3 of dimension 3 is a subgroup of D_3 of dimension 6.

In ChaosBook *Chapter 10. Flips, slides and turns* cosets are absolutely essential. The significance of the coset is that if a solution has a symmetry, then the elements in a coset act on the solution the same way, and generate all equivalent copies of this solution. Example 10.7. *Subgroups, cosets of* D_3 should help you understand that.

Henriette Roux writes: When talking about the cosets of a subgroup we demonstrated multiplication between cosets with a specific example, but this wasn't leading to something along the lines of that the set of all left cosets of a subgroup (or the set of all the right cosets of a subgroup) form a group, correct? It didn't appear so in the example since the "unit" $\{E, A\}$ we looked appears to only have the properties of an identity with multiplication from one direction (the direction depending on if it is the set of left cosets or the set of right cosets). In the context of the lecture I think this point was related to Lagrange's theorem (although we didn't call it that) and I vaguely remember cosets being used in the proof of Lagrange's theorem but I wasn't connecting it today. Are we going to cover that in a future lecture?

Predrag You are right - Lagrange's theorem (see the wiki) simply says the order of a subgroup has to be a divisor of the order of the group. We used cosets to partition elements of G to prove that. But what we really need cosets for is to define (see Dresselhaus *et al.* [4] Sect. 1.7) *Factor Groups* whose elements are cosets of a self-conjugate subgroup (click here). I will not cover that in a subsequent lecture, so please read up on it yourself.

Henriette Roux You talked about the period of an element X, and said that that *period* is the *set*

$$\{E, X, \cdots, X^{n-1}\}, \tag{2.27}$$

where n is the *order* of the element X. I had thought that set was the subgroup generated by the element X and that the period of the element X was a synonym for the order of the element X? Is that incorrect?

Predrag To keep things as simple as possible, in Thursday's lecture I followed Sect. 1.3 *Basic Definitions* of Dresselhaus *et al.* textbook [4], to the letter. In Def. 3 the *order* of an element X is the smallest n such that $X^n = E$, and they call the set (2.27) the *period* of X. I do not like that usage (and do not remember seeing it anywhere else). As you would do, in ChaosBook.org Chap. *Flips, slides and turns* I also define the smallest n to be the *period* of X and refer to the set (2.27) as the *orbit* generated by X. When we get to compact continuous groups, the orbit will be a (great) circle generated by a given Lie algebra element, and look more like what we usually think of as an orbit.

I am not using my own ChaosBook.org here, not to confuse things further by discussing both time evolution and its discrete symmetries. Here we focus on the discrete group only (typically spatial reflections and finite angle rotations).

References

[1] G. B. Arfken, H. J. Weber, and F. E. Harris, *Mathematical Methods for Physicists: A Comprehensive Guide*, 7th ed. (Academic, New York, 2013).

[2] R. Brauer, "On algebras which are connected with the semisimple continuous groups", Ann. Math. **38**, 857 (1937).

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[5] R. Penrose, "Angular momentum: An approach to combinatorical space-time", in *Quantum Theory and Beyond*, edited by T. Bastin (Cambridge Univ. Press, Cambridge, 1971).

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[7] R. Penrose, *The Road to Reality - A Complete Guide to the Laws of the Universe* (A. A. Knopf, New York, 2005).

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[10] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge Univ. Press, Cambridge UK, 2009).

[11] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

[12] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1931).

Exercises

2.1. **$G_x \subset G$.** The maximal set of group actions which maps a state space point x into itself,

$$G_x = \{g \in G : gx = x\}\,, \tag{2.28}$$

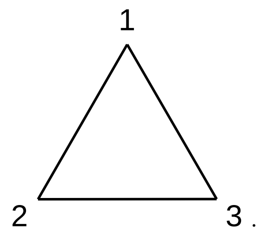
is called the *isotropy group* (or *stability subgroup* or *little group*) of x. Prove that the set G_x as defined in (2.28) is a subgroup of G.

2.2. **Transitivity of conjugation.** Assume that $g_1, g_2, g_3 \in G$ and both g_1 and g_2 are conjugate to g_3. Prove that g_1 is conjugate to g_2.

2.3. **Isotropy subgroup of gx.** Prove that for $g \in G$, x and gx have conjugate isotropy subgroups:

$$G_{gx} = g\,G_x\,g^{-1}$$

2.4. **D$_3$: symmetries of an equilateral triangle.** Consider group D$_3$, the symmetry group of an equilateral triangle:



(a) List the group elements and the corresponding geometric operations

(b) Find the subgroups of the group D$_3$.

(c) Find the classes of D$_3$ and the number of elements in them, guided by the geometric interpretation of group elements. Verify your answer using the definition of a class.

(d) List the conjugacy classes of subgroups of D$_3$. (continued as exercise 4.1)

2.5. **C$_4$ invariant potential.** Consider the Schrödinger equation for a particle moving in a two-dimensional bounding potential V, such that the spectrum is discrete. Assume that V is C$_N$-invariant (in some literature, Z$_N$-invariant), i.e., V remains invariant under the rotation R by the angle $2\pi/N$. For $N = 3$ case, figure 2.6 (a), the spectrum of the system can be split into two sectors: $\{E_n^0\}$ non-degenerate levels corresponding to symmetric eigenfunctions $\phi_n(Rx) = \phi_n(x)$ and doubly degenerate levels $\{E_n^\pm\}$ corresponding to non-symmetric eigenfunctions $\phi_n(Rx) = e^{\pm 2\pi i/3}\phi_n(x)$.

(a) What is the spectral structure in the case of $N = 4$, figure 2.6 (b)? How many sectors appear and what are their degeneracies?

(b) What is the spectral structure for general N?

(c) A constant magnetic field normal to the $2D$ plane is added to V. How will it affect the spectral structure?

(d) (bonus question) Figure out the spectral structure if the symmetry group of potential is D$_3$ (also includes 3 reflections), figure 2.6 (c).

(Boris Gutkin)

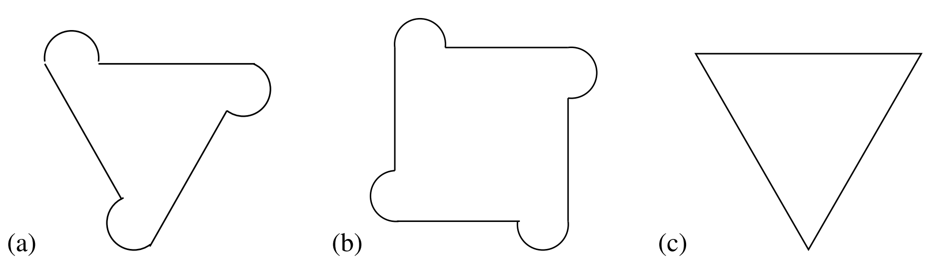


Figure 2.6: Hard wall potential with (a) symmetry C$_3$, (b) symmetry C$_4$, and (c) symmetry D$_3$.

2.6. **Permutation of three objects.** Consider S_3, the group of permutations of 3 objects.

(a) Show that S_3 is a group.

(b) List the conjugacy classes of S_3.

(c) Give an interpretation of these classes if the group elements are substitution operations on a set of three objects.

(c) Give a geometrical interpretation in case of group elements being symmetry operations on equilateral triangle.

2.7. **Three masses on a loop.** Three identical masses, connected by three identical springs, are constrained to move on a circle hoop as shown in figure 2.7. Find the normal modes. Hint: write down coupled harmonic oscillator equations, guess the form of oscillatory solutions. Then use basic matrix methods, i.e., find zeros of a characteristic determinant, find the eigenvectors, etc.. (K. Y. Short)

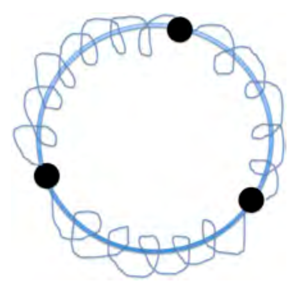


Figure 2.7: Three identical masses are constrained to move on a hoop, connected by three identical springs such that the system wraps completely around the hoop. Find the normal modes.

group theory - week 3

Group representations

Homework HW3

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 3.1 *1-dimensional representation of anything*	1 point
Exercise 3.2 *2-dimensional representation of* S_3	4 points
Exercise 3.3 *3-dimensional representations of* D_3	5 points

Bonus points

Exercise 3.4 *Abelian groups*	1 point
Exercise 3.5 *Representations of* C_N	1 point

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

3.1 Week's videos, reading

3.1.1 Matrix representations, Schur's Lemma

2021-05-25 Predrag Lecture 3

Irreps, unitary reps and Schur's Lemma.

- Dresselhaus *et al.* [1] Sect. 2.4 *The Unitarity of Representations*.
- Dresselhaus *et al.* [1] Sects. 2.5 and 2.6 *Schur's Lemma*.
- *Lecture 3* (Unedited) Unitarity of reps; Schur's Lemma (2:04:56 h)

3.1.2 Wonderful Orthogonality Theorem

2021-05-27 Predrag Lecture 4

- *Section playlist*
- (extra) *Recap of lect. 3* (5:36 min)
- ○ Sect. 3.2 It's all about class
- Dresselhaus *et al.* [1] Sect. 2.7 *'Wonderful' Orthogonality Theorem*, sect. 2.8 *Representations and vector spaces*.
- *Whence "orthogonality"?* The ideas: observables are Hermitian; matrix reps are unitary; average over the group to extract invariants. A matrix rep forms a complex unit vector, hence "orthogonality". (9:47 min)
- *Character orthogonality theorem* (X:XX min)
- Dresselhaus *et al.* [1] Sects. 3.1 *Characters and Class* to 3.5 *The number of irreducible representations*.
- *Characters. Character orthogonality.* (X:XX min)
- *Class* (17:56 min)
- *Number of classes equals the number of irreps* (6:24 min)
- (extra) *Discussion:* Irrep dimension; Are classes subgroups, cosets? Week's homework. Classes and irreps of D_3. N-gon intuition. A LaTeX template. (26:39 min)

Tinkham [3] covers the same material in Chapter 3 *Theory of Group Representations*, in a more compact way.

3.2 It's all about class

The essential group theory notions introduced here are the notion of irreducible representations (irreps) and their orthogonality

The Group Orthogonality Theorem: Let D_μ, $D_{\mu'}$ be two irreducible matrix representations of a compact group G of dimensions d_μ, $d_{\mu'}$, where the sum is over all elements of the group, $G = \{g\}$, and $|G|$ is their number, or the order of the group:

$$\frac{1}{|G|}\sum_g^G D^{(\mu)}(g)_a{}^b D^{(\mu')}(g^{-1})^{a'}{}_{b'} = \frac{1}{d_\mu}\delta_{\mu\mu'}\delta_a^{a'}\delta_{b'}^b\,.$$

This is a remarkable formula, one relation for each of the $d_\mu^2 + d_{\mu'}^2$ matrix entries. Still, the explicit matrix entries reflect largely arbitrary coordinate choices - there should a more compact statement of irreducibility, and there is: the "character orthogonality theorem" (3.1).

Consider a reducible representation $D(g)$, i.e., a representation of group element g that after a suitable similarity transformation takes form

$$D(g) = \begin{pmatrix} D^{(a)}(g) & 0 & 0 & 0 \\ 0 & D^{(b)}(g) & 0 & 0 \\ 0 & 0 & D^{(c)}(g) & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix},$$

with character for class $\mathcal{C}$ given by

$$\chi(\mathcal{C}) = c_a\,\chi^{(a)}(\mathcal{C}) + c_b\,\chi^{(b)}(\mathcal{C}) + c_c\,\chi^{(c)}(\mathcal{C}) + \cdots\,,$$

where c_a, the multiplicity of the ath irreducible representation (colloquially called "irrep"), is determined by the character orthonormality relations,

$$c_a = \overline{\chi^{(a)*}\,\chi} = \frac{1}{h}\sum_k^{class} N_k\chi^{(a)}(\mathcal{C}_k^{-1})\,\chi(\mathcal{C}_k)\,. \tag{3.1}$$

Knowing characters is all that is needed to figure out what any reducible representation decomposes into! Work out exercise 4.2 as an example.

3.3 Other sources (optional)

3.3.1 Hard work builds character

Irreps, unitary reps, Schur's Lemma.

Section playlist

Chapter 2 *Representation Theory and Basic Theorems*
Dresselhaus *et al.* [1], up to and including
Sect. 2.4 *The Unitarity of Representations* (click here)

- *This requires character* (1:23 min)
- *Hard work builds character* (15:05 min)
- *The symmetry group of a propeller* (6:13 min)
- *Irreps of* C_3 (14:52 min)
- *Rotation and reflections in a plane: irreps of* D_3 (13:38 min)
  - (extra) *Discussion: more symmetries, fewer invariant subspaces* (2:10 min)
  - (extra) *Discussion: abelian vs. nonabelian* (2:11 min)

This week's Dresselhous exposition (or the corresponding chapter in Tinkham [3]) comes from Wigner's classic [4] *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra*, which is a harder going, but the more group theory you learn the more you'll appreciate it. Eugene Wigner got the 1963 Nobel Prize in Physics, so by mid 60's gruppenpest was accepted in finer social circles.

In this course, we learn about full reducibility of finite and compact continuous groups in two parallel ways. On one hand, I personally find the multiplicative *projection operators* (1.25), coupled with the notion of class algebras (Harter [2] (click here) appendix C) most intuitive - a block-diagonalized subspace for each distinct eigenvalue of a given all-commuting matrix.

On the other hand, the character weighted sums (here related to the multiplicative projection operators as in ChaosBook Example A24.2 *Projection operators for discrete Fourier transform*) offer a deceptively 'simple' and elegant formulation of full-reducibility theorems, preferred by all standard textbook expositions.

3.3.2 History (optional)

The structure of finite groups was understood by late 19th century. A full list of finite groups was another matter. The complete proof of the classification of all finite groups takes about 3 000 pages, a collective 40-years undertaking by over 100 mathematicians, read the wiki.

Alex Kontorovich, Rutgers MAT 640:503 Complex Analysis. A wonderful lecturer, here he diverges into the story of Cardano and cubics. They are *cube*-ic for a reason. Did you know people learned to use $\sqrt{-1}$ before they understood that a number can be negative, like -1? Listen to his first lecture. Oh no! He just made me solve the cubic, something I had avoided my entire life. So far. You'll love it.

According to Kevin Hartnett, *The 'Useless' Perspective That Transformed Mathematics*: Representation theory was initially dismissed. Today, it's central to much of mathematics.

Groups are complicated collections of mathematical objects – like numbers or symmetries – that stand in a particular structured relationship with each other. Representation theory is a way of taking such complicated objects and "representing" them with simpler objects. It converts the sometimes mysterious world of groups into the well-trammeled territory of linear algebra, the study of simple transformations performed on objects called vectors, which are effectively directed line segments. These objects are defined by coordinates, which can be displayed in the form of a matrix, the core element of linear algebra, an array of numbers. While groups are abstract and often difficult to get a handle on, matrices and linear algebra are elementary.

Geordie Williamson, *Mathematics in light of representation theory*. October 16, 2015 at Urania: Symmetry is all around us. The mathematical study of symmetry becomes simpler when we linearize, and in doing so we enter the realm of representation theory. Representation theory has applications throughout mathematics (the Fourier transform, monstrous moonshine, the Langlands program, the proof of Fermat's last theorem, ...) and science (crystallography and spectroscopy in chemistry, signal processing in engineering, the standard model in physics, ...). The lecture is an introduction to the representation theory of finite groups, both over the complex numbers and over fields of positive characteristic (so-called modular representation theory). Williamson discusses Frobenius' discovery of the character table in Berlin in 1896, Brauer's first steps in modular representation theory in the 1930s, and the role of the character table in the discovery of the monster simple group in the 1980s. Williamson finishes with a discussion of recent developments in the modular representations of symmetric and finite general linear groups.

From Emory Math Department: A pariah is real! The simple finite groups fit into 18 families, except for the 26 sporadic groups. 20 sporadic groups AKA the Happy Family are parts of the Monster group. The remaining six loners are known as the pariahs. (Check the notes sect. 5.2 *Literature* for links to the Ree group and the whole classification.)

References

[1] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

[2] W. G. Harter and N. dos Santos, "Double-group theory on the half-shell and the two-level system. I. Rotation and half-integral spin states", Amer. J. Phys. **46**, 251–263 (1978).

[3] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

[4] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1931).

Exercises

3.1. **1-dimensional representation of anything.** Let $D(g)$ be a representation of a group G. Show that $d(g) = \det D(g)$ is one-dimensional representation of G as well.

(B. Gutkin)

3.2. **2-dimensional representation of S_3.**

(i) Show that the group S_3 can be generated by two permutations:

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

(ii) Show that matrices:

$$D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ D(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ D(d) = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix},$$

with $z = e^{i2\pi/3}$, provide proper (faithful) representation for these elements and find representation for the remaining elements of the group.

(iii) Is this representation irreducible?

(B. Gutkin)

3.3. **3-dimensional representations of D_3.** The dihedral group D_3 is the symmetry group of the equilateral triangle. It has 6 elements

$$\mathrm{D}_3 = \{E, C, C^2, \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}\},$$

where C is rotation by $2\pi/3$ and $\sigma^{(i)}$ is reflection along one of the 3 symmetry axes.

(i) Prove that this group is isomorphic to S_3

(ii) Show that matrices

$$D(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, D(C) = \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^2 \end{pmatrix}, D(\sigma^{(1)}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.2}$$

generate a 3-dimensional representation $D(g)$ of D_3. Hint: Calculate products for representations of group elements and compare with the group table (see lecture).

(iii) Show that this is a reducible representation which can be split into one dimensional A and two-dimensional representation Γ. In other words find a matrix R such that

$$RD(g)R^{-1} = \begin{pmatrix} A(g) & 0 \\ 0 & \Gamma(g) \end{pmatrix}$$

for all elements g of D_3. (Might help: D_3 has only one (non-equivalent) 2-dim irreducible representation).

(B. Gutkin)

3.4. **Abelian groups.** Let G be a group with only one-dimensional irreducible representations. Show that G is Abelian.

(B. Gutkin)

3.5. **Representations of C_N.** Find all irreducible representations of C_N.

(B. Gutkin)

group theory - week 4

Hard work builds character

Homework HW4

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 4.3 *All irreducible representations of* D_4	10 points

Bonus points

Exercise 4.4 *Irreducible representations of dihedral group* D_n	2 points
Exercise 4.5 *Perturbation of* T_d *symmetry*	6 points
Exercise 4.7 *Two particles in a potential*	4 points

Total of 10 points = 100 % score. Bonus points accumulate, can help you later if you miss a few problems.

Table 4.1: The D_3 group multiplication table. The same as table 2.1, but written as a class operator multiplication table.

D_3	1	r	r^2	σ_1	σ_2	σ_3
1	1	r	r^2	σ_1	σ_2	σ_3
r	r	r^2	1	σ_3	σ_1	σ_2
r^2	r^2	1	r	σ_2	σ_3	σ_1
σ_1	σ_1	σ_2	σ_3	1	r	r^2
σ_2	σ_2	σ_3	σ_1	r^2	1	r
σ_3	σ_3	σ_1	σ_2	r	r^2	1

D_3	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$\mathcal{C}_1$	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$
$\mathcal{C}_2$	$\mathcal{C}_2$	$2\mathcal{C}_1+\mathcal{C}_2$	$2\mathcal{C}_3$
$\mathcal{C}_3$	$\mathcal{C}_3$	$2\mathcal{C}_3$	$3\mathcal{C}_1+3\mathcal{C}_2$

2021-06-01 Lecture 5

Character orthogonality theorem

- *Character orthogonality relations.* (10:53 min)
  Character defined. Character of identity = dimension of the representation. Character orthogonality stated as an average of the group over irrep characters (but not derived). Special cases checked. Completeness verified. Example: Reflection group in 1 dimension. Characters and their orthogonality checked.

- *A summary: it is all about class and character* (18:50 min)
  Presumes knowledge of C_N irreps, argues that a reflection (D_N) mixes them up, thus reducing the number of irreps. 3-disk classes. Character is labelled by the class and the irrep label. Example: discrete Fourier transform is an $[N \times N]$ unitary matrix. D_4 character table.

  - (extra) *Discussion: class and character* (7:01 min)

2021-06-01 Predrag Lecture 6

Hard work builds character

Complete Dresselhaus *et al.* [1] sects. 3.3 “Wonderful Orthogonality Theorem for Characters” to 3.8 “Setting up Character Tables” (click here). This material is also covered in Tinkham [7] Chapter 3 *Theory of Group Representations*.

1. theory of finite groups are a natural generalization of discrete Fourier representations

2. it is all about class and character. “Character", in particular, I find very surprising - one complex number suffices to characterize a matrix!

4.1 Other sources (optional)

Group theory? It is all about class & character.
— Predrag Cvitanović, *One minute elevator pitch*

For a continuous group version of the character orthogonality theorem, see sect. 9.4. In particular, the replacement of an irrep matrix representation $D^{(\mu)}(g)_a{}^b$ by its character $\chi^{(\mu)}(g)$ (a single scalar quantity) leads to no loss of any of the matrix indices structure.

I enjoyed reading Mathews and Walker [6] Chap. 16 *Introduction to groups*. You can download it from here. Goldbart writes that the book is "based on lectures by Richard Feynman at Cornell University." Very clever. Try working through the example of fig. 16.2: deadly cute, you get explicit eigenmodes from group theory alone. The main message is that if you think things through first, you never have to construct the representation matrices in explicit form - recasting the calculation in terms of invariants, such characters, will get you there much faster.

You might find Gutkin notes useful:

Lect. 4 *Representation Theory II*, up to Sect. 4.5 *Three types of representations*: Character tables. Dual character orthogonality. Regular representation. Indicators for real, pseudo-real and complex representations. See example 4.3 "Irreps for quaternion multiplication table."

- Oliver Pierson *ChaosBook.org chapter Discrete factorization - Character tables* (10:05 min)

- Oliver Pierson *ChaosBook.org chapter Discrete factorization - Projection into invariant subspaces* (5:31 min)

Lect. 5 *Applications I. Vibration modes* go through Wigner's theorem, C_n symmetry and D_3 symmetry. Study Example 5.1. C_n symmetry. More quantum mechanics applications follow in

sect. 6.2 *Applications II. Quantum Mechanics*, Sect. 2. *Perturbation theory*.

Does the proof in the Lect. 4 *Representation Theory II Appendix* that the number of irreps equals the number of classes make sense to you? For an easy argument, see Vedensky Theorem 5.2 *The number of irreducible representations of a group is equal to the number of conjugacy classes of that group.* For a proof, work though Murnaghan Theorem 7. If you prefer a proof that your professor cannot understand, click here.

For the record (I retract the heady claim I made in class):
Mathworld.Wolfram.com: "A character table often contains enough information to identify a given abstract group and distinguish it from others. However, there exist nonisomorphic groups which nevertheless have the same character table, for example D_4 (the symmetry group of the square) and Q_8 (the quaternion group)."

exercise 4.3

Fun read along these lines: Hart and Segerman [2] discuss the distinction between abstract groups and symmetry groups of objects. They exhibit two very different objects with

$$D_4 = \langle r, \sigma \,|\, \sigma r \sigma = r^{-1}, r^4 = \sigma^2 = e \rangle \tag{4.1}$$

symmetry (describing the group this way is called a *presentation* of D_4), and explain the Cayley graph for D_4 (its edges with arrows correspond to rotations, the other edges correspond to reflections). For quaternions they discuss a 1-dimensional space group built of "monkey blocks" (but do not identify its crystallographic name). Q_8 is a subgroup of the symmetries of the 3-dimensional sphere S^3 , the unit sphere in $\mathbb{R}^4$. They offer a visualisation of the action of Q_8 on a hypercube and construct a sculpture whose symmetry group is Q_8, using stereographic projection from the unit sphere in 4-dimensional space. Q_8 is discussed here in example 4.3.

Simon Berman You would think that the analysis of three masses connected by harmonic strings, see figure 4.1, is a simple exercise finding irreps of D_3 symmetry, but no, it merits a 2019 Phys. Rev. Lett., see Katz and Efrati [3] *Self-driven fractional rotational diffusion of the harmonic three-mass system*. The article even starts with our figure 4.1. We continue the discussion in sect. 6.4.

Example 4.1. ***D_3* symmetry:** *Reflections and rotations of a triangle, figure 2.5 (c)*

$$D(T) = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}, \qquad D(\sigma_1) = \begin{pmatrix} -1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&1\\ 0&0&-1&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix} \tag{4.2}$$

$$D(\sigma_2) = \begin{pmatrix} 0&0&0&0&-1&0\\ 0&0&0&0&0&1\\ 0&0&-1&0&0&0\\ 0&0&0&1&0&0\\ -1&0&0&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix}, \qquad D(\sigma_3) = \begin{pmatrix} 0&0&-1&0&0&0\\ 0&0&0&1&0&0\\ -1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&1 \end{pmatrix} \tag{4.3}$$

$$G = \{[e];[g,g^2];[\sigma_1,\sigma_2,\sigma_3]\}, \qquad \chi^{(1)} = \{1,1,1\}, \chi^{(2)} = \{1,1,-1\}, \chi^{(3)} = \{2,-1,0\}$$

$$r_i = \chi(e)\chi^{(i)}(e)/6; \qquad r_i = \{1,1,2\} \quad \Longrightarrow \quad D = 2E \oplus A_1 \oplus A_2 .$$

$$P_i = \frac{1}{3}\sum_{g\in G} \chi^{(i)}(g) D(g)$$

$$P_1 = \frac{1}{3}\begin{pmatrix} 0&0&0&0&0&0\\ 0&1&0&1&0&1\\ 0&0&0&0&0&0\\ 0&1&0&1&0&1\\ 0&0&0&0&0&0\\ 0&1&0&1&0&1 \end{pmatrix}, \qquad P_2 = \frac{1}{3}\begin{pmatrix} 1&0&1&0&1&0\\ 0&0&0&0&0&0\\ 1&0&1&0&1&0\\ 0&0&0&0&0&0\\ 1&0&1&0&1&0\\ 0&0&0&0&0&0 \end{pmatrix} \tag{4.4}$$

The 3 equal masses connected by harmonic springs system of figure 4.1 is a textbook example of such system, see for example problems 6.37 and 9.16 in Kotkin and Serbo [4] Collection of Problems in Classical Mechanics.

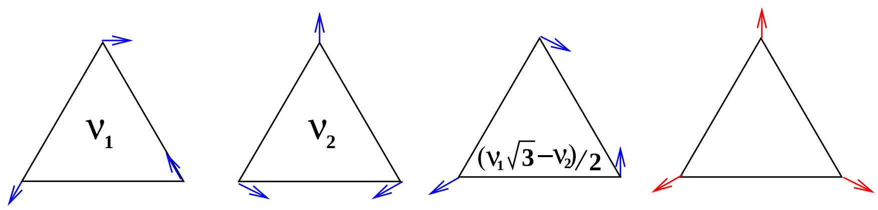


Figure 4.1: Modes of a molecule with D_3 symmetry. (B. Gutkin)

The vibrational modes associated with the two 1-dimensional representations are given by

$$P_{1V} = \alpha \begin{pmatrix} 0\\1\\0\\1\\0\\1 \end{pmatrix} \qquad \textit{and} \qquad P_{2V} = \beta \begin{pmatrix} 1\\0\\1\\0\\1\\0 \end{pmatrix},$$

respectively. Here P_{1V} represents symmetric mode shown in figure 4.1 (red). The second mode P_{2V} corresponds to the rotations of the whole system. The projection operator for the two-dimensional representation is

$$P_3 = \frac{2}{6}(2D(I) - D(T) - D(T^2)) = \frac{1}{3} \begin{pmatrix} 2 & 0 & -1 & 0 & -1 & 0\\ 0 & 2 & 0 & -1 & 0 & -1\\ -1 & 0 & 2 & 0 & -1 & 0\\ 0 & -1 & 0 & 2 & 0 & -1\\ -1 & 0 & -1 & 0 & 2 & 0\\ 0 & -1 & 0 & -1 & 0 & 2 \end{pmatrix} \tag{4.5}$$

From this we have to separate two vectors corresponding to shift in x and y directions.

$$\eta_x = \begin{pmatrix} 1\\0\\-1/2\\ \sqrt{3}/2\\ -1/2\\ -\sqrt{3}/2 \end{pmatrix}, \qquad \eta_y = \begin{pmatrix} 0\\1\\-\sqrt{3}/2\\ -1/2\\ \sqrt{3}/2\\ -1/2 \end{pmatrix}$$

$$P_{3V} = \left\{ \underbrace{\alpha \frac{1}{\sqrt{6}} \begin{pmatrix} 2\\0\\-1\\0\\-1\\0 \end{pmatrix}}_{\xi_1} \underbrace{+\beta \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\0\\1\\0\\-1\\0 \end{pmatrix}}_{\xi_2} \underbrace{+\gamma \frac{1}{\sqrt{6}} \begin{pmatrix} 0\\2\\0\\-1\\0\\-1 \end{pmatrix}}_{\xi_3} \underbrace{+\delta \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\0\\0\\1\\0\\-1 \end{pmatrix}}_{\xi_4} \right\},$$

where $\eta_x = \sqrt{3/2}(\xi_4 + \xi_1)$, $\eta_y = \sqrt{3/2}(\xi_3 - \xi_2)$. Vectors ξ_i are columns of P_3 and their *linear combinations. The orthogonal vectors are given by*

$$\nu_1 = \sqrt{3/2}(\xi_1 - \xi_4) = \begin{pmatrix} 1 \\ 0 \\ -1/2 \\ -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \qquad \nu_2 = \sqrt{3/2}(\xi_2 + \xi_3) = \begin{pmatrix} 0 \\ 1 \\ \sqrt{3}/2 \\ -1/2 \\ -\sqrt{3}/2 \\ -1/2 \end{pmatrix}.$$

(B. Gutkin)

Example 4.2. ***(Pseudo)real and complex representations.*** *There are three types of representation: real, pseudo-real and complex (see Montaldi for details). For real representations matrices $D(g)$ can be brought into real form such that $D_{ij}(g) = \bar{D}_{ij}(g)$. This implies in particular that all the characters are real. For pseudo-real representation the characters are also real but matrices $D(g)$ cannot be brought into real form. Finally, for complex representations the characters are complex. In the last case $D(g)$ and the conjugate $\bar{D}(g)$ constitute two different representation (since they characters are different), while in the real and pseudo-real case both representations are equivalent, i.e., $\bar{D}(g) = UD(g)U^\dagger$.*

Indicator. *To distinguish between three types of representations one looks at the indicator:*

$$\textit{Ind}(\alpha) = \frac{1}{|G|}\sum_{g\in G} \chi^{(\alpha)}(g^2) \in \{1, 0, -1\}\,, \tag{4.6}$$

where 1, -1, 0 are obtained for real, complex and pseudo-real representations, respectively.

Proof: For a general irreducible representation we have

$$D^{(\alpha)}(g) = U\bar{D}^{(\beta)}(g)U^\dagger, \tag{4.7}$$

where $\alpha \neq \beta$ for a complex representation (since $\chi^{(\alpha)}(g) \neq \bar{\chi}^{(\alpha)}(g)$) and $\alpha = \beta$ for real and pseudo-real representations. From $D^{(\alpha)}(g^2) = D^{(\alpha)}(g)D^{(\alpha)}(g)$ follows

$$\textit{Ind}(\alpha) = \sum_{i,j=1}^{m_\alpha}\sum_{k,n=1}^{m_\alpha}\sum_{g\in G}\frac{1}{|G|}\sum_{g\in G} U_{k,j}D^{(\alpha)}_{i,k}(g)\bar{D}^{(\beta)}_{j,n}(g)U^\dagger_{ni},$$

with m_α being dimension of α. By the orthogonality theorem this expression is zero for $\alpha \neq \beta$ which is the case of complex α. For real and pseudo-real representations we have

$$\textit{Ind}(\alpha) = \frac{1}{m_\alpha}\mathrm{tr}\,\left(U\bar{U}\right).$$

Now note, that for $\alpha = \beta$ eq. (4.7) yields

$$D^{(\alpha)}(g)U\bar{U} = U\bar{U}D^{(\alpha)}(g).$$

By the first Schur's lemma it follows then that $U\bar{U} = \gamma I$, or $U = \gamma U^\top$ which also implies $\gamma^2 = 1$. This leaves only two possibilities $\gamma = 1$ for real and $\gamma = -1$ for pseudo-real representations. In the first case we have $UU^\top = I$ and Ind$(\alpha) = 1$, while in the second one $UU^\top = -I$ and Ind$(\alpha) = -1$. Note finally, that $1 = \det\left(U\bar{U}\right) = \gamma^{m_\alpha}$. So $\gamma = -1$ might appear only if m_α is even. In other words, a pseudo-real irreducible representation must be of even dimension.

Example 4.3. ***Quaternions:*** *Quaternion multiplication table is*

$$\{\pm 1, \pm i, \pm j, \pm k\} \qquad i^2 = j^2 = k^2; \quad ij = k.$$

This group has five conjugate classes:

$$\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}.$$

The only possible solution for the equation $\sum_{i=1}^{5} m_i^2 = 8$ *is* $m_i = 1, i = 1, \dots 4, m_5 = 2$. *In addition to fully symmetric representation, the other three one-dimensional representations are easy to find:* $\chi(1) = 1, \chi(-1) = 1$*, while* $\chi(i) = -1, \chi(j) = -1, \chi(k) = 1$*;* $\chi(i) = -1, \chi(k) = -1, \chi(j) = 1$ *or* $\chi(k) = -1, \chi(j) = -1, \chi(i) = 1$. *The two-dimensional representation can be find by the orthogonality relation:*

$$2 + \chi(-1) \pm \chi(k) \pm \chi(i) \pm \chi(j) = 0, \Longrightarrow \chi(-1) = -2, \chi(k) = \chi(i) = \chi(j) = 0\,.$$

Since the indicator equals

$$Ind = (2\chi(1) + 6\chi(-1))/8 = -1,$$

the last representation is pseudo-real. Note that this representation can be realized using Pauli matrices:

$$\{\pm I, \pm\sigma_x, \pm\sigma_y, \pm\sigma_z\}.$$

(B. Gutkin)

References

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[3] O. Katz-Saporta and E. Efrati, "Self-driven fractional rotational diffusion of the harmonic three-mass system", Phys. Rev. Lett. **122**, 024102 (2019).

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Exercises

4.1. **Characters of D$_3$.** (continued from exercise 2.4) D$_3$, the group of symmetries of an equilateral triangle: has three irreducible representations, two one-dimensional and the other one of multiplicity 2.

(a) All finite discrete groups are isomorphic to a permutation group or one of its subgroups, and elements of the permutation group can be expressed as cycles. Express the elements of the group D$_3$ as cycles. For example, one of the rotations is (123), meaning that vertex 1 maps to 2, $2 \to 3$, and $3 \to 1$.

(b) Use your representation from exercise 2.4 to compute the D$_3$ character table.

(c) Use a more elegant method from the group-theory literature to verify your D$_3$ character table.

(d) Two D$_3$ irreducible representations are one dimensional and the third one of multiplicity 2 is formed by $[2\times 2]$ matrices. Find the matrices for all six group elements in this representation.

4.2. **Decompose a representation of S_3.** As an illustration of the utility of the character orthonormality relations (3.1), let's work out the reduction of the matrix representation of S_3 permutations. The identity element acting on three objects $[a\ b\ c]$ is a 3×3 identity matrix,

$$D(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transposing the first and second object yields $[b\ a\ c]$, represented by the matrix

$$D(A) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

since

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a \\ c \end{pmatrix}$$

1. Find all six matrices for this representation.
2. Split this representation into its conjugacy classes.
3. Evaluate the characters $\chi(\mathcal{C}_j)$ for this representation.
4. Determine multiplicities c_a of irreps contained in this representation.
5. (bonus) Construct explicitly all irreps.
6. (bonus) Explain whether any irreps are missing in this decomposition, and why.

4.3. **All irreducible representations of D$_4$.** Dihedral group D$_4$, the symmetry group of a square, consists of 8 elements: identity, rotations by $\pi/2$, π, $3\pi/2$, and 4 reflections across symmetry axes: D$_4 = \langle g, \sigma | g^4 = \sigma^2 = e, g\sigma = \sigma g^3 \rangle$

(a) Find all conjugacy classes.

(b) Determine the dimensions of irreducible representations using the relationship

$$\sum_i d_i^2 = |G|, \tag{4.8}$$

where d_i is the dimension of ith irreducible representation.

(c) Determine the remaining items of the character table.

(d) Compare with the character table of quaternions, example 4.3. Are they the same or different?

(e) Determine the indicators for all irreps of D_4. Are they the same as for the irreps of the quaternion group?

If you are at loss how to proceed, take a look at Landau and Lifschitz [5] *Vol.3: Quantum Mechanics*

(Boris Gutkin)

4.4. **Irreducible representations of dihedral group D_n.**

(a) Determine the dimensions of all irreps of dihedral group D_n, n odd.

(b) Determine the dimensions of all irreps of dihedral group D_n, n even.

This exercise is meant to be easy - guess the answer from the irreps dimension sum rule (4.8), and what you already know about D_1, D_3 and D_4. Working out also D_2 case (cut a disk into two equal halves) might be helpful. A more serious attempt would require counting conjugacy classes first. This exercise might help you later, when you are looking at irreps of the orthogonal groups $O(n)$; turns out they are different for n odd or even n, and that has physical consequences: what you learn by working out a problem in 2 dimensions might be misleading for working it out in 3 dimensions.

4.5. **Perturbation of T_d symmetry.**

A non-relativistic charged particle moves in an infinite bound potential $V(x)$ with T_d symmetry. Consult exercise 5.1 *Vibration Modes of CH_4* for the character table and other T_d details.

(a) What are the degeneracies of the quantum energy levels? How often do they appear relative to each other (i.e., what is the level density)?

A weak constant electric field is now added now along one of the $2\pi/3$ rotation axes, splitting energy levels into multiplets.

(b) What is the symmetry group of the system now?

(c) How are the levels of the original system split? What are the new degeneracies?

(Boris Gutkin)

4.6. **Selection rules for T_d symmetry.**

The setup is the same as in exercise 4.5, but now assume that instead of a constant field, a time dependent electric field $\boldsymbol{E}_0 \cos(\omega t)$ is added to the system, with $\boldsymbol{E}_0$ not necessarily directed along any of the symmetry axes. In general, when $|E_n - E_m| = \hbar\omega$, such time-dependent perturbation induces transitions between energy levels E_n and E_m.

(a) What are the selection rules? Between which energy levels of the system are transitions possible?

(b) Would the answer be different if a magnetic field $\boldsymbol{B}_0 \cos(\omega t)$ is added instead? Explain how and why.

4.7. **Two particles in a potential.**
Two distinguishable particles of the same mass move in a 2-dimensional potential $V(r)$ having D_4 symmetry. In addition they interact with each other with the term $\lambda W(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$.

(a) What is the symmetry group of the Hamiltonian if $\lambda = 0$? If $\lambda \neq 0$?.

(b) What are the degeneracies of the energy levels if $\lambda = 0$?

(c) Assuming that $\lambda \ll 1$ (weak interaction), describe the energy level structure, i.e., degeneracies and quasi-degeneracies of the energy levels. What will be the answer if the interaction is strong?

Hint: when interaction is weak we can think about it as perturbation. (Boris Gutkin)

group theory - week 5

It takes class

Homework HW5

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

Exercise 5.1 *Vibration modes of CH_4, parts (a) (b) (c) i*	8 points
Exercise 5.2 *Keep it classy (a)*	2 points

Bonus points

Exercise 5.1 *Vibration modes of CH_4, part (c) ii*	2 points
Exercise 5.2 *Keep it classy (b)*	2 points
Exercise 5.2 *Keep it classy (c)*	4 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

Show class, have pride, and display character. If you do, winning takes care of itself.
— Paul Bryant

2021-06-03 Predrag Lecture 9 It takes class

Section playlist

- Sect. 5.1 It's all about class

Dresselhaus *et al.* [6] Sect. 3.6 *Second Orthogonality Relation for Characters*.

6.1 A character table is a unitary matrix from classes to irreps (9:12 min)

6.2 Projection operator perspective (11:35 min)

Harter's Sect. *3.2 First stage of non-Abelian symmetry analysis*
group multiplication table (3.1.1); class operators; class multiplication table (3.2.1b); all-commuting or central operators;

6.3 Example: the projection operator reduction of D_3 (23:00 min)

Harter's Sect. *3.3 Second stage of non-Abelian symmetry analysis*
projection operators (3.2.15); 1-dimensional irreps (3.3.6); 2-dimensional irrep (3.3.7); Lagrange irreps dimensionality relation (3.3.17)

6.4 Example: A reduction by two commuting operators (Harter problem 1.2.6). What comes next: a nonlinear symmetry reduction; translations; Fourier series. (35:14 min)

2021-06-08 Predrag Lecture 10 It takes grit

Gutkin notes, Lect. 5 *Applications I. Vibration modes*: Example 5.1. C_n symmetry completed.

5.1 It's all about class

In week 1 we introduced projection operators (1.27). How are they related to the character projection operators constructed in the previous lecture? While the character orthogonality might be wonderful, it is not very intuitive - it's a set of solutions to a set of symmetry-consistent orthogonality relations. You can learn a set of rules that enables you to construct a character table, but it does not tell you what it means. Similar thing will happen again when we turn to the study of continuous groups: all semisimple Lie groups will be classified by Killing and Cartan by a more complex set of orthogonality and integer-dimensionality (Diophantine) constraints. You obtain all possible Lie algebras, but have no idea what their geometrical significance is.

In my own Group Theory book [4] I (almost) get all simple Lie algebras using projection operators constructed from invariant tensors. What that means is easier to understand for finite groups, and here I like the Harter's exposition [8] best. Harter constructs 'class operators', shows that they form a basis for the algebra of 'central' or 'all-commuting' operators, and uses their characteristic equations to construct the projection operators (1.27) from the 'structure constants' of the finite group, i.e., its class multiplication tables. Expanded, these projection operators are indeed the same as the ones obtained from character orthogonality.

I find Harter's Sect. *3.3 Second stage of non-Abelian symmetry analysis* particularly illuminating. It shows how physically different (but mathematically isomorphic) higher-dimensional irreps are constructed corresponding to different subgroup embeddings. One chooses the irrep that corresponds to a particular sequence of physical symmetry breakings.

You might want to have a look at Harter [9] *Double group theory on the half-shell* (click here). Read appendices B and C on spectral decomposition and class algebras. Article works out some interesting examples.

See also remark 1.1 *Projection operators* and perhaps watch Harter's online lecture from Harter's online course.

There is more detail than what we have time to cover here, but I find Harter's Sect. *3.3 Second stage of non-Abelian symmetry analysis* particularly illuminating. It shows how physically different (but mathematically isomorphic) higher-dimensional irreps are constructed corresponding to different subgroup embeddings. One chooses the irrep that corresponds to a particular sequence of physical symmetry breakings.

5.1.1 Dirac characters, Burnside's method (optional)

I told you that everybody who understands anything about group theory, writes a book. This weeks winner is Daniel Arovas, who is writing up his *Lecture Notes on Group Theory in Physics*. Check them out - they are cute, and even contain !jokes!.

For example, here I learn for the first time that Harter's central operators (Harter's Sect. *3.2 First stage of non-Abelian symmetry analysis*) are in condensed matter physics known as 'Dirac characters'.

Dirac characters were introduced by Dirac [5] in *The Principles of Quantum Mechanics* (1930) (click here). He refers to them as "[...] what is called in group theory a character of the group of permutations." Corson [3] *Note on the Dirac character operators* (1948) writes:

> [...] the evaluation of Dirac and similar character operators is all that is required for the solution of the standard molecular problems in the spirit of Dirac's original program which avoids appeal to formal group theory.

Dirac characters use not only the abstract group information, but also account for the symmetry information contained in the basis set used. The diagonalization of Dirac characters has three main advantages:

1. It can be realized by means of a quite simple and general algorithm.
2. The projective irreps obtained are just the ones that are needed to reduce the starting basis set into irreducible sets.
3. No tabulated quantities are required to construct the projective irreps.

The scheme is completely general, in the sense that it applies to all space groups.

Ananda Dasgupta had 1.68K followers on YouTube, now he has one more:

- playlist for his *Symmetries in Physics* course:
- *Lecture 15* (start at about 35 min into the lecture) has a nice discussion of Dirac characters, their relation to characters, and motivates the algorithmic Burnside's method for computing characters via class multiplication tables $(H_i)_{jk}$.
- *PH4213 Discussion Class 8* applies Burnside's method to D_4.

More generally, the whole course is of interest, if covers most topics of our course in greater depth:

- *PH4213 Discussion Class 9* gets projection operators out of characters.
- *Lecture 16 Projection operators* attempts to give you an appreciation of the power of the Wigner Eckart theorem (what in my book is described as all calculations being 'vacuum bubbles', maybe not precisely in these words).

A few textbooks that use Dirac characters:

- Cini [2] *Topics and Methods in Condensed Matter Theory* (2007) (click here)
- Jacobs [10] *Group Theory with Applications in Chemical Physics*, (click here) (2005)
- El-Batanouny and Wooten [1] *Symmetry and Condensed Matter Physics: A Computational Approach* (2008) (click here). In sect. 4.3 they describe the Burnside's method. They give an example of Mathematica code that constructs the character table. If needed, on might use Dixon's method, which is more clever for numerical computations.
- Big Chemical Encyclopedia: Dirac character.
- The *CRYSTAL* package performs ab initio calculations of the ground state energy, energy gradient, electronic wave function and properties of periodic systems. Uses Dirac characters.
- For a bit of history, see J. E. Humphreys review of Pioneers of representation theory.

5.1.2 William G. Harter (optional)

Who is Bill Harter? He is a prodigy who at age 16 taught himself group theory by reading Hamermesh [7]. He was a graduate student at Caltech (1964-65), together with Ron Fox. They hated the atmosphere there and the teaching was terrible (Feynman did not teach that year but Harter and Feynman were good friends). Harter and Fox shared an interest in group theory and discovered that most of the group theory books in the physics library had been checked out in 1960-62 by Gell-Mann, Zweig and Glashow. That only half of the entering students were meant to complete their PhD's there led to lots of ugly competition. Harter transferred to UC Irvine, and, upon graduation, got a job at USC in LA. After a few years he suggested in a faculty meeting that the way they could improve their quality as a department was "to get rid of all the old farts." These same "old farts" soon voted to deny him tenure. He ended up in Campinas, Brazil. Fox rescued him from there by bringing him for an interview at Georgia Tech, where he was hired in late 1970's. He was brilliant, an asset for teaching, making all sorts of demonstration devices. He built a giant rotating table upon which he placed billiard balls, a wonderful demonstration of mechanical analogues for charged particle motion in crossed E and B fields. Everyone (except for one nefarious character) liked him, his work, and especially his devices. The faculty unanimously supported his promotion to tenure. He did not, however, think much of the Director of School of Physics, and made that clear. After an argument with the Director, he stormed out, offended. So, he was denied tenure and moved in 1985 to University of Arkansas where he is a professor today.

In 1987 Harter and Weeks used Harter's theory of the rotational dynamics of molecules to calculate the rotational-vibrational spectra of the soccer ball-shaped molecule Buckminsterfullerene, C60, or "buckyball." C60 had been proposed in 1985 by chemists, who had seen a mass-spectra peak of atomic mass 720. By 1989 the Harter theory calculations led to a realization that chemists had been making C60 since the early 1970s. In 1992 Science named C60 "Molecule of the Year," and in 1996 Curl, Kroto and Smalley were awarded the Nobel Prize in Chemistry for their discovery of fullerenes.

You can find here many Soft Elegant Educational tools developed by Harter, and follow his lectures on line. He is a great teacher. Georgia Tech's loss.

5.2 Other sources (optional)

Continuing reading Mathews and Walker [11], now Chap. 14. Porter works out nicely the normal modes of the D_3 springs and masses (again!).

Not all finite groups are as simple or easy to figure out as D_3. For example, the order of the Ree group ${}^2F_4(2)'$ is $212(26 + 1)(24 - 1)(23 + 1)(2 - 1)/2 = 17\,971\,200$.

5.3 Discussion

Henriette Roux I have a few questions about the exercise 5.1 part (d) *Vibration modes of CH_4: Find all modes of the methane molecule.*

1. When we use the angle of improper rotation, is it true that reflection equals to the π improper rotation?
2. I assume it is π and it gives me other characters are zero. In the case of all symmetry, this will give the , which we usually get non-negative integer. As a result, I'm not perfectly sure that the character formulas you give are correct.

3. Moreover seems it's in the representation of $[12 \times 12]$ matrices instead of $[24 \times 24]$ matrices.

Predrag The solution set is very detailed, so how about waiting Tuesday afternoon, when it gets posted on T-square? Then –if it is still unclear– we continue the discussion.

1. If $g \in \mathrm{SO}(3)$ is a rotation, and $D(i)\boldsymbol{r} = -\boldsymbol{r}$ is the inversion transformation, then rotation combined with the inversion $g\,i$ is an improper rotation $g\,i \in \mathrm{O}(3)$. If $g \in T$ (a discrete tetrahedron rotation) then $g\,i$ is an improper element of T_d.
2. ? (check the solution set).
3. The proper rotations group T of order 12 is a normal subgroup. However, I do not think you can have an improper rotations subgroup of T_d, as $g_i\,i g_j\,i$ is a proper rotation.

References

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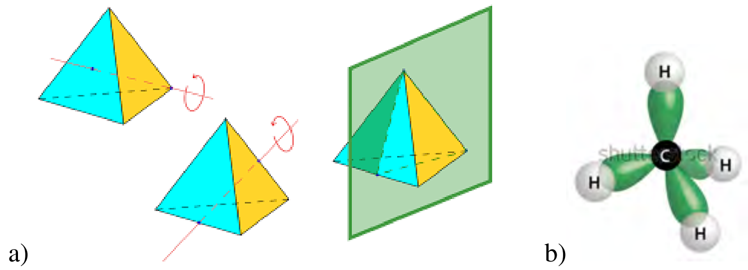

Figure 5.1: a) Two classes of rotational symmetries, and a class of reflection symmetries of a tetrahedron. (left) Hold the Tetra Pak by a tip, turn it by a third. (middle) Hold the Tetra Pak by the midpoints of a pair of opposing edges, make a half-turn. (right) Exchange the vertices outside the reflection plane. b) Methane molecule with the symmetry T_d.

Exercises

5.1. **Vibration modes of CH_4.**
Tetrahedral group T describes rotational symmetries of a tetrahedron. The order of the group is $|\mathrm{T}| = 12$, and its conjugacy classes are:

- The identity mapping.
- Four rotations by $\varphi = 2\pi/3$, with each of the four rotation axes going through a vertex, and piercing the midpoint of the triangle opposite.
- Four inverse rotations by $\varphi = -2\pi/3$.
- Three rotations by $\varphi = \pi$, one for each of the three rotation axes going through midpoints of opposing edges.

The full group of tetrahedron symmetries T_d includes also reflections. This is the symmetry group of molecules such as methane CH_4, see figure 5.1).

(a) What is the order of the group T_d? Show that the group is isomorphic to i) the group of permutations S_4; ii) to the group O of rotational symmetries of the cube. iii) Show that T is normal subgroup of T_d.

(b) Find all conjugacy classes of the group. Which of these classes correspond to proper (det $R(\varphi) = +1$), improper (det $R(\varphi) = -1$) rotations ?
Information on T might help. Note that φ might be also 0.

(c) i) Find all irreducible representations of the group & build the character table.
A shortcut: find all one-dimensional representations, assume that characters are integers, then use the orthogonality relationship between characters.
ii) Really compute the character table, without assuming that characters are integers (2 bonus points).
One-dimensional representations + orthogonality of characters is not enough to build the whole character table for T_d. One needs more black magic, such as representation of permutation group by matrices.

(d) Find all modes of the methane molecule. Which of them correspond to vibrations, translations and rotations? What are the degeneracies?

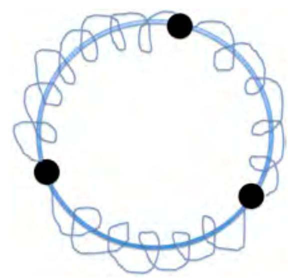

Figure 5.2: Three identical masses are constrained to move on a hoop, connected by three identical springs such that the system wraps completely around the hoop. Find all symmetries of the equations of motion.

Path: Find characters of the full (reducible) representation by using formulas from the lecture:

$$\chi(g) = \begin{cases} n_g(1 + 2\cos(\varphi)) & \textit{rotation}, \\ n_g(-1 + 2\cos(\varphi)) & \textit{improper rotation}. \end{cases}$$

Here n_g is the number of atoms staying at the same place under the action of g, φ is the rotation angle corresponding to $g = R(\varphi)$. Then decompose this representation into irreducible representations. Identify the rotational and translational parts.

(e) To what representation corresponds the most symmetric "breathing" mode and why? Is it infrared active, i.e., can this mode can be excited by electromagnetic field?

(B. Gutkin)

5.2. **Keep it classy.** Check out Harter's PowerPoint presentation :)

(a) Go through the derivation of the three projection operators for D_3.

(b) Decompose $\mathbf{P}^{(3)} = \mathbf{P}_1^{(3)} + \mathbf{P}_2^{(3)}$. Construct $\mathbf{P}_{ij}^{(3)}$. Verify that they are idempotent.

(d) Compute the $[2 \times 2]$ irreducible matrix representation $D^{(3)}(g)_{ij}$ for every group element g, in the spirit of Harter's slides 13-8 and 13-9.

5.3. **Three masses on a loop.** (Exercise 2.7 revisited.) Three identical masses, connected by three identical springs, are constrained to move on a circle hoop as shown in figure 5.2.

(a) Find all symmetries of the equations of motion.

(b) Find the normal modes using group-theoretic decompositions to irreps and character orthonormality.

(c) How many eigenvalues are there in all?

(d) Interpret the eigenvalues and eigenvectors from a group-theoretic, symmetry point of view.

group theory - week 6

For fundamentalists

Homework HW6

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 6.1 *3-disk symbolic dynamics*	2 bonus points
Exercise 6.2 *Reduction of 3-disk symbolic dynamics to binary*	3 bonus points
Exercise 6.3 *3-disk fundamental domain cycles*	2 bonus points
Exercise 6.4 *C_2-equivariance of Lorenz system*	3 points
Exercise 6.5 *Proto-Lorenz system* parts 1.-5.	7 points
Exercise 6.5 *Proto-Lorenz system* parts 6.-8.	6 bonus points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2021-06-08 Predrag Lecture 11 True grit

Lecture 7 (Unedited)

- We work out the symmetry reduction and a breaking of the D_3 symmetry in the $[3 \times 3]$ permutation matrices representation
- A preview; reduction of a cyclic C_N symmetric molecule chain to a $[2 \times 2]$ frequency matrix calculation. It will take a couple of weeks of discrete Fourier lectures to fill in the details. (2:07:51 h)

So far we have covered what any QM fixated Group Theory textbook since 1930's and on covers. Today to turn to what we actually use group theory for *today*, *here*, in Howey, and for that there is no book but ChaosBook.org.

Many fundamental problems of fluid dynamics and more generally non-linear field theories are studied in experimental settings equipped with symmetries. That is the subject of *dynamical systems theory* (of which classical, quantum and stochastic mechanics and field theories are but specialized branches). We start gently, with the famed Lorenz butterfly.

2021-06-10 Predrag Lecture 12 Nonlinear symmetry reduction

Lecture 8 (Unedited) Symmetries and nonlinear systems. First a mini-course on nonlinear dynamics and chaos. I use the symmetry reduction on the Lorenz, and turn it into Van Gogh. Conclusion: can use either a fundamental domain, or invariant polynomial bases to reduce symmetries of a nonlinear system. (2:28:00 h)

- Lorenz flow example. Read ChaosBook Chapter 11. World in a mirror ChaosBook.org Chapter 11 *World in a mirror*. Maybe start with ChaosBook example 10.6 *Equivariance of the Lorenz flow*, example 11.8 *Desymmetrization of Lorenz flow*, and then work your way back if needed.
- example 6.1 *Equivariance of the Lorenz flow*
- example 6.2 *Desymmetrization of Lorenz flow*

The reading and the homework for this week, is augmented by - if you find that helpful - by 'live' online blackboard lectures: click here.

6.1 Other sources (optional)

An example: a 1-dimensional system with a symmetry

Fundamental domain

Tiling of state space by a finite group

Make the "fundamental tile" your hood

Symmetry-reduced dynamics

Regular representation of permuting tiles

6.2 Thoughts (optional)

How I think of the fundamental domain is explained in my online lectures, Week 14, in particular the snippet *Regular representation of permuting tiles*.

Unfortunately - if I had more time, that would have been shorter, this goes on and on, Week 15, lecture *29. Discrete symmetry factorization*, and by the time the dust settles, I do not have a gut feeling for the boundary conditions when it comes to higher-dimensional irreps (see also last week's sect. 6.2 *Discussion*).

The basic insight is that if the symmetry and dynamics commute, one can implement the stratification of the state space by the symmetry first, paying no heed to the dynamics. In arbitrary coordinates, the state space is a stratified by a jumble of group orbits. It is an 'orbitfold', in the sense that in general it contains subspaces on which group orbits are of the dimension of a symmetry subgroup, with the group action on invariant subspaces trivial, and on which group orbits are points.

On the linear level, the natural stratification is implemented by decomposing the state space into irreps of the symmetry group. This is a linear reshuffling of coordinates that makes the action of the symmetry operators as simple as possible. You can think of the new basis vectors as eigenvectors of the symmetry operators (Fourier modes, spherical eigenfunctions, etc.). The nonlinear terms in dynamical equations jumble everything up. They are re-expressed in this basis using Kronecker-product decompositions into sums over products of irreps.

Unfortunately –if I had more time, that would have been shorter– this goes on and on, ChaosBook course 2, Week 15, lecture *29. Discrete symmetry factorization*.

Henriette Roux What do the parameters σ, ρ and b stand for in the Lorenz equations (6.4)?

Predrag The short answer is the truncation of the Navier-Stokes that leads to Lorenz equations is so drastic that they have no longer any physical meaning; in his 1963 paper [17] Lorenz played with the parameters until he empirically found an interesting example of deterministic chaos. Since then, applied mathematicians have reverse-engineered various physical systems to find situations where parameters σ, ρ and b mean something, see remark 6.1 (copied to here from ChaosBook.org). The discrete symmetry of the original Navier-Stokes system ('left' is as good as 'right') happened to survive the drastic truncation from 10^5 Fourier modes (for physically accurate simulations) to 3. I prefer to teach non-linear dynamics using the Rössler system, precisely because it has no discrete symmetry, just chaos.

6.3 ChaosBook notes

Copied here are a few snippets from this week's lecture notes, needed here just because exercises refer to them - read the full lecture notes instead.

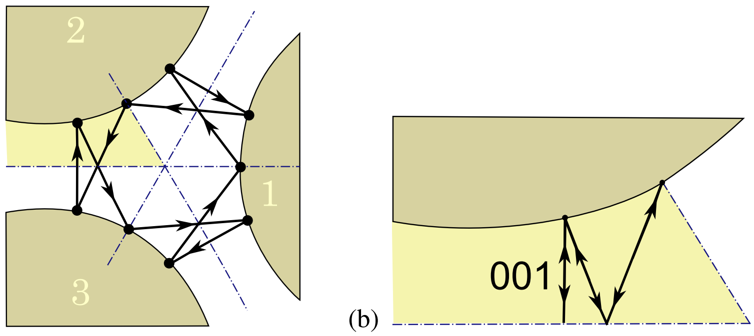

Figure 6.1: (a) The pair of full-space 9-cycles, the counter-clockwise $\overline{121232313}$ and the clockwise $\overline{131323212}$ correspond to (b) one fundamental domain 3-cycle $\overline{001}$.

Definition: Flow invariant subspace. A typical point in fixed-point subspace $\mathcal{M}_H$ moves with time, but, due to equivariance

$$f(gx) = g f(x)\,, \tag{6.1}$$

its trajectory $x(t) = f^t(x)$ remains within $f(\mathcal{M}_H) \subseteq \mathcal{M}_H$ for all times,

$$h f^t(x) = f^t(hx) = f^t(x)\,, \quad h \in H\,, \tag{6.2}$$

i.e., it belongs to a *flow invariant subspace*. This suggests a systematic approach to seeking compact invariant solutions. The larger the symmetry subgroup, the smaller $\mathcal{M}_H$, easing the numerical searches, so start with the largest subgroups H first.

We can often decompose the state space into smaller subspaces, with group acting within each 'chunk' separately:

Definition: Invariant subspace. $\mathcal{M}_\alpha \subset \mathcal{M}$ is an *invariant* subspace if

$$\{\mathcal{M}_\alpha \mid gx \in \mathcal{M}_\alpha \text{ for all } g \in G \text{ and } x \in \mathcal{M}_\alpha\}\,. \tag{6.3}$$

$\{0\}$ and $\mathcal{M}$ are always invariant subspaces. So is any $\mathrm{Fix}\,(H)$ which is point-wise invariant under action of G.

Definition: Irreducible subspace. A space $\mathcal{M}_\alpha$ whose only invariant subspaces under the action of G are $\{0\}$ and $\mathcal{M}_\alpha$ is called *irreducible*.

Example 6.1. ***Equivariance of the Lorenz flow.*** *The velocity field in Lorenz equations [17]*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y-x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ -xz \\ xy \end{bmatrix} \tag{6.4}$$

is equivariant under the action of cyclic group $C_2 = \{e, r\}$ acting on $\mathbb{R}^3$ by a π rotation about the z axis,

$$r(x, y, z) = (-x, -y, z)\,. \tag{6.5}$$

Table 6.1: D_3 correspondence between the binary labeled fundamental domain prime cycles $\tilde{p}$ and the full 3-disk ternary labeled cycles p, together with the D_3 transformation that maps the end point of the $\tilde{p}$ cycle into the irreducible segment of the p cycle. White spaces in the above ternary sequences mark repeats of the irreducible segment; for example, the full space 12-cycle 1212 3131 2323 consists of 1212 and its symmetry related segments 3131, 2323. The multiplicity of p cycle is $m_p = 6n_{\tilde{p}}/n_p$. The shortest pair of fundamental domain cycles related by time reversal (but no spatial symmetry) are the 6-cycles $\overline{001011}$ and $\overline{001101}$.

$\tilde{p}$	p	$\mathbf{g}_{\tilde{p}}$	$\tilde{p}$	p	$\mathbf{g}_{\tilde{p}}$
0	1 2	σ_{12}	000001	121212 131313	σ_{23}
1	1 2 3	C	000011	121212 313131 232323	C^2
01	12 13	σ_{23}	000101	121213	e
001	121 232 313	C	000111	121213 212123	σ_{12}
011	121 323	σ_{13}	001011	121232 131323	σ_{23}
0001	1212 1313	σ_{23}	001101	121231 323213	σ_{13}
0011	1212 3131 2323	C^2	001111	121231 232312 313123	C
0111	1213 2123	σ_{12}	010111	121312 313231 232123	C^2
00001	12121 23232 31313	C	011111	121321 323123	σ_{13}
00011	12121 32323	σ_{13}	0000001	1212121 2323232 3131313	C
00101	12123 21213	σ_{12}	0000011	1212121 3232323	σ_{13}
00111	12123	e	0000101	1212123 2121213	σ_{12}
01011	12131 23212 31323	C	0000111	1212123	e
01111	12132 13123	σ_{23}	···	···	···

Example 6.2. ***Desymmetrization of Lorenz flow:*** *(continuation of example 6.1) Lorenz equation (6.4) is equivariant under (6.5), the action of order-2 group $C_2 = \{e, r\}$, where r is $[x, y]$-plane, half-cycle rotation by π about the z-axis:*

$$(x, y, z) \to r(x, y, z) = (-x, -y, z)\,. \tag{6.6}$$

$(r)^2 = 1$ condition decomposes the state space into two linearly irreducible subspaces $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$, the z-axis $\mathcal{M}^+$ and the $[x, y]$ plane $\mathcal{M}^-$, with projection operators onto the two subspaces given by

$$\mathbf{P}^+ = \frac{1}{2}(1 + r) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}^- = \frac{1}{2}(1 - r) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.7}$$

so

$$\begin{aligned} \begin{pmatrix} \dot{x}_- \\ \dot{y}_- \end{pmatrix} &= \begin{pmatrix} -\sigma & \sigma \\ \rho & -1 \end{pmatrix} \begin{pmatrix} x_- \\ y_- \end{pmatrix} + \begin{pmatrix} 0 \\ -z\, x_- \end{pmatrix} \\ \dot{z}_+ &= -b\, z_+ + \frac{1}{4}(x_+ + x_-)(y_+ + y_-)\,, \end{aligned} \tag{6.8}$$

where $z_+ = z$. As $(\dot{x}_+, \dot{y}_+) = (0, 0)$, values of (x_+, y_+) are conserved parts of the initial condition. We define the fundamental domain by the (arbitrary) condition $\hat{x}_- \geq 0$, and whenever exits the domain,we replace the function dependence by the corresponding fundamental domain coordinates,

$$(x_-, y_-) = r(\hat{x}_-, \hat{y}_-) = (-\hat{x}_-, -\hat{y}_-) \quad \textit{if} \quad x_- < 0\,,$$

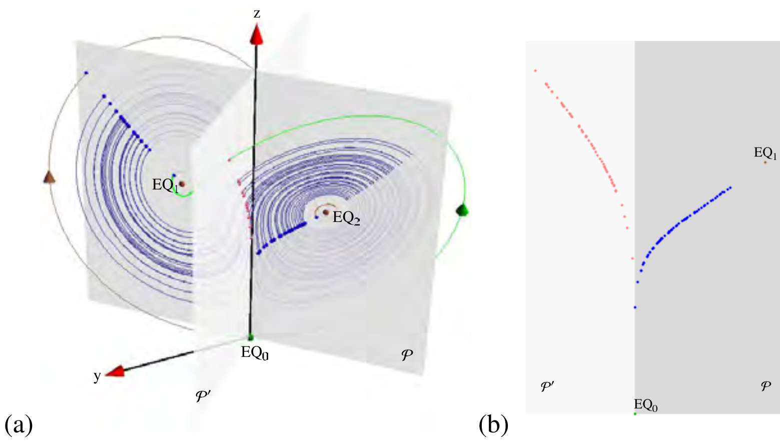

Figure 6.2: (a) Lorenz flow cut by $y = x$ Poincaré section plane $\mathcal{P}$ through the z axis and both $E_{1,2}$ equilibria. Points where flow pierces into section are marked by dots. To aid visualization of the flow near the E_0 equilibrium, the flow is cut by the second Poincaré section, $\mathcal{P}'$, through $y = -x$ and the z axis. (b) Poincaré sections $\mathcal{P}$ and $\mathcal{P}'$ laid side-by-side. (E. Siminos)

and record that we have applied r (that is the 'reconstruction equation' in the case of a discrete symmetry). When we integrate (6.8), the trajectory coordinates $(\hat{x}_-(t), \hat{y}_-(t))$ are discontinuous whenever the trajectory crosses the fundamental domain border. That, however, we do not care about - the only thing we need are the Poincaré section points and the Poincaré return map in the fundamental domain.

Poincaré section hypersurface can be specified implicitly by a single condition, through a function $U(x)$ that is zero whenever a point x is on the Poincaré section,

$$\hat{x} \in \mathcal{P} \quad \textit{iff} \quad U(\hat{x}) = 0\,. \tag{6.9}$$

In order that there is only one copy of the section in the fundamental domain, this condition has to be invariant, $U(g\hat{x}) = U(\hat{x})$ for $g \in G$, or, equivalently, the normal to it has to be equivariant

$$\partial_j U(g\hat{x}) = g\partial_j U(\hat{x}) \quad \textit{for} \quad g \in G\,. \tag{6.10}$$

There are two kinds of compact (finite-time) orbits. Periodic orbits $x(T_p) = x(T_p)$ are either self dual under rotation r, or appear in pairs related by r; in the fundamental domain there is only one copy $\hat{x}(T_p) = \hat{x}(T_p)$ of each. Relative periodic orbits (or 'pre-periodic orbits') $\hat{x}(T_p) = rx(T_p)$ they are periodic orbits.

As the flow is C_2-invariant, so is its linearization $\dot{x} = Ax$. Evaluated at E_0, A commutes with r, and the E_0 stability matrix A decomposes into $[x, y]$ and z blocks.

The 1-dimensional $\mathcal{M}^+$ subspace is the fixed-point subspace, with the z-axis points left point-wise invariant *under the group action*

$$\mathcal{M}^+ = \mathrm{Fix}\,(C_2) = \{x \in \mathcal{M} \mid g\,x = x \textit{ for } g \in \{e, r\}\} \tag{6.11}$$

(here $x = (x, y, z)$ is a 3-dimensional vector, not the coordinate x). A C_2-fixed point $x(t)$ in $\mathrm{Fix}\,(C_2)$ moves with time, but according to (6.2) remains within $x(t) \in \mathrm{Fix}\,(C_2)$ for all times; the subspace $\mathcal{M}^+ = \mathrm{Fix}\,(C_2)$ is flow invariant. *In case at hand this jargon is a bit*

of an overkill: clearly for $(x, y, z) = (0, 0, z)$ the full state space Lorenz equation (6.4) is reduced to the exponential contraction to the E_0 equilibrium,

$$\dot{z} = -b\,z\,. \tag{6.12}$$

However, for higher-dimensional flows the flow-invariant subspaces can be high-dimensional, with interesting dynamics of their own. Even in this simple case this subspace plays an important role as a topological obstruction: the orbits can neither enter it nor exit it, so the number of windings of a trajectory around it provides a natural, topological symbolic dynamics.

The $\mathcal{M}^-$ subspace is, however, not *flow-invariant, as the nonlinear terms $\dot{z} = xy - bz$ in the Lorenz equation (6.4) send all initial conditions within $\mathcal{M}^- = (x(0), y(0), 0)$ into the full, $z(t) \neq 0$ state space $\mathcal{M}/\mathcal{M}^+$.*

By taking as a Poincaré section any r-equivariant, non-self-intersecting surface that contains the z axis, the state space is divided into a half-space fundamental domain $\tilde{\mathcal{M}} = \mathcal{M}/C_2$ and its 180^o rotation $r\tilde{\mathcal{M}}$. An example is afforded by the $\mathcal{P}$ plane section of the Lorenz flow in figure 6.3. Take the fundamental domain $\tilde{\mathcal{M}}$ to be the half-space between the viewer and $\mathcal{P}$. Then the full Lorenz flow is captured by re-injecting back into $\tilde{\mathcal{M}}$ any trajectory that exits it, by a rotation of π around the z axis.

As any such r-invariant section does the job, a choice of a 'fundamental domain' is here largely mater of taste. For purposes of visualization it is convenient to make the double-cover nature of the full state space by $\tilde{\mathcal{M}}$ explicit, through any state space redefinition that maps a pair of points related by symmetry into a single point. In case at hand, this can be easily accomplished by expressing (x, y) in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, and then plotting the flow in the 'doubled-polar angle representation:'

$$(\hat{x}, \hat{y}, z) \;=\; (r\cos 2\theta, r\sin 2\theta, z) \;=\; ((x^2 - y^2)/r, 2xy/r, z)\,, \tag{6.13}$$

*as in figure 6.4 (a). In contrast to the original G-equivariant coordinates $[x, y, z]$, the Lorenz flow expressed in the new coordinates $[\hat{x}, \hat{y}, z]$ is G-*invariant*. In this representation the $\tilde{\mathcal{M}} = \mathcal{M}/C_2$ fundamental domain flow is a smooth, continuous flow, with (any choice of) the fundamental domain stretched out to seamlessly cover the entire $[\hat{x}, \hat{y}]$ plane.*

(E. Siminos and J. Halcrow)

6.4 Chaotic 3-spring, integrable 3-vortex systems (optional)

Continued from sect. 4.1.

Simon Berman According to the 2019 Phys. Rev. Lett., of Katz-Saporta and Efrati [15], *Self-driven fractional rotational diffusion of the harmonic three-mass system*, a system of three masses connected by harmonic springs might be the simplest mechanical system (homonuclear triatomic molecule, such as ozone, except the three couplings are not the same) that exhibits a *geometric phase*. Away from its resting configuration the system is nonlinear, and once its rotational SO(2) symmetry is reduced, and as its energy is increased, it exhibits all kinds of shape-dependent chaotic geometric phases. Katz and Efrati [15] mostly do numerical simulations and plot displacement vs. time diffusion plots in its 6D phase space, like this is still early 1960's. The earlier arXiv:1706.09868 version has more information than the PRL (no doubt thanks to impatience of the referees, plus space constraints of a PRL). One suspects that a bit of thinking along periodic orbit theory lines could yield some insight into the diffusive properties of its shape-changing dynamics.

In the symmetry-reduced or the 'shape' state space there is a D_3 symmetry. One sees it in their [15] Hamiltonian (2): the b^{ij} vectors can be viewed as the three coordinates of an equilateral triangle in the $w_1 - w_2$ plane. Since the Hamiltonian only depends on $|w|$ and in a symmetric way on $w \cdot b^{ij}$, it has a D_3 symmetry for (w_1, w_2) components of the w vector, and a reflection symmetry for w_3. So the total symmetry group is $D_3 \times C^{1/2}$.

Predrag As the system is D_3 symmetric, the symmetry should be quotiented as in (this week's lectures) and ChaosBook.org. The students from Weizmann (as well as all our local plumber apprentices) believe they have been born knowing everything, and thus they do not need to take ChaosBook.org/course1, so they would have no idea that

- they are supposed to quotient the symmetry
- probability densities (eigenfunctions of the evolution operator; Perron--Frobenius and its generalizations) block diagonalize as irreps of D_3, and
- that makes all calculations, numerical and periodic orbit-type more transparent and more convergent.

By going to relative w's coordinates, one has quotiented only the $2D$ Euclidean translations and $SO(2)$ rotations, no discrete symmetries, so D_3 still remains. Now, anyone who has taken ChaosBook.org/course1 knows that the next step is to quotient D_3, and do the calculation in the 1/6th of the phase space, i.e., the fundamental domain.

I'm curious whether I'm right, because soon we'll look at space groups (infinite lattices with discrete symmetries) and there I have confused understanding of how to quotient the space group, but that is related to diffusion in space, rather than the angular diffusion, as in this 3-springs system.

We can make this a course project for a student in this course (a project instead of taking the final). To be especially pedagogical, we'll ask them to do it in Julia (there is one potential candidate on Piazza).

Predrag proposal: 2-body, 3-spring system We need the *simplest* illustration of a geometric phase, and its diffusion along the continuous symmetry direction induced by chaotic ("turbulent") shape-changing dynamics. So let's take one of the masses infinite. Still 3 springs, but only 2 bodies moving in a plane. We still have $SO(2)$ continuous symmetry to reduce. What remains is the $D_2 = \{e, \sigma\}$ symmetry of exchanging the two particles, with two irreps, the symmetric and the antisymmetric normal modes. There is shape-changing dynamics, with the potential a nonlinear function of w_j's, so for larger energies we expect angular geometric phase diffusion, but in a lower-dimensional phase space than that of the free 3-springs system. Easier to work out and look at Poincaré sections, search for relative equilibria and relative periodic orbits, compute the angular diffusion constant from its cycle expansion formulation.

Predrag: N **vortex system** Went to hear Tomoki Ohsawa (Google Scholar), talk about *Symplectic reduction and the Lie–Poisson dynamics of point vortices on the plane*, arXiv:1808.01769 .

I had previously written to Tomoki's friend Molei about how much I had already suffered through Weinstein, Marsden, etc. moment maps, for decades. We all have to do symmetry reductions, but with Marsden it is always the moment map, and then the climax is the rigid 3D body example which is the end-all of every article and book. Perhaps due to my pleas, Ohsawa gave us a gentle, sensible seminar, Weinstein-Marsden for humans, where he explained why moment map is called 'moment,' etc.. As nice a birthday present one could hope for, see the slides here.

Ohsawa develops a Hamiltonian formulation of the dynamics of the "shape" of N point vortices on the plane and the sphere. If $N = 3$, it is the dynamics of the shape of the triangle formed by three point vortices, regardless of the position and orientation of the triangle on the plane/sphere.

For the planar case, reducing the basic equations of point vortex dynamics by the special Euclidean group SE(2) yields a Lie-Poisson equation for relative configurations of the vortices. The shape dynamics is periodic in certain cases. The approach can be extended to the spherical case by first lifting the dynamics from the two-sphere to C^2 and then performing reductions by symmetries.

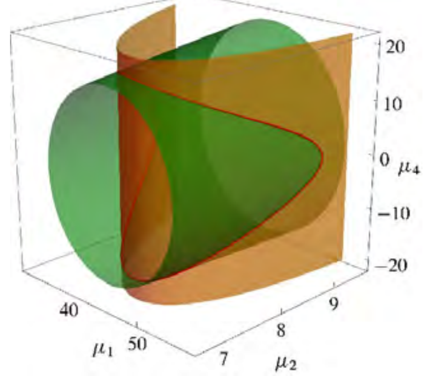

Figure 6.3: (green) Level set of quadratic Casimir C_2 ellipsoid. (orange) Level set of Hamiltonian h. The intersection is the unique periodic orbit of the symmetry-reduced $N = 3$ vortex system. See also the corresponding figure for 3 vortices on the sphere on p. 46 of the slide presentation. T. Ohsawa

I think Ohsawa discovery is that the system has a previously un-noted quartic Casimir, whose invariance reduces the dimension of the symmetry reduced phase space by one degree of freedom (dof). The implies that the symmetry-reduced dynamics in the $N = 3$ case is 1 dof, i.e., integrable, see figure 6.3. In addition, if the sum of the vortex circulations is 0, the $N = 4$ case is integrable. This fact is not yet explained - my intuition is that the zero total circulation implies extra rotational symmetry. For more vortices I expect the usual Hamiltonian mixed phase space.

Parenthetically, statement that there is quartic order Casimir that is invariant under the symmetry group can probably be written as a syzygy constraint on the invariant polynomial basis in (Hilbert's) theory of invariant polynomial bases.

In the Katz-Saporta and Efrati [15] example there is no quartic Casimir, so one ends with a generic chaotic system. Ohsawa's geometric technique works because of the simple symplectic structure on the point-vortex problem (there is no 'momentum'), whereas Katz-Saporta and Efrati problem is a standard classical-mechanical one on the cotangent bundle of a configuration space, with momentum there. Ohsawa believes that one can apply the techniques developed by Richard Montgomery to this setting as well. (Montgomery's paper motivated him to work on the point-vortex problem).

There are also examples in cardiac (!) dynamics where one must reduce 2D Euclidean symmetry first, with similar outcome to yours, but no moment maps, as such PDEs have no variational formulation (that I am aware of). Googling "Barkley model" might do the trick. I do not think there is a variational (Lagrangina) formulation.

But that is the whole point - *any* flow with a symmetry has to have the symmetry quotiented out. It's easier to understand this for flows which are not symplectic - in that case, every continuous symmetry parameter reduces the dimension of the symmetry-reduced state space by one. The Hamiltonin case is a pain (or bliss, if you love moment maps) because every continuous symmetry reduces the dimension of the phase space by one degree of freedom (ie, by 2). Also variational problems obey Noether's theorem, our (dissipative) problems usually do not. If I understand this right...

Kimberly Short Foulkes PhD thesis [7] *Drift and Meander of Spiral Waves* (2009) might be an user friendly introduction for students that need to understand Euclidean symmetry? Covers refs. [2, 8]. Page 14 solves a differential equation with $SE(2)$ symmetry. Appendix 8.9 discusses symmetries, and gives the condition for equivariance.

6.5 Eigenfunctions (optional)

What follows is an inconclusive discussion of eigenfunctions over fundamental domains - feel free to ignore...

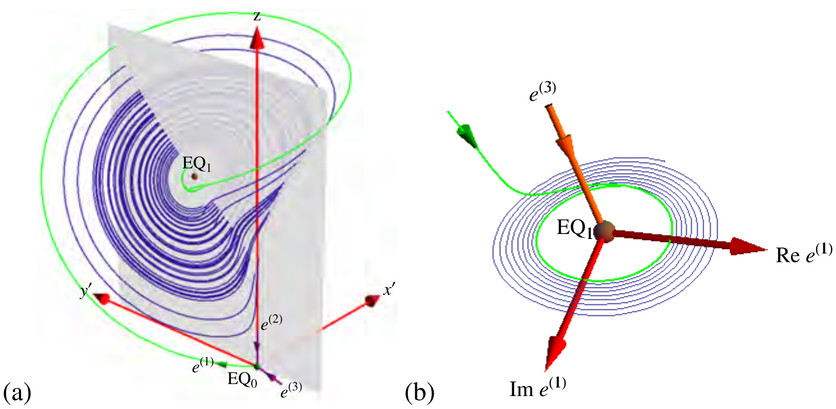

Figure 6.4: (a) Lorenz attractor plotted in $[\hat{x}, \hat{y}, z]$, the doubled-polar angle coordinates (6.13), with points related by π-rotation in the $[x, y]$ plane identified. Stable eigenvectors of E_0: $\mathbf{e}^{(3)}$ and $\mathbf{e}^{(2)}$, along the z axis (6.12). Unstable manifold orbit $W^u(\mathrm{E}_0)$ (green) is a continuation of the unstable $\mathbf{e}^{(1)}$ of E_0. (b) Blow-up of the region near E_1: The unstable eigenplane of E_1 defined by $\mathrm{Re}\,\mathbf{e}^{(2)}$ and $\mathrm{Im}\,\mathbf{e}^{(2)}$, the stable eigenvector $\mathbf{e}^{(3)}$. The descent of the E_0 unstable manifold (green) defines the innermost edge of the strange attractor. As it is clear from (a), it also defines its outermost edge. (E. Siminos)

Predrag Heilman and Strichartz [13] *Homotopies of Eigenfunctions and the Spectrum of the Laplacian on the Sierpinski Carpet*, arXiv:0908.2942, is not an obvious read for us, but they compute a spectrum on a square domain, and we might have to be mindful of it: " Since all of our domains are invariant under the D_4 symmetry group, we can simplify the eigenfunction computations by reducing to a fundamental domain. On this domain we impose appropriate boundary conditions according to the rep-resentation type. For the 1-dimensional representation, we restrict to the sector $0 \leq \theta \leq \pi/4$. Recall that the functions will extend evenly when reflected about $\theta = 0$ in the 1++ and 1– cases, and oddly in the 1-+ and 1+- cases. Note that performing an even extension across a ray is equivalent to imposing Neumann boundary conditions on that ray. Similarly, the odd extension is equivalent to Dirichlet conditions. For the 2-dimensional representation our fundamental domain is the sector $0 \leq \theta \leq \pi/2$, and we impose Neumann boundary conditions on the ray $\theta = 0$ and Dirichlet conditions on the ray $\theta = \pi/2$. Note that our fundamental domains are simply connected. "

This seems to be saying that one gets the 2-dimensional representation by doubling the fundamental domain and mixing boundary conditions. Do you understand that?

Boris Here is my present understanding of the fundamental domains issue: If you want simple boundary conditions like Dirichlet or Neumann you stick to 1d representations only. They connect eigenfunction to itself at the fundamental domain boundaries – otherwise you would need to connect pair of functions (would be something like boundary conditions for spinor in case of 2d representations.) So what you do is the following: take the largest abelian subgroup $Z_2 \times Z_2$ (for D_4) and split its spectrum with respect to its fundamental domain defined as 1/4 of the square (twice the fundamental domain of the full group). Then your see that Dirichlet-Dirichlet and Neumann-Neumann Hamiltonians still have Z_2 symmetry so your split them further into the Hamiltonians of the 1/8 fundamental domain. But Dirichlet-Neumann remains 1/4th of the square.

Predrag Your argument is in the spirit of Harter's class operators construction (see week 5) of higher-dimensional representations by using particular chains of subgroups, but I am not able to visualize how that larger fundamental domain (of the lower-order subgroup) folds back into the small fundamental domain of the whole group. By the time the dust settles, I have the symmetry factorization of the determinants that we need, but I do not have a gut feeling for the boundary conditions that you do, when it comes to higher-dimensional irreps.

Commentary

Remark 6.1. Lorenz equation. The Lorenz equation (6.4) is the most celebrated early illustration of "deterministic chaos" [17] (but not the first - that honor goes to Dame Cartwright [3] in 1945. Amusingly, Denisov and Ponomarev [6] argue that Ben F. Laposky might have been the first to observe chaotic attractors as early as 1953, which, strictly speaking falls after 1945, even in Russia). Lorenz's 1963 paper, which can be found in reprint collections refs. [5, 12], is a pleasure to read, and it is still one of the best introductions to the physics motivating such models (read more about Lorenz here). The equations, a set of ODEs in $\mathbb{R}^3$, exhibit strange attractors. W. Tucker [24–26] has proven rigorously (via interval arithmetic) that the Lorenz attractor is strange for the original parameters (no stable orbits) and that it has a long stable periodic orbit for slightly different parameters. In contrast to the hyperbolic strange attractors such as the weakly perturbed cat map [4], the Lorenz attractor is structurally unstable. Frøyland [9] has a nice brief discussion of Lorenz flow. Frøyland and Alfsen [10] plot many periodic and heteroclinic orbits of the Lorenz flow; some of the symmetric ones are included in ref. [9]. Guckenheimer-Williams [11] and Afraimovich-Bykov-Shilnikov [1] offer an in-depth discussion of the Lorenz equation. The most detailed study of the Lorenz equation was undertaken by Sparrow [22]. For a geophysics derivation, see Rothman course notes [20]. For a physical interpretation of ρ as "Rayleigh number," see Jackson [14] and Seydel [21]. The Lorenz truncation to 3 modes, however, is so drastic that the model bears no relation to the geophysical hydrodynamics problem that motivated it. Just for fun, as Lorentz was such a lovable weatherman, in 1972 Willem Malkus constructed [18], by a feat of reverse engineering, a physical system, a "water wheel", popularized by Strogatz [23], that is described by Lorentz equations. You can see it simulated on wolfram.com, and tested experimentally at http://www.ace.gatech.edu. There is no deep physics in this lovely game, it is but a cute distraction. For detailed pictures of Lorenz invariant manifolds consult Vol II of Jackson [14] and "Realtime visualization of invariant manifolds" by Ronzan. The Lorenz attractor is a very thin fractal – as we shall see, stable manifold thickness is of the order 10^{-4} – whose fractal structure has been accurately resolved by D. Viswanath [27, 28]. If you wonder what analytic function theory has to say about Lorenz, check ref. [29]. Modular flows are your thing? E. Ghys and J. Leys have a beautiful tale for you. Refs. [16, 19] might also be of interest.

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Exercises

6.1. **3-disk symbolic dynamics.** As periodic trajectories will turn out to be our main tool to breach deep into the realm of chaos, it pays to start familiarizing oneself with them now by sketching and counting the few shortest prime cycles. Show that the 3-disk pinball has $3 \cdot 2^{n-1}$ itineraries of length n. List periodic orbits of lengths 2, 3, 4, 5, $\cdots$. Verify that the shortest 3-disk prime cycles are 12, 13, 23, 123, 132, 1213, 1232, 1323, 12123, $\cdots$. Try to sketch them. (continued in exercise 6.3)

A comment about exercise 6.1, exercise 6.2, and exercise 6.3: If parts of these problems seem trivial - they are. The intention is just to check that you understand what these symbolic dynamics codings are - the main message is that the really smart coding (fundamental domain) is 1-to-1 given by the group theory operations that map a point in the fundamental domain to where it is in the full state space. This observation you might not find deep, but it is - instead of *absolute* labels, in presence of symmetries one only needs to keep track of *relative* motions, from domain to domain, does not matter which domain in absolute coordinates - that is what group actions do. And thus the word '*relative*' creeps into this exposition.

6.2. **Reduction of 3-disk symbolic dynamics to binary.** (continued from exercise 6.1)

(a) Verify that the 3-disk cycles
$\{\overline{1\,2}, \overline{1\,3}, \overline{2\,3}\}$, $\{\overline{1\,2\,3}, \overline{1\,3\,2}\}$, $\{\overline{12\,13}$ + 2 perms.$\}$,
$\{\overline{121\,232\,313}$ + 5 perms.$\}$, $\{\overline{121\,323}$+ 2 perms.$\}$, $\cdots$,
correspond to the fundamental domain cycles $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$ respectively.

(b) Check the reduction for short cycles in table 6.1 by drawing them both in the full 3-disk system and in the fundamental domain, as in figure 6.3.

(c) Optional: Can you see how the group elements listed in table 6.1 relate irreducible segments to the fundamental domain periodic orbits?

(continued in exercise 6.3)

6.3. **3-disk fundamental domain cycles.** Try to sketch $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$. in the fundamental domain, and interpret the symbols $\{0, 1\}$ by relating them to topologically distinct types of collisions. Compare with table 6.1. Then try to sketch the location of periodic points in the Poincaré section of the billiard flow. The point of this exercise is that while in the configuration space longer cycles look like a hopeless jumble, in the Poincaré section they are clearly and logically ordered. The Poincaré section is always to be preferred to projections of a flow onto the configuration space coordinates, or any other subset of state space coordinates which does not respect the topological organization of the flow.

6.4. **C_2-equivariance of Lorenz system.** Verify that the vector field in Lorenz equations (6.4)

$$\dot{x} = v(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y-x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix} \tag{6.14}$$

is equivariant under the action of cyclic group $C_2 = \{e, r\}$ acting on $\mathbb{R}^3$ by a π rotation about the z axis,

$$r(x, y, z) = (-x, -y, z)\,,$$

as claimed in example 6.1.

6.5. **Proto-Lorenz system.** Here we quotient out the C_2 symmetry by constructing an explicit "intensity" representation of the desymmetrized Lorenz flow.

1. Rewrite the Lorenz equation (6.4) in terms of variables

$$(u, v, z) = (x^2 - y^2, 2xy, z)\,, \tag{6.15}$$

show that it takes form

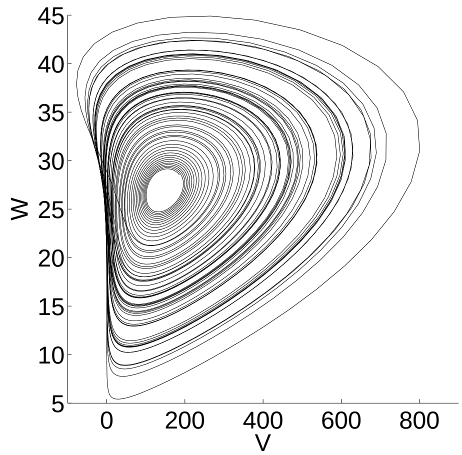
$$\begin{aligned} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} -(\sigma+1)u + (\sigma-r)v + (1-\sigma)N + vz \\ (r-\sigma)u - (\sigma+1)v + (r+\sigma)N - uz - Nz \\ v/2 - bz \end{bmatrix} \\ N &= \sqrt{u^2+v^2}\,. \end{aligned} \tag{6.16}$$

2. Show that this is the (Lorenz)/C_2 quotient map for the Lorenz flow, i.e., that it identifies points related by the π rotation (6.6).

3. Show that (6.15) is invertible. Where does the inverse not exist?

4. Compute the equilibria of proto-Lorenz and their stabilities. Compare with the equilibria of the Lorenz flow.

5. Plot the strange attractor both in the original form (6.4) and in the proto-Lorenz form (6.16)



for the Lorenz parameter values $\sigma = 10$, $b = 8/3$, $\rho = 28$. Topologically, does it resemble more the Lorenz, or the Rössler attractor, or neither? (plot by J. Halcrow)

6. Show that a periodic orbit of the proto-Lorenz is either a periodic orbit or a relative periodic orbit of the Lorenz flow.

7. Show that if a periodic orbit of the proto-Lorenz is also periodic orbit of the Lorenz flow, their Floquet multipliers are the same. How do the Floquet multipliers of relative periodic orbits of the Lorenz flow relate to the Floquet multipliers of the proto-Lorenz?

8. Show that the coordinate change (6.15) is the same as rewriting

$$\begin{aligned}
\dot{r} &= \frac{r}{2}(-\sigma - 1 + (\sigma + \rho - z)\sin 2\theta \\
&\quad +(1-\sigma)\cos 2\theta) \\
\dot{\theta} &= \frac{1}{2}(-\sigma + \rho - z + (\sigma - 1)\sin 2\theta \\
&\quad +(\sigma + \rho - z)\cos 2\theta) \\
\dot{z} &= -bz + \frac{r^2}{2}\sin 2\theta\,. \qquad (6.17)
\end{aligned}$$

in variables

$$(u, v) = (r^2 \cos 2\theta, r^2 \sin 2\theta)\,,$$

i.e., squaring a complex number $z = x + iy$, $z^2 = u + iv$.

group theory - week 7

Discrete Fourier representation

Homework HW7

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 7.1 *Am I a group?*	2 points
Exercise 7.2 *Product of two groups*	2 points
Exercise 7.3 *Laplacian is a non-local operator*	4 points
Exercise 7.4 *Lattice Laplacian diagonalized*	8 points
Exercise 7.5 Work through ChaosBook Example A24.2	
− *Projection operators for discrete Fourier transform.*	6 bonus points

Total of 16 points = 100 % score. Bonus points accumulate, can help you later if you miss a few problems.

2021-06-15 Predrag Lecture 13 Fundamentalist vision

How I think of the fundamental domain is explained in my online lectures, Week 14, in particular the snippet *Regular representation of permuting tiles*.

2021-06-15 Predrag Lecture 14 Diffusion confusion

You also might find my online lectures, Week 13 helpful.

Discretization of continuum, lattices, discrete derivatives, discrete Fourier representations.

The fastest way to watch any week's lecture videos is by letting YouTube run *the course playlist* (2h 50 min + 45 min for extras).

- *Symmetry is your friend - overview. The power of thinking.* (9 min)
- Applied math version: how to discretize derivatives:
  ChaosBook Appendix A24 Deterministic diffusion
  Sects. A24.1 to A24.1.1 *Lattice Laplacian*.
- *Lattice discretization, lattice state* (7 min)
- *Lattice derivative* (6 min)
- *Shift operator: the generator of discrete translations* (15 min)
  - (extra) *Discussion: Shift matrix must have the periodic b.c.; Derivative being nonlocal is easiest to grasp on discrete lattice. It's so easy to make errors in the continuum formulation.* (14 min)
- *Derivative is a linear operator* (15 min)
- *Lattice Laplacian* (5 min)
- *Derivative is a non-local operator* (6 min)
  - (extra) *Discussion: Lattice discretization; What if geometry is not flat in all directions, but spherical? What about General Relativity? Life's persistent questions, skated around.* (14 min)
  - (extra) *Discussion: What is a derivative? Hypercubic lattice is a graph, with nodes connected by links. Every graph has a notion of derivative associated with it; in particular a Laplacian. I was not allowed to say "Laplacian" here, as I have not gotten to defining it in my lecture at that point...* (2 min)
- A periodic lattice as the simplest example of the theory of finite groups:
  ChaosBook Sects. A24.1.2 to A24.3.1.
  ChaosBook Example A24.2 *Projection operators for discrete Fourier representation.*
  ChaosBook Example A24.3 *'Configuration-momentum' Fourier space duality.*

Have symmetry? Use it! (14 min)

(extra) *Rant: Symmetrize you must. Karl Schwarzschild found his exact solution in 1915, a month after the publication of Einstein's theory of general relativity, while serving on a World War I front.* (3 min)

Have symmetry? Go to "eigen"subspace! Fourier decomposition of a 2-sites periodic lattice. (7 min)

Periodic lattices (5 min)

Fourier eigenvalues (9 min)

Discrete Fourier representation (6 min)

Laplacian in Fourier representation (9 min)

Propagator in Fourier representation (6 min)

A meta truth; We live in The Matrix; Fourier transformation is just a matrix (10 min)

7.1 Optional reading

A theoretical physicist's version of the above notes: *Quantum Field Theory - a cyclist tour*, Chapter 1 *Lattice field theory* motivates discrete Fourier representations by computing a free propagator on a lattice.

(extra) *Quantum Mechanics in a box: Sometimes it is simplest to impose the periodic b.c. on a localized solution, than relax it towards the correct (infinite extent) continuum solution.* (5 min)

(extra) *Rocket science needs complex numbers; Why Fourier? Digital image processing!* (8 min)

Exercises

7.1. **Am I a group?** Show that multiplication table

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	b	f	c
b	b	d	e	f	c	a
c	c	b	f	e	a	d
d	d	f	c	a	e	b
f	f	c	a	d	b	e

describes a group. Or does it? (Hint: check whether this table satisfies the group axioms.)

7.2. **Product of two groups.** Let G_1 and G_2 be two finite groups. The elements of the product set $G = G_1 \times G_2$ are defined as pairs (g_1, g_2), $g_1 \in G_1$ $g_2 \in G_2$.

(a) Show that G is a group with the multiplication operation $(g_1, g_2) \cdot (g_1', g_2') = (g_1 g_1', g_2 g_2')$.

Let D_1 be an irreducible representation of G_1 and let D_2 be an irreducible representation of G_2. For each $g = (g_1, g_2) \in G$ define $D(g) = D_1(g_1) \times D_2(g_2)$

(b) Show that $D = D_1 \times D_2$ is an irreducible representation of G. What are the characters of D?

7.3. **Laplacian is a non-local operator.**
While the Laplacian is a simple tri-diagonal difference operator, its inverse (the "free" propagator of statistical mechanics and quantum field theory) is a messier object. A way to compute is to start expanding propagator as a power series in the Laplacian

$$\frac{1}{m^2\mathbf{1} - \Delta} = \frac{1}{m^2}\sum_{n=0}^{\infty}\frac{1}{m^{2n}}\Delta^n \,. \tag{7.1}$$

As Δ is a finite matrix, the expansion is convergent for sufficiently large m^2. To get a feeling for what is involved in evaluating such series, show that Δ^2 is:

$$\Delta^2 = \frac{1}{a^4}\begin{bmatrix} 6 & -4 & 1 & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & \ddots & & & \\ & & & & & 6 & -4 \\ -4 & 1 & & & 1 & -4 & 6 \end{bmatrix} . \tag{7.2}$$

What Δ^3, Δ^4, $\cdots$ contributions look like is now clear; as we include higher and higher powers of the Laplacian, the propagator matrix fills up; while the *inverse* propagator is differential operator connecting only the nearest neighbors, the propagator is integral operator, connecting every lattice site to any other lattice site.

This matrix can be evaluated as is, on the lattice, and sometime it is evaluated this way, but in case at hand a wonderful simplification follows from the observation that the lattice action is translationally invariant, exercise 7.4.

7.4. **Lattice Laplacian diagonalized.** Insert the identity $\sum \mathbf{P}^{(k)} = \mathbf{1}$ wherever you profitably can, and use the shift matrix eigenvalue equation to convert shift σ matrices into scalars. If $\mathbf{M}$ *commutes* with σ, then $(\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) = \tilde{M}^{(k)}\delta_{kk'}$, and the matrix $\mathbf{M}$ acts as a multiplication by the scalar $\tilde{M}^{(k)}$ on the kth subspace. Show that for the 1-dimensional lattice, the projection on the kth subspace is

$$(\varphi_k^\dagger \cdot \Delta \cdot \varphi_{k'}) = \frac{2}{a^2}\left(\cos\left(\frac{2\pi}{N}k\right) - 1\right)\delta_{kk'} \,. \tag{7.3}$$

In the kth subspace the propagator is simply a number, and, in contrast to the mess generated by (7.1), there is nothing to evaluating it:

$$\varphi_k^\dagger \cdot \frac{1}{m^2\mathbf{1} - \Delta} \cdot \varphi_{k'} = \frac{\delta_{kk'}}{m^2 - \frac{2}{a^2}\left(\cos 2\pi k/N - 1\right)} \,, \tag{7.4}$$

where k is a site in the N-dimensional dual lattice, and $a = L/N$ is the lattice spacing.

group theory - week 8

Space groups

Homework HW8

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 8.1 *Space group*	2 points
Exercise 8.2 *Band structure of a square lattice*	8 points

Bonus points

Exercise 8.3 *Tight binding model*	8 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2021-06-17 Predrag Lecture 15 Space groups **2019-02-21 Claire Berger** has no time to teach this lecture. But if she did, she would: (i) Start with 2D square lattice. (ii) Define Bravais lattice unit cell. (iii) Show that rotation symmetries compatible with a 2D lattice are (none), 2-, 3-, 4-, or 6-fold. (iv) Sketch the resulting 17 wallpaper groups, sect. 8.3.1.

2021-06-17 Predrag Lecture 16 Reciprocal lattice

2019-02-21 Claire Berger has no time to teach this lecture. But if she did, she would: (i) Start with Bragg diffraction off 2-layer square lattice to motivate the reciprocal lattice. (ii) Show her group's graphene diffraction measurements that identify and distinguish the one- and the two-layer graphene. Reciprocal lattice is not a mathematical construct - it *is* what experimentalists see. (iii) Construct the reciprocal lattice and the first Brillouin zone. (iv) Show the Brillouin zone for graphene, explain what is seen in experiments.

Lecture 10 (Unedited) Space groups. Bravais lattice. Reciprocal lattice. Brillouin cell. Fundamental domain. (2:29:20 h)

9.1 Space groups (24:49 min; included in the above)

Gutkin lecture notes Lect. 7 *Applications III. Energy Band Structure*, Sect. *7.2 Lattice symmetries*.

8.1 Other sources (optional)

Gutkin lecture notes Lect. 7 *Applications III. Energy Band Structure*, Sect. *7.2 Band structure*.

If you are curious about graphene, work out Gutkin lecture notes sect. 7.3 *Applications III. Energy Band Structure*

Liang and Cvitanović [23] *A chaotic lattice field theory in one dimension* (2022).

Also good reads: Dresselhaus *et al.* [11] chapter *9. Space Groups in Real Space* (click here), and Cornwell [9] chapter *7. Crystallographic Space Groups* (click here).

Walt De Heer learned this stuff from Herzberg [15] *Molecular Spectra and Molecular Structure*. Condensed matter people like Kittel [21] *Introduction to Solid State Physics*, but I am not a fan, because simple group theoretical facts are there presented as condensed matter phenomena.

Quinn and Yi [25] *Solid State Physics: Principles and Modern Applications* introduction to space groups looks compact and sensible. *Band structure of graphene*.

Martin Mourigal found the Presqu'île Giens, May 2009 *Contribution of Symmetries in Condensed Matter* Summer School very useful. Villain [30] *Symmetry and group theory throughout physics* gives a readable overview. The overheads are here, many of them are of potential interest (Mourigal recommended).

Canals and Schober [8] *Introduction to group theory*. It is very concise and precise, a bastard child of Bourbaki and Hamermesh [13]. Space groups show up only once, on

p. 24: "By working with the cosets we have effectively factored out the translational part of the problem."

Ballou [1] *An introduction to the linear representations of finite groups* appears rather formal (and very erudite).

Grenier, B. and Ballou [12] *Crystallography: Symmetry groups and group representations*.
The word crystal stems from Greek 'krustallas' and means "solidified by the cold."

Schober [28] *Symmetry characterization of electrons and lattice excitations* gives an eminently readable discussion of space groups.

Rodríguez-Carvajal and Bourée [26] *Symmetry and magnetic structures*

Schweizer [29] *Conjugation and co-representation analysis of magnetic structures* deals with black, white and gray groups that Martin tries not to deal with, so all Mourigal groups are gray.

Villain discusses graphene in the Appendix A of *Symmetry and group theory throughout physics* [30].

8.2 Thoughts (optional)

This week's notes are long, because I'm fascinated why –of all fields of physics where problems are formulated on lattices– only condensed matter utilizes the theory of irreps of space groups. For the course itself, read sect. 8.3 *Space groups* and sect. 8.3.1 *Wallpaper groups* - the rest is speculations, mostly.

Why do I care? In this course we are learning theory of space groups as applied to quantum mechanics of crystals - rather than diagonalizing the Hamiltonian and computing energy levels, one works on the reciprocal lattice, and computes energy bands (continuum limit of finely spaced discrete eigenvalues of finite, periodic lattices). If fluctuations from strict periodicity are small, one can often identify the crystal by measuring the intensities of Bragg peaks.

Then there are other kinds of lattices. In computational field theory (classical and quantum) one discretizes the space-time, often on a cubic lattice; one example is worked out here in sect. 8.4 *Elastodynamic equilibria of 2D solids*. The there are Ising models in one, two, three dimensions, problems like deterministic diffusion on periodic lattices of scatterers, coupled maps lattices. None of that literature *ever* (to best of my knowledge) reduces the computations to the reciprocal space Brilluion zone. Why?

The funny thing is - I *know* the answer since 1976, but the siren song of classical crystallography is so enchanting that it has blinded me with science. I think that is due to a deep and under-appreciated "chaos / turbulence" physics underlying these problems. If deviations from the strict periodic structure are *small* (the basic "long wavelength" assumption of sect. 8.4), the "integrable" thinking in terms of normal modes applies, and you should use the crystallography described here. If the symmetry of the law you are studying is a space group, but the deviations of typical solutions are *large* (our deterministic diffusion, Ising models, ...), we have to think again. One fundamental thing we learned in studies of transitions to chaos is that the traditional Fourier analysis is useless - it just yields broad, shapeless continuous spectra. The powerful way to think about these problems is Poincaré's qualitative theory of solutions of differential equations : analyse the geometry of their flows in their *state space*. I know for a fact (from a study of cat maps and spatiotemporal cat maps - see links to talks in ChaosBook.org/overheads/spatiotemporal; the papers are slowly being written up) that in that case the translational eigenfunctions are hyperbolic sinhes and coshes, rather than the sines and cosines we are used to as C_n eigenfunctions. For finite discrete symmetries you saw that irreps were fine for linear problems, like coupled arrays of springs, but symmetry reduction for a nonlinear problem like Lorenz equations required quite different techniques. For space group symmetries the analogous nonlinear problems seem still quite unexplored.

8.3 Space groups

Kepler noted that there are only three regular tessellations (tilings) of the plane, by triangles, by squares, and by hexagons. In 1900 David Hilbert posed his 23 problems, including "Is there in n-dimensional Euclidean space only a finite number of essentially different kinds of groups of motions with a fundamental region?" In 1910 Bieberbach solved this problem and proved that in dimension n there were only finitely many *Bieberbach groups*, extensions of the translation group, which is isomorphic to $\mathbb{Z}^n$, by a finite subgroup of $GL(n,\mathbb{Z})$. In 1948 Zassenhaus gave an algorithm to determine a complete set of representatives of the types of n-dimensional space groups. In the mid 1970's computers helped to determine that there are 4783 four-dimensional groups.

For the above history and references, see David E. Joyce.

A space group, a subgroup of the group of rotations and translations in three dimensions, is the set of transformations that leave a crystal invariant. A space group operator is commonly denoted as

$$\{R|\boldsymbol{t}\}\,, \tag{8.1}$$

where $\boldsymbol{t}$ belongs to the infinite set of discrete translations, and R is one of the finite number of discrete orientation (point group) symmetries. Translation symmetry, i.e., the periodicity of a crystal, manifests itself physically through phonons, magnons, and other smooth, long-wavelength deformations. Discrete orientation symmetry manifests itself through macroscopic anisotropies of crystals, and its natural faces. The experimental challenge is to determine the crystal structure, typically by diffraction (study of the *reciprocal lattice*). It is a challenge, as one measures only the intensities of Bragg peaks, not their phases, but the answer should be one of the 230 space groups listed in the *International Tables for Crystallography*, the "Bible" of crystallographers.

Unless you have run into a quasicrystal :). In that case Claire has a story to tell, but it will have to remain private.

Understanding the Bible requires much more detail than what we can cover in a week or two (it could take a lifetime), and has been written up many places. I found Dresselhaus *et al.* [11] Chapter *9. Space Groups in Real Space* quite clear on matrix representation of space groups (click here). (The MIT course 6.734 online version contains much of the same material.) I also found Béatrice Grenier's overview over crystallography helpful. Many online tools are available to ease the task, for example the FullProf suite of crystallographic programs. The Bible was completed in 19th century, but the field is undergoing a revival, as the study of topological insulators requires diving deeper into crystallography than simply looking up the tables.

The translation group T, the set of translations $\boldsymbol{t}$ that put the crystallographic structure in coincidence with itself, constitutes the *lattice*. T is a normal subgroup of G. It defines the *Bravais lattice*. Translations are of the form

$$\boldsymbol{t} = \boldsymbol{t}_{\boldsymbol{n}} = n_1\boldsymbol{a}_1 + n_2\boldsymbol{a}_2 + n_3\boldsymbol{a}_3 \,, \qquad n_j \in \mathbb{Z} \,.$$

The basis vectors $\boldsymbol{a}_j$ span the *unit cell*. There are 6 simple (or primitive) unit cells that contain a single point, specified by the lengths of the unit translations a, b, c and pairwise angles α, β, γ between them. The most symmetric among them is the *cubic* cell, with $a = b = c$ and $\alpha = \beta = \gamma = 90^o$.

The lattice unit cell (*primitive cell*) is always a *generating region* (a tile that tiles the entire space). The smallest generating region –*the fundamental domain*– is a minimal region that generates the whole pattern through its images under all symmetries. At each lattice point the identical group of "atoms" constitutes the *motif*. The lattice and the motif completely characterize the crystal.

The cosets by translation subgroup T (the set all translations) form the *factor* (AKA *quotient*) group G/T, isomorphic to the point group g (rotations). All irreducible representations of a space group G can be constructed from irreducible representations of g and T. This step, however, is tricky, as, due to the non-commutativity of translations and rotations, the quotient group G/T is not a normal subgroup of the space group G.

The quantum-mechanical calculations are executed by approximating the infinite crystal by a triply-periodic one, and going go to the *reciprocal* space by deploying C_{N_j} discrete Fourier transforms. This implements the G/T quotienting by translations and reduces the calculation to a finite *Brilluoin zone*. That is the content of the *'Bloch theorem'* of condensed matter physics. Further work is then required to reduce the calculations to the point group irreps.

Point symmetry operations leave at least one point fixed. They are (a) inversion through a point, (b) rotation around an axis, (c) roto-inversion around an axis and through a point and (d) reflection through a mirror plane. The rotations have to be compatible with the translation symmetry: in 3 spatial dimensions they can only be of orders 1, 2, 3, 4, or 6. They can be proper (det $= +1$) or improper (det $= -1$).

The spectroscopists' Schoenflies notation labels point groups as: cyclic C_n, dihedral $C_{n'}$, tetrahedral T and octahedral O rotation point groups, of order $n = 1, 2, 3, 4, 6$, respectively. The superscript $'$ refers to either v (parallel mirror plane) or h (perpendicular mirror plane). The crystallographer's preferred classification is, however, the international crystallographic (Hermann-Mauguin) notation.

8.3.1 Wallpaper groups

Pedagogically, it pays to start with a discussion of two-dimensional space groups. In 1924 George Pólya and Paul Niggli proved that there are exactly 17 different symmetry types of'wallpaper' pattern (says Twitter). The 17 *wallpaper groups*, classify the distinct systems of symmetries that can occur in a periodic tiling of the plane. In a wallpaper pattern, there are translational symmetries in two independent directions,

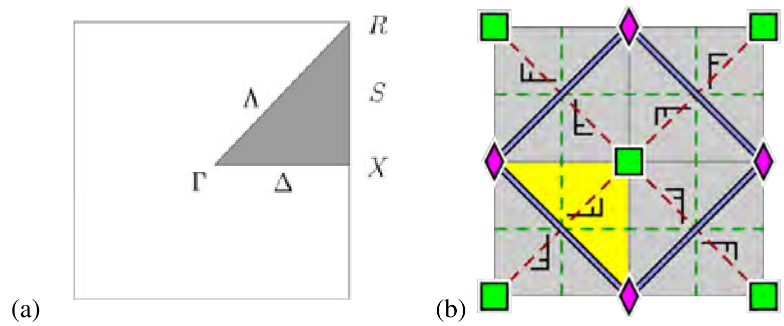


Figure 8.1: The shaded (or yellow) area indicates a *fundamental domain*, i.e., the smallest part of the pattern whose repeats tile the entire plane. (a) For the most symmetric 2D square lattice, with point group $p4mm$, the fundamental domain is indicated by the shaded triangle $\Gamma\Lambda RSX\Delta\Gamma$ which constitutes 1/8 of the Brillouin zone, and contains the basic wave vectors and the high symmetry points (Fig. 10.2 of Dresselhaus *et al.* [11]). (b) For the 2D square lattice with the glide and reflect point group $p4g$ the fundamental domain is indicated by the yellow triangle (Figure drawn by M. von Gagern).

forming a normal subgroup. Rotations can only have order 1,2,3,4 or 6 (crystallographic restriction).

For wallpaper groups the Hermann-Mauguin notation begins with either p or c, for a primitive cell or a face-centred cell. This is followed by a digit, n, indicating the highest order of rotational symmetry: 1-fold (none), 2-fold, 3-fold, 4-fold, or 6-fold. The first, resp. second of the next two symbols indicates the symmetry relative to one translation axis of the pattern, referred to as the main, resp. second one. The symbols are either m, g, or 1, for mirror, glide reflection, or none.

Section *9.3 Two-Dimensional Space Groups* of Dresselhaus *et al.* [11] discusses the most symmetric of the wallpaper groups, the tiling of a plane by squares, which in the international crystallographic notation is denoted by #11, with point group $p4mm$. We work out this space group in exercise 8.2. The largest invariant subgroup of C_{4v} is C_4. In that case, the space group is $p4$, or #10. Prefix p indicates that the unit cell is primitive (not centered). This is a ‘simple’, or *symmorphic* group, which makes calculations easier. There is, however, the third, non-symmorphic two-dimensional square space group $p4g$ or #12 ($p4gm$), see Table B.10 of ref. [11]. If someone can explain its ‘Biblical’ diagram to me, I would be grateful. The wiki explanation, reproduced here as figure 8.1 (b), is the best one that I have found so far, but I’m still scratching my head:) The Bravais lattice ‘unit cell’ is a square in all three cases. In the crystallographic literature the ChaosBook’s ‘fundamental domain’ makes an appearance only in the reciprocal lattice, as the Brilloun zone depicted for $p4mm$ in figure 8.1 (a). However, the ‘wallpaper groups’ wiki does call ‘fundamental domain’ the smallest part of the configuration pattern that, when repeated, tiles the entire plane.

The quantum-mechanical calculations are carried out in the reciprocal space, in our case with the full Γ point, $k = 0$, wave vector symmetry (see Table 10.1 of ref. [11]), and ‘Large Representations’.

Sect. *10.5 Characters for the Equivalence Representation* look like those for the point group, sort of.

8.3.2 One-dimensional line groups

One would think that the one-dimensional *line groups*, which describe systems exhibiting translational periodicity along a line, such as carbon nanotubes, would be simpler still. But even they are not trivial – there are 13 of them.

The normal subgroup of a line group L is its translational subgroup T, with its factor group L/T isomorphic to the *isogonal point group* P of discrete symmetries of its 1-dimensional unit cell $x \in (-a/2, a/2]$. In the reciprocal lattice k takes on the values in the first Brillouin zone interval $(-\pi/a, \pi/a]$e. In *Irreducible representations of the symmetry groups of polymer molecules. I*, Božović, Vujičić and Herbut [7] construct all the reps of the line groups whose isogonal point groups are $\mathrm{C}_n, \mathrm{C}_{nv}, \mathrm{C}_{nh}, S_{2n}$, and D_n. For some of these line groups the irreps are obtained as products of the reps of the translational subgroup and the irreps of the isogonal point group.

According to W. De Heer, the Mintmire, Dunlap and White [24] paper *Are Fullerene tubules metallic?* which took care of chiral rotations for nanotubes by a tight-binding calculation, played a key role in physicists’ understanding of line groups.

8.3.3 Time reversal symmetry

Consequences of time-reversal symmetry on line groups are discussed by Božović [6]; In the case when the Hamiltonian is invariant under time reversal [14], the symmetry group is enlarged: $L + \theta L$. It is interesting to learn if the degeneracy of the levels is doubled or not.

Johnston [19] *Group theory in solid state physics* is one of the many reviews that discusses Wigner’s time-reversal theorems for a many-electron system, including the character tests for time-reversal degeneracy, the double space groups, and the time-reversal theorems (first discussed by Herring [14] in *Effect of time-reversal symmetry on energy bands of crystals*).

For a very different take on reflection symmetries on spatiotemporal lattices, of which time reversal on a temporal lattice is the simplest example, see Liang and Cvitanović [23] *A chaotic lattice field theory in one dimension* (2022).

8.4 Elastodynamic equilibria of 2D solids (optional)

Artificial lattices are often introduced to formulate classical field theories (described by partial differential equations) and quantum field theories (described by path integrals) as finite-dimensional problems, either for theoretical reasons (QM in a periodic box), or in order to port them to computers. For example, lattice QCD approximates Quantum Chromodynamics by a 4-dimensional cubic crystal. What follows is a simple example of such formulation of a classical field theory, taken from Mehran Kardar’s MIT course, Lect. 23.

Consider a perfect two-dimensional solid at $T = 0$. The equilibrium configuration of atoms forms a lattice,

$$\boldsymbol{r}_0(m,n) = m\boldsymbol{e}_1 + n\boldsymbol{e}_2\,,$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are basis vectors, $a = |\boldsymbol{e}_j|$ is the lattice spacing, and $\{m,n\}$ are integers. At finite temperatures, the atoms fluctuate away from their equilibrium position, moving to

$$\boldsymbol{r}(m,n) = \boldsymbol{r}_0(m,n) + \boldsymbol{u}(m,n)\,,$$

As the low temperature distortions do not vary substantially over nearby atoms, one can define a coarse-grained distortion field $\boldsymbol{u}(\boldsymbol{x})$, where $\boldsymbol{x} = (x_1, x_2)$ is treated as continuous, with an implicit short distance cutoff of the lattice spacing a. Due to translational symmetry, the elastic energy depends only on the strain matrix,

$$u_{ij}(\boldsymbol{x}) = \tfrac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)\,.$$

Kardar picks the triangular lattice, as its elastic energy is isotropic (i.e., invariant under lattice rotations, see Landau and Lifshitz [22]). In terms of the Lamé coefficients λ and μ,

$$\begin{aligned}\beta H &= \frac{1}{2}\int d^2\boldsymbol{x}\,(2\mu\, u_{ij}u_{ij} + \lambda\, u_{ii}u_{jj}) \\ &= -\frac{1}{2}\int d^2\boldsymbol{x}\, u_i[2\mu\,\Box\,\delta_{ij} + (\mu+\lambda)\,\partial_i\partial_j]\,u_j\,.\end{aligned} \tag{8.2}$$

(here we have assumed either infinite or doubly periodic lattice, so no boundary terms from integration by parts), with the equations of motion something like (FIX!)

$$\partial_t^2 u_i = [2\mu\,\Box\,\delta_{ij} + (\mu+\lambda)\,\partial_i\partial_j]\,u_j\,. \tag{8.3}$$

(Note that Kardar keeps time continuous, but discretizes space. In numerical computations time is discretized as well.) The symmetry of a square lattice permits an additional term proportional to $\partial_x^2 u_x^2 + \partial_y^2 u_y^2$. In general, the number of independent elastic constants depends on the dimensionality and rotational symmetry of the lattice in question. In two dimensions, square lattices have three independent elastic constants, and triangular lattices are "elastically isotropic" (i.e., elastic properties are independent of direction and thus have only two [22]).

The Goldstone modes associated with the broken (PC: why "broken"?) translational symmetry are *phonons*, the normal modes of vibrations. Eq. (8.3) supports two types of lattice normal modes, transverse and longitudinal.

The order parameter describing broken translational symmetry is

$$\rho_{\boldsymbol{G}}(\boldsymbol{x}) = e^{i\boldsymbol{G}\cdot\boldsymbol{r}(\boldsymbol{x})} = e^{i\boldsymbol{G}\cdot\boldsymbol{u}(\boldsymbol{x})}\,,$$

where $\boldsymbol{G}$ is any reciprocal lattice vector. Since, by definition, $\boldsymbol{G}\cdot\boldsymbol{r_0}$ is an integer multiple of 2π, $\rho_{\boldsymbol{G}} = 1$ at zero temperature. Due to the fluctuations,

$$\langle\rho_{\boldsymbol{G}}(\boldsymbol{x})\rangle = \langle e^{i\boldsymbol{G}\cdot\boldsymbol{u}(\boldsymbol{x})}\rangle$$

decreases at finite temperatures, and its correlations decay as $\langle \rho_{\boldsymbol{G}}(\boldsymbol{x})\rho^*_{\boldsymbol{G}}(\boldsymbol{0})\rangle$. This is the order parameter ChaosBook and Gaspard use in deriving formulas for deterministic diffusion. Kardar computes this in Fourier space by approximating $\boldsymbol{G}\cdot\boldsymbol{q}$ with its angular average $G^2q^2/2$, ignoring the rotationally symmetry-breaking term $\cos\boldsymbol{q}\cdot\boldsymbol{x}$, and getting only the asymptotics of the correlations right (the decay is algebraic).

The translational correlations are measured in diffraction experiments. The scattering amplitude is the Fourier transform of $\rho_{\boldsymbol{G}}$, and the scattered intensity at a wavevector q is proportional to the structure factor. At zero temperature, the structure factor is a set of delta-functions (Bragg peaks) at the reciprocal lattice vectors.

The orientational order parameter that characterizes the broken rotational symmetry of the crystal can be defined as

$$\Psi(\boldsymbol{x}) = e^{6i\theta(\boldsymbol{x})}\,,$$

where $\theta(\boldsymbol{x})$ is the angle between local lattice bonds and a reference axis. The factor of 6 accounts for the equivalence of the 6 possible D_3 orientations of the triangular lattice. (Kardar says the appropriate choice for a square lattice is $\exp(4i\theta(\boldsymbol{x}))$ - shouldn't the factor be 8, the order of C_{4v}?) The order parameter has unit magnitude at $T=0$, and is expected to decrease due to fluctuations at finite temperature. The distortion $u(\boldsymbol{x})$ leads to a change in bond angle given by

$$\theta(\boldsymbol{x}) = -\tfrac{1}{2}\left(\partial_x u_y - \partial_y u_x\right)\,.$$

(This seems to be dimensionally wrong? For detailed calculations, see the above Kardar lecture notes.)

8.5 Literature, reflections (optional)

Predrag The story of quantum scattering off crystals, I believe, starts with the Bouckaert, Smoluchowski and Wigner (1936) paper [5].

To understand the order of the full group O_h of symmetries of the cube, exercise 5.1 a.ii, it is instructive to look at figure 8.2 (figs. 8.8 and 8.12 in Joshi [20]). When a cube is a building block that tiles a $3D$ cubic lattice, it is referred to as the 'elementary' or 'Wigner-Seitz' cell, and its Fourier transform is called 'the first Brillouin zone' in 'the reciprocal space'. The special points and the lines of symmetry in the Brillouin zone are shown in figure 8.2 (a). The tetrahedron ΓXMR, an $1/48$th part of the Brillouin zone, is the fundamental domain, as the action of the 48 elements of the point group O_h on it tiles the Brillouin zone without any gaps or overlaps.

Predrag OK, I'll confess. The reason why it is lovely to teach graduate level physics is that one is allowed to learn new things while doing it. I'll now sketch one, perhaps wild, direction that you are completely free to ignore.

Here is the problem of space groups in the nutshell. The Euclidean invariance on Newtonian space-time (including its subgroups, such as the discrete space groups), and the Poincaré invariance of special-relativistic space-time is a

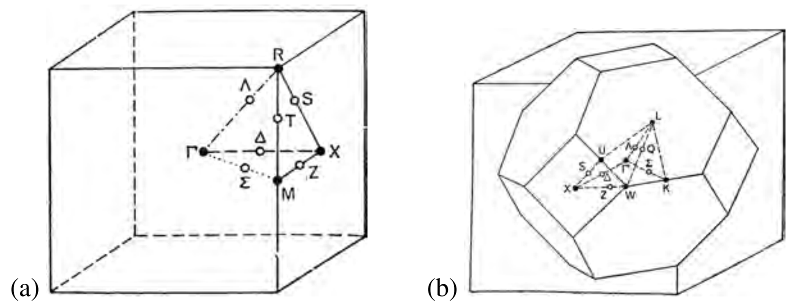


Figure 8.2: (a) The special points and the lines of symmetry in the first Brillouin zone of a simple cubic lattice define its fundamental domain, the tetrahedron ΓXMR. (b) Just not to get any ideas that this is easy: the fundamental domain for the first Brillouin zone of a bcc lattice. (From Joshi [20].)

strange brew: the space is non-compact (homogeneity), while rotations are compact (isotropy). That leads to the conceptually awkward situation of mixing a group of additions (translations) with a group of multiplications (rotations). To work with such group we *first* translate objects to the *origin* and *then* rotate them with the respect to the origin. That's not nice, because by translation invariance any point is as good as any other, there is no preferred origin. There is no reason why one should translate first, rotate second. What one needs is a formalism that implements translations and rotations on the same footing.

If I understand Hestenes [16] right (also David Finkelstein and perhaps Holger Beck Nielsen have told me things in this spirit) a way to accomplish that is to replace the flat translational directions by a compact manifold where translations and rotations are non-commuting multiplicative group operations.

A part of the Hestenes program is redoing crystallography. I have read Hestenes [17] paper (but not the Hestenes and Holt [18] follow up). It looks very interesting, but I will spare you from my comments here, as I do not know how to make this formalism work for our purposes (character; explicit computations), so I should not waste your time on that. If you do have a look at his, or at Coxeter [10] discussion of planar tilings, please do report back to me.

Predrag Graphene is a two-dimensional sheet of carbon in which the carbon atoms are arranged in a honeycomb lattice: each carbon atom is connected to three neighbors. It was exfoliated by Schafhaeutl [4, 27] in 1840 (more recently, a con man got a Nobel Prize for that), and formally defined for chemists by Boehm [3] in 1986. In 1947 Wallace [31] calculated the electronic structure of graphene, as a preliminary exercise to calculating electronic structure of graphite, and noted that the velocity of the electrons was independent of their energies: they all travel at the same speed (about 100 km per second, about 1/3000 of the speed of light): plot of the energy of the electrons in graphene as a function of its momentum (which is inversely proportional to its wavelength) is V shaped since the energy of the electron is linearly proportional to its momentum (Wallace [31] Eq. 3.1).

The energy of a free electron is proportional to the square of its momentum, but not so in a crystal. As this is reminiscent of massless elementary particles like photons and neutrino's, it has been renamed since 'Dirac cones', but Dirac has nothing whatsoever to do with that. To learn more, talk to people from the Claire Berger and Walt De Heer's group [2] - I have extracted above history of graphene from De Heer's notes (the "con man" is my own angle on what went down with this particular Nobel prize).

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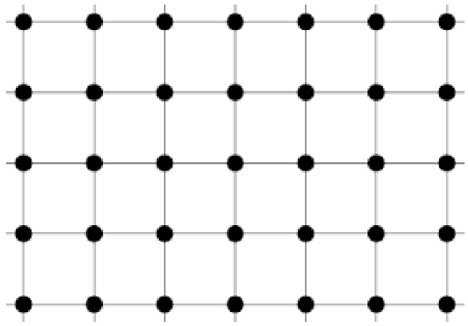


Figure 8.3: Square lattice of atoms

Exercises

8.1. **Space group.**

(a) Show that for any space group, the translations by vectors from Bravais lattice form a normal subgroup.

(b) Can rotations of the lattice at a fixed point constitute a normal subgroup of a space group?

(B. Gutkin)

8.2. **Band structure of a square lattice.** A charged particle (without spin) moves in a potential created by an infinite square lattice of atoms, see figure 8.3.

(a) What are the symmetry groups of the Bravais and reciprocal lattices?

(b) Plot the 1st Brillouin zone. What is its symmetry? What is the corresponding fundamental domain?

Let $\boldsymbol{k}$ be quasi-momentum and $E_n(\boldsymbol{k})$ the energy of the nth band.

(c) At which points of the Brillouin zone is the group $G^{(\boldsymbol{k})}$ (the group which leaves vector $\boldsymbol{k}$ invariant) nontrivial? What is it?

(d) What is the symmetry of $E_n(\boldsymbol{k})$ as a function of $\boldsymbol{k}$? At which points of the Brillouin zone is the group velocity $\nabla E_n(\boldsymbol{k})$ equal 0?

(e) At which points of the Brillouin zone neighboring bands (generically) stick to each other? How many bands can stick? Explain from the group theory prospective.

(f) Assume now that the lattice is slightly squeezed along one of the axis. What will be the new symmetry of the system and its 1st Brillouin zone? Will the sticking between bands be lifted or persiss?

(B. Gutkin)

8.3. **Tight binding model.** Verify your solution of exercise 8.2 within the 2-state tight binding model. Assume that particle can hop either from corner to corner of the square lattice with coefficient t_1 or from corner to the middle of the square with coefficient t_2 (and vice versa).

(a) Show the obtained energy bands $E_i(\boldsymbol{k})$ as both contour- and 3-dimensional plots.

(b) Compare with the results from exercise 8.2.

group theory - week 9

Continuous groups

Homework HW9

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 9.1 *Irreps of* $SO(2)$	2 points
Exercise 9.2 *Reduction of product of two* $SO(2)$ *irreps*	1 point
Exercise 9.3 *Irreps of* $O(2)$	2 points
Exercise 9.4 *Reduction of product of two* $O(2)$ *irreps*	1 point

Bonus points

Exercise 9.5 *A fluttering flame front*	4 points
Exercise 9.6 $O(2)$ *fundamental domain for a PDE*	(difficult) 10 points

Total of 6 points = 100 % score.

2022-11-15 Predrag Lecture 17 Continuous groups

2022-11-15 Predrag Lecture 18 Lie groups

The fastest way to watch any week's lecture videos is by letting YouTube run the

lecture playlist

These lectures are about the basic ideas of how one goes from finite groups to the continuous ones. We have worked one example out in week 2, the discrete Fourier transform of example 2.6 *Projection operators for cyclic group C_N*. The cyclic group C_N is generated by the powers of the rotation by $2\pi/N$, and in the $N \to \infty$ limit one only needs to understand the algebra of T_ℓ, generators of infinitesimal transformations, $D(\theta) = 1 + i \sum_\ell \theta_\ell T_\ell$. Applied to functions, they turn out to be partial derivatives.

Continuous symmetries - an introduction (2 min)

They still do not get it! (6 min)

- Lie groups, sect. 9.3: Definition of a Lie group; Cyclic group $C_N \to$ continuous $SO(2)$ plane rotations; Infinitesimal transformations; $SO(2)$ generator of rotations.

What is a symmetry? (8 min)

Group element; transformation generator (8 min)

What is a symmetry group? (7 min)

What is a group orbit? (3 min)

What is dynamics? (2 min)

Group SO(2) (3 min)

Unitary groups are mothers of all finite / compact symmetries.
(1 h 4 min)

- The $N \to \infty$ limit of C_N gets you to the continuous Fourier transform as a representation of $SO(2)$, but from then on this way of thinking about continuous symmetries gets to be increasingly awkward. A fresh restart is afforded by matrix groups, and in particular the unitary group $U(n) = U(1) \otimes SU(n)$, which contains all other compact groups, finite or continuous, as subgroups.

Special orthogonal group SO(n) (9 min)

Symplectic group Sp(n) (9 min)

9.1 Other sources (optional)

Do not get intimidated by this week's lectures notes.

- What's the payback? While for you the geometrically intuitive representation is the set of rotation [2×2] matrices, group theory says no! They split into pairs of 1-dimensional irreps, and the basic building blocks of *our* 2-dimensional rotations on our kitchen table (forget quantum mechanics!) are the $U(1)$ $[1 \times 1]$ complex unit vector phase rotations.

  - Reading: C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, Sects. 6.1-6.3 Irreps of $SO(2)$. In particular, note that while geometrically intuitive representation is the set of rotation $[2 \times 2]$ matrices, they split into pairs of 1-dimensional irreps.
  - Reading: C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, Sect. 6.6 completes discussion of Fourier analysis as continuum limit of cyclic groups C_n, compares $SO(2)$, $O(2)$, discrete translations group, and continuous translations group.
  - Chen, Ping and Wang [2] *Group Representation Theory for Physicists*, Sect 5.2 *Definition of a Lie group, with examples* (click here).
  - AWH Chapter 17 *Group Theory, Sect. 17.7 Continuous groups* (click here).

- Sect. 9.4 *Character orthogonality theorem*

Infinitesimal symmetries: Lie derivative (8 min)

Tell no Lie to plumbers (39 sec)

It's a matter of no small pride for a card-carrying dirt physics theorist to claim full and total ignorance of group theory (XX min)

9.2 Discussion (optional)

Calligraphic M denotes the state space manifold as well as any subspace, such as a group orbit (3:38 min)

Why are continuous transformation group elements represented by exponentials? (5:39 min)

How did we get the Lie algebra? Why is (almost) every symmetry we care about a subgroup of an unitary group? (9 min)

How did we get the SO(2) generator? (2 min)

Orthogonal and unitary transformations (7 min)

Fourier modes are so simple, that no one calls them irreps. But add more symmetries, and there have to be fewer irreps. (11 min)

Why did we move from orthogonal group O(n) to special orthogonal group SO(n)? (3:32 min)

Why SU(n) rather than U(n)? (6:30 min)

Why is SU(n) dimension $n^2 - 1$? (56 sec)

What are these "characters"? And why is there a Journal of Linear Algebra*, today? Inconclusive blah blah.* (12 min)

Rant 1 - Is beauty symmetry? The first piece of art found in China is a perfect disk carved out of jade. All of Bach is symmetries. (9 min)

Rant 2 - students find letter A beautifully symmetric, but Predrag finds zero 'O' the most beautiful grade. (1 min)

Rant 3 - SO(3) & SU(2) preview and a long rant - listen to it at your own risk. Roger Penrose thoughts on quantum spacetime and quantum brain. Are laws of physics time invariant? Waiting for dark energy to go away. Arrow of time. (17 min)

Rant 4 - SO(3) & SU(2) preview and a long rant - listen to it at your own risk. Get this: math uses 2d complex vectors (spinors) to build our real 3d space. And all we see - starlight, graphene, greenhouse effect, helioseismography, gravitational wave detectors - it is all irreps! (12 min)

Rant 5 - Help me, I'm bullied by a mathematician. (3 min)

Rant 6 - you can always count on Prof. Z. (1/2 min)

Question 9.1. Henriette Roux, pondering exercise 4.2, writes

Q I want to make sure I understand the concept of irreducible representations.

1. If a representation (which can be thought of as a sort of basis) is reducible, all group element matrices can be simultaneously diagonalized. I want to be able to see how this definition of reducibility matches with the notion of block diagonalizability of an overall representation D(g).
2. AWH p. 822-823 has a discussion of this, but I'm wondering if there's an intuitive way to connect these two definitions or if it's just linear algebra.
3. We have familiarized ourselves with the concept of (conjugacy) classes. Here, we now add in the concept of character, which is just the trace of any matrix in a given class (and every matrix of the same class will have the same trace b/c of the properties of classes/traces).
4. So to find the characters for a given representation, we just need to find the classes and then take the trace of a matrix representation in each class?
5. My next and related question then concerns what character means conceptually. Does it relate classes to other classes within a given representation, or different representations (whether reducible or not), or both? AWH says that "the set of characters for all elements and irreducible representations of a finite group defines an orthogonal finite-dimensional vector space."

6. How does a vector space come about from a set of traces, each of which I normally think of as just a number, like the determinant? And finally,
7. How can we use our knowledge of classes/character to find irreducible representations, since that seems to be an important goal in examining a group.
8. Exercise 4.2 (c) says to find the characters for this representation, which seems to imply that character depends on representation. But I would've thought that character, which is a trace of a matrix, is invariant under any similarity transform, which is how you get from a reducible representation to an irreducible representation.
9. Do the multiplicities of irreducible representations correspond to the multiplicity of characters (i.e. the number of elements in each class)? If so, why? (Or if not, why not?)
10. The same thing for classes, correct? Classes shouldn't depend on representation b/c they can be thought of as corresponding to a physical operation (e.g. transposition or cyclic permutation), something which is independent of basis.

.

A Great framing for a discussion, thanks! I'll probably reedit this post several times, everybody's input is very welcome. Items numbered as in above:

(2) My favorite step-by-step, pedagogical exposition are the chapters 2 *Representation Theory and Basic Theorems* and 3 *Character of a Representation* of Dresselhaus *et al.* [6]. There is too much material for our course, but if you want to understand it once for all times, it's worth your time.

(3) Correct.

(4) Correct. Note, however, that while every matrix representation has a trace, and thus a character, you want to decompose this character into the sum of irrep characters, as it is obvious after the block diagonalization has been attained.

(5) The unitary diagonalization matrix, whose entries are characters, takes character-weighted sums of classes in order to project them onto irreps, just like what the Fourier representation does. The result (as we know from projection operators analysis), are mutually orthogonal sub-spaces.

(6) Whenever you do not understand something about finite groups, ask yourself - how does it work for finite lattice Fourier representation?

There the vector space comes via a unitary transformation from the configuration coordinates (where each group element is represented by a full matrix) to the diagonalized, irreducible subspaces coordinates (Fourier modes).

The unitary $\mathcal{F}$ matrix is full of ω^{ij}, ie, characters of the cyclic group C_n. That's where the characters come from.

Now mess up C_3 by adding a reflection. Dihedral group D_3, the group of rotations and reflections, has more symmetry constrains, it cannot have 6 irreps, as reflection invariance mixes together the two senses of rotation. Now there are 3 classes, ie, kinds of things the group does: nothing, flip, rotate. The unitary transformation that diagonalizes group element matrices is now morally a smaller unitary $[3\times 3]$ matrix from 'classes' in configuration space to 'irreps' in the diagonalized representation, where some sub-spaces must have dimension higher than one.

The surprise, for me, is that the entries in the unitary diagonalization matrix can still be written as traces of irreps, ie, characters. For me it is a calculation, a beautiful example of mathematics leading us somewhere where our intuition falls short. If you find a good intuitive explanation somewhere, please let us all know.

(7) That's automatic, now. Each irrep has the projection operator associated with it; we construct it as a sub-product of factors in Hamilton-Cayley formula. Now we know we can write it -just as we did with the Fourier representation- as sum over all class group actions, each weighted by a the irrep's character.

(8) Characters are elements of the unitary matrix with one index running over classes, the other over irreps. So you expect character to differ from representation to representation; very clear from D_3 character table. As always, you already know that from the Fourier representation example.

(9) They do not. Dresselhaus *et al.* [6] has the answer - enter it here once you understand it.

(10) Correct.

Question 9.2. Henriette Roux, digesting sect. 10.7.1, asks

Q Please explain when one keeps track of the order of tensorial indices?

A In a tensor, upper, lower indices are separately ordered - and that order matters. The simplest example: if some indices form an antisymmetric pair, writing them in wrong order gives you a wrong sign. In a matrix representation of a group action, one has to distinguish between the "in" set of indices – the ones that get contracted with the initial tensor, and the "out" set of indices that label the tensor after the transformation. If you understand Eq. (3.22) in birdtracks.eu, you get it. Does that answer your question?

Question 9.3. Henriette Roux asks

Q Please explain the $M_{\mu\nu,\delta\rho}$ generators of $\mathrm{SO}(n)$.

A Let me know if you understand the derivation of Eqs. (4.51) and (4.52) in birdtracks.eu. Does that answer your question?

9.3 Continuous symmetries: unitary and orthogonal

This week's lectures are not taken from any particular book, they are about basic ideas of how one goes from finite groups to the continuous ones that any physicist should know. We have worked one example out earlier, in week 9 and ChaosBook Sect. A24.4. It gets you to the continuous Fourier transform as a representation of $\mathrm{U}(1) \simeq \mathrm{SO}(2)$, but from then on this way of thinking about continuous symmetries gets to be increasingly awkward. So we need a fresh restart; that is afforded by matrix groups, and in particular the unitary group $\mathrm{U}(n) = \mathrm{U}(1) \otimes \mathrm{SU}(n)$, which contains all other compact groups, finite or continuous, as subgroups.

The main idea in a way comes from discrete groups: the cyclic group C_N is generated by the powers of the smallest rotation by $\Delta\theta = 2\pi/N$, and in the $N \to \infty$ limit one only needs to understand the commutation relations among T_ℓ, generators of infinitesimal transformations,

$$D(\Delta\theta) = 1 + i \sum_\ell \Delta\theta_\ell T_\ell + O(\Delta\theta^2)\,. \tag{9.1}$$

These thoughts are spread over chapters of my book *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [5] that you can steal from my website, but the book itself is too sophisticated for this course.

9.4 Character orthogonality theorem

You might like my intuitive derivation [5] of the character orthogonality theorem for continuous compact lie groups, birdtracks.eu sect. 8.2 *Characters*.

Note that the replacement of an irrep matrix representation $D^{(\mu)}(g)_a{}^b$ by its character $\chi^{(\mu)}(g)$ (a single scalar quantity) does not mean that any of the matrix indices structure is lost; the full $D^{(\mu)}(g)_a{}^b$ can be recovered by differentiation, as in birdtracks.eu eq. (8.27).

9.5 Reps of compact groups are fully reducible

(copied from *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [5])

The objective of physicists' group-theoretic calculations is a description of the spectroscopy of a given theory. This entails identifying the levels (irreducible multiplets), the degeneracy of a given level (dimension of the multiplet) and the level splittings (eigenvalues of various casimirs). The basic idea that enables us to carry this program through is extremely simple: a hermitian matrix can be diagonalized. This fact has many names: Schur's lemma, Wigner-Eckart theorem, full reducibility of unitary reps, and so on (see sect. 9.5.1). We exploit it by constructing invariant hermitian matrices M from the primitive invariant tensors. The M's have collective indices and act on tensors. Being hermitian, they can be diagonalized

$$CMC^\dagger = \begin{pmatrix} \lambda_1 & 0 & 0 & & \cdots \\ 0 & \lambda_1 & 0 & & \\ 0 & 0 & \lambda_1 & & \\ & & & \lambda_2 & \\ \vdots & & & & \ddots \end{pmatrix},$$

and their eigenvalues can be used to construct projection operators that reduce multiparticle states into direct sums of lower-dimensional reps (see sect. 9.5.1):

$$\mathbf{P}_i = \prod_{j\neq i} \frac{M - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} = C^\dagger \left(\begin{array}{ccc} \begin{array}{cc|} \ddots & \vdots \\ \cdots & 0 \\ \hline \end{array} & \cdots & 0 \\ \vdots & \boxed{\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 1 \end{array}} & \vdots \\ 0 & \cdots & \begin{array}{|cc} \hline 0 & \cdots \\ \vdots & \ddots \end{array} \end{array} \right) C\,. \qquad (9.2)$$

An explicit expression for the diagonalizing matrix C (Clebsch-Gordan coefficients or *clebsches,* sect. 9.6) is unnecessary — it is in fact often more of an impediment than an aid, as it obscures the combinatorial nature of group-theoretic computations (see sect. 9.7).

All that is needed in practice is knowledge of the characteristic equation for the invariant matrix M (see sect. 9.5.1). The characteristic equation is usually a simple consequence of the algebraic relations satisfied by the primitive invariants, and the eigenvalues λ_i are easily determined. The λ_i' s determine the projection operators $\mathbf{P}_i$, which in turn contain all relevant spectroscopic information: the rep dimension is given by $\mathrm{tr}\,\mathbf{P}_i$, and the casimirs, 6-$j$'s, crossing matrices, and recoupling coefficients) are traces of various combinations of $\mathbf{P}_i$'s. All these numbers are *combinatoric*; they can often be interpreted as the number of different colorings of a graph, the number of singlets, and so on.

The key results are the construction of projection operators from invariant matrices, the Clebsch-Gordan coefficients rep of projection operators (9.26), the invariance conditions (10.41) and the Lie algebra relations (10.45).

The basic idea is simple: a hermitian matrix can be diagonalized. If this matrix is an invariant matrix, it decomposes the reps of the group into direct sums of lower-dimensional reps. Most of computations to follow implement the spectral decomposition

$$\mathbf{M} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \cdots + \lambda_r \mathbf{P}_r \,,$$

which associates with each distinct root λ_i of invariant matrix $\mathbf{M}$ a projection operator (9.5):

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} \,.$$

The exposition given here in sect. 9.5.1 is taken from refs. [3, 4]. Who wrote this down first I do not know, but I like Harter's exposition [9–11] best.

9.5.1 Projection operators

For $\mathbf{M}$, a hermitian matrix, there exists a diagonalizing unitary matrix C such that

$$C\mathbf{M}C^{\dagger} = \left(\begin{array}{ccc} \begin{array}{ccc} \lambda_1 & \ldots & 0 \\ & \ddots & \\ 0 & \ldots & \lambda_1 \end{array} & 0 & 0 \\ 0 & \begin{array}{cccc} \lambda_2 & 0 & \ldots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots & \lambda_2 \end{array} & 0 \\ 0 & 0 & \begin{array}{cc} \lambda_3 & \ldots \\ \vdots & \ddots \end{array} \end{array} \right) \,. \tag{9.3}$$

Here $\lambda_i \neq \lambda_j$ are the r distinct roots of the minimal characteristic polynomial

$$\prod_{i=1}^{r} (\mathbf{M} - \lambda_i \mathbf{1}) = 0 \tag{9.4}$$

(the characteristic equations are discussed in sect. 2.10).

In the matrix $C(\mathbf{M}-\lambda_2\mathbf{1})C^\dagger$ the eigenvalues corresponding to λ_2 are replaced by zeroes:

$$
\left(\begin{array}{ccc|c|ccc}
\lambda_1-\lambda_2 & & & & & & \\
 & \lambda_1-\lambda_2 & & & & & \\
 & & \lambda_1-\lambda_2 & & & & \\
\hline
 & & & \begin{array}{ccc} 0 & & \\ & \ddots & \\ & & 0 \end{array} & & & \\
\hline
 & & & & \lambda_3-\lambda_2 & & \\
 & & & & & \lambda_3-\lambda_2 & \\
 & & & & & & \ddots
\end{array}\right),
$$

and so on, so the product over all factors $(\mathbf{M}-\lambda_2\mathbf{1})(\mathbf{M}-\lambda_3\mathbf{1})\ldots$, with exception of the $(\mathbf{M}-\lambda_1\mathbf{1})$ factor, has nonzero entries only in the subspace associated with λ_1:

$$
C\prod_{j\neq 1}(\mathbf{M}-\lambda_j\mathbf{1})C^\dagger = \prod_{j\neq 1}(\lambda_1-\lambda_j)
\left(\begin{array}{ccc|cccc}
1 & 0 & 0 & & & & \\
0 & 1 & 0 & & 0 & & \\
0 & 0 & 1 & & & & \\
\hline
 & & & 0 & & & \\
 & 0 & & & 0 & & \\
 & & & & & 0 & \\
 & & & & & & \ddots
\end{array}\right).
$$

In this way, we can associate with each distinct root λ_i a *projection operator* $\mathbf{P}_i$,

$$
\mathbf{P}_i = \prod_{j\neq i}\frac{\mathbf{M}-\lambda_j\mathbf{1}}{\lambda_i-\lambda_j}\,, \tag{9.5}
$$

which acts as identity on the ith subspace, and zero elsewhere. For example, the projection operator onto the λ_1 subspace is

$$
\mathbf{P}_1 = C^\dagger
\left(\begin{array}{ccc|cccc}
1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & & & & \\
\hline
 & & & 0 & & & \\
 & & & & 0 & & \\
 & & & & & \ddots & \\
 & & & & & & 0
\end{array}\right) C\,. \tag{9.6}
$$

The matrices $\mathbf{P}_i$ are *orthogonal* ◇

$$
\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_j\,, \qquad (\text{no sum on } j)\,, \tag{9.7}
$$

and satisfy the *completeness relation*

$$\sum_{i=1}^{r} \mathbf{P}_i = \mathbf{1}\,. \tag{9.8}$$

As tr $(C\mathbf{P}_i C^\dagger) = \mathrm{tr}\,\mathbf{P}_i$, the dimension of the ith subspace is given by

$$d_i = \mathrm{tr}\,\mathbf{P}_i\,. \tag{9.9}$$

It follows from the characteristic equation (9.4) and the form of the projection operator (9.5) that λ_i is the eigenvalue of $\mathbf{M}$ on $\mathbf{P}_i$ subspace:

$$\mathbf{M}\mathbf{P}_i = \lambda_i \mathbf{P}_i\,, \qquad (\text{no sum on } i)\,. \tag{9.10}$$

Hence, any matrix polynomial $f(\mathbf{M})$ takes the scalar value $f(\lambda_i)$ on the $\mathbf{P}_i$ subspace

$$f(\mathbf{M})\mathbf{P}_i = f(\lambda_i)\mathbf{P}_i\,. \tag{9.11}$$

This, of course, is the reason why one wants to work with irreducible reps: they reduce matrices and "operators" to pure numbers.

9.5.2 Spectral decomposition

Suppose there exist several linearly independent invariant $[d \times d]$ hermitian matrices $\mathbf{M}_1, \mathbf{M}_2, \ldots$, and that we have used $\mathbf{M}_1$ to decompose the d-dimensional vector space $\tilde{V} = \Sigma \oplus V_i$. Can $\mathbf{M}_2, \mathbf{M}_3, \ldots$ be used to further decompose V_i? This is a standard problem of quantum mechanics (simultaneous observables), and the answer is that further decomposition is possible if, and only if, the invariant matrices commute:

$$[\mathbf{M}_1, \mathbf{M}_2] = 0\,, \tag{9.12}$$

or, equivalently, if projection operators $\mathbf{P}_j$ constructed from $\mathbf{M}_2$ commute with projection operators $\mathbf{P}_i$ constructed from $\mathbf{M}_1$,

$$\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_j\mathbf{P}_i\,. \tag{9.13}$$

Usually the simplest choices of independent invariant matrices do not commute. In that case, the projection operators $\mathbf{P}_i$ constructed from $\mathbf{M}_1$ can be used to project commuting pieces of $\mathbf{M}_2$:

$$\mathbf{M}_2^{(i)} = \mathbf{P}_i\mathbf{M}_2\mathbf{P}_i\,, \qquad (\text{no sum on } i)\,.$$

That $\mathbf{M}_2^{(i)}$ commutes with $\mathbf{M}_1$ follows from the orthogonality of $\mathbf{P}_i$:

$$[\mathbf{M}_2^{(i)}, \mathbf{M}_1] = \sum_j \lambda_j [\mathbf{M}_2^{(i)}, \mathbf{P}_j] = 0\,. \tag{9.14}$$

Now the characteristic equation for $\mathbf{M}_2^{(i)}$ (if nontrivial) can be used to decompose V_i subspace.

An invariant matrix $\mathbf{M}$ induces a decomposition only if its diagonalized form (9.3) has more than one distinct eigenvalue; otherwise it is proportional to the unit matrix and commutes trivially with all group elements. A rep is said to be *irreducible* if all invariant matrices that can be constructed are proportional to the unit matrix.

An invariant matrix $\mathbf{M}$ commutes with group transformations $[G, \mathbf{M}] = 0$, see (10.30). Projection operators (9.5) constructed from $\mathbf{M}$ are polynomials in $\mathbf{M}$, so they also commute with all $g \in \mathcal{G}$:

$$[G, \mathbf{P}_i] = 0 \tag{9.15}$$

(remember that $\mathbf{P}_i$ are also invariant $[d \times d]$ matrices). Hence, a $[d \times d]$ matrix rep can be written as a direct sum of $[d_i \times d_i]$ matrix reps:

$$G = \mathbf{1}G\mathbf{1} = \sum_{i,j} \mathbf{P}_i G \mathbf{P}_j = \sum_i \mathbf{P}_i G \mathbf{P}_i = \sum_i G_i \,. \tag{9.16}$$

In the diagonalized rep (9.6), the matrix G has a block diagonal form:

$$CGC^\dagger = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix} , \qquad G = \sum_i C^i G_i C_i \,. \tag{9.17}$$

The rep G_i acts only on the d_i-dimensional subspace V_i consisting of vectors $\mathbf{P}_i q$, $q \in \tilde{V}$. In this way an invariant $[d \times d]$ hermitian matrix $\mathbf{M}$ with r distinct eigenvalues induces a decomposition of a d-dimensional vector space $\tilde{V}$ into a direct sum of d_i-dimensional vector subspaces V_i:

$$\tilde{V} \overset{\mathbf{M}}{\to} V_1 \oplus V_2 \oplus \ldots \oplus V_r \,. \tag{9.18}$$

The theory of class algebras [9–11] offers a more elegant and systematic way of constructing the maximal set of commuting invariant matrices $\mathbf{M}_i$ than the sketch offered in this section.

9.6 Clebsch-Gordan coefficients

(copied from *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [5])

Consider the product

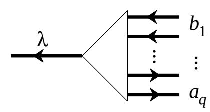
$$\left(\begin{array}{cc|ccc|cccc} 0 & & & & & & & & \\ & 0 & & & & & & & \\ \hline & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ \hline & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & \ddots \end{array} \right) C \tag{9.19}$$

of the two terms in the diagonal representation of a projection operator (9.6). This matrix has nonzero entries only in the d_λ rows of subspace V_λ. We collect them in a $[d_\lambda \times d]$ rectangular matrix $(C_\lambda)^\alpha_\sigma$, $\alpha = 1, 2, \ldots d$, $\sigma = 1, 2, \ldots d_\lambda$:

$$C_\lambda = \underbrace{\begin{pmatrix} (C_\lambda)^1_1 & \ldots & (C_\lambda)^d_1 \\ \vdots & & \vdots \\ & & (C_\lambda)^d_{d_\lambda} \end{pmatrix}}_{d} \Bigg\} d_\lambda \,. \tag{9.20}$$

The index α in $(C_\lambda)^\alpha_\sigma$ stands for all tensor indices associated with the $d = n^{p+q}$-dimensional tensor space $V^p \otimes \bar{V}^q$. In the birdtrack notation these indices are explicit:

$$(C_\lambda)_{\sigma,}{}^{b_p \ldots b_1}_{a_q \ldots a_2 a_1} = \tag{9.21}$$



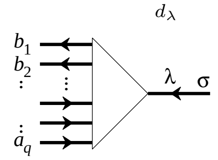
Such rectangular arrays are called *Clebsch-Gordan coefficients* (hereafter referred to as *clebsches* for short). They are explicit mappings $V \to V_\lambda$. The conjugate mapping $V_\lambda \to \bar{V}$ is provided by the product

$$C^\dagger \begin{pmatrix} \begin{matrix} 0 & \\ & 0 \end{matrix} & & \\ & \boxed{\begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix}} & \\ & & \begin{matrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{matrix} \end{pmatrix} , \tag{9.22}$$

which defines the $[d \times d_\lambda]$ rectangular matrix $(C^\lambda)^\sigma_\alpha$, $\alpha = 1, 2, \ldots d$, $\sigma = 1, 2, \ldots d_\lambda$:

$$C^\lambda = \underbrace{\begin{pmatrix} (C^\lambda)^1_1 & \ldots & (C^\lambda)^{d_\lambda}_1 \\ \vdots & & \vdots \\ & & (C^\lambda)^{d_\lambda}_d \end{pmatrix}}_{d_\lambda} \Bigg\} d$$

$$(C^\lambda)^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}{}_{,}{}^{\sigma} = \tag{9.23}$$



The two rectangular Clebsch-Gordan matrices C^λ and C_λ are related by hermitian conjugation.

The tensors transform as tensor products of the defining rep. In general, tensors transform as tensor products of various reps, with indices running over the corresponding rep dimensions:

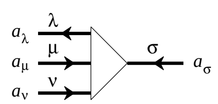
$$x^{a_{p+1}\dots a_{p+q}}_{a_1 a_2 \dots a_p} \qquad \text{where} \qquad \begin{aligned} a_1 &= 1,2,\dots,d_1 \\ a_2 &= 1,2,\dots,d_2 \\ &\vdots \\ a_{p+q} &= 1,2,\dots,d_{p+q}\,. \end{aligned} \tag{9.24}$$

The action of the transformation g on the index a_k is given by the $[d_k \times d_k]$ matrix rep G_k.

Clebsches are notoriously index overpopulated, as they require a rep label and a tensor index for each rep in the tensor product. Diagrammatic notation alleviates this index plague in either of two ways:

1. One can indicate a rep label on each line:

$$C_{a_\lambda}^{a_\mu a_\nu},{}^{a_\sigma} \quad = \tag{9.25}$$

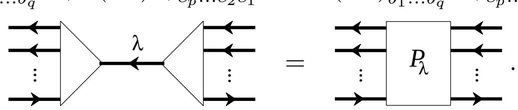


   (An index, if written, is written at the end of a line; a rep label is written above the line.)

2. One can draw the propagators (Kronecker deltas) for different reps with different kinds of lines. For example, we shall usually draw the adjoint rep with a thin line.

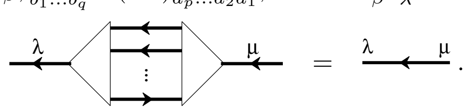
By the definition of clebsches (9.6), the λ rep projection operator can be written out in terms of Clebsch-Gordan matrices $C^\lambda C_\lambda$:

$$\begin{aligned} C^\lambda C_\lambda &= \mathbf{P}_\lambda\,, \qquad \text{(no sum on } i) \\ (C^\lambda)_{b_1\dots b_q}^{a_1 a_2 \dots a_p},{}^{\alpha}\,(C_\lambda)_{\alpha},{}^{d_q\dots d_1}_{c_p\dots c_2 c_1} &= (\mathbf{P}_\lambda)_{b_1\dots b_q}^{a_1 a_2\dots d_p},{}^{d_q\dots d_1}_{c_p\dots c_2 c_1} \end{aligned} \tag{9.26}$$



A specific choice of clebsches is quite arbitrary. All relevant properties of projection operators (orthogonality, completeness, dimensionality) are independent of the explicit form of the diagonalization transformation C. Any set of C_λ is acceptable as long as it satisfies the orthogonality and completeness conditions. From (9.19) and (9.22) it follows that C_λ are *orthonormal*:

$$\begin{aligned} C_\lambda C^\mu &= \delta^\mu_\lambda \mathbf{1}\,, \\ (C_\lambda)_\beta,{}^{a_1 a_2\dots a_p}_{b_1\dots b_q}(C^\mu)_{a_p\dots a_2 a_1}^{b_q\dots b_1},{}^{\alpha} &= \delta^\alpha_\beta \delta^\mu_\lambda \end{aligned} \tag{9.27}$$

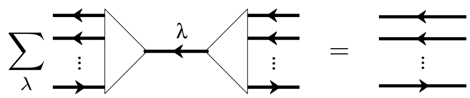


Here $\mathbf{1}$ is the $[d_\lambda \times d_\lambda]$ unit matrix, and C_λ's are multiplied as $[d_\lambda \times d]$ rectangular matrices.

The *completeness relation* (9.8)

$$\sum_\lambda C^\lambda C_\lambda = \mathbf{1}\,, \qquad ([d \times d] \text{ unit matrix})\,,$$

$$\sum_\lambda (C^\lambda)_{b_1 \ldots b_q}^{a_1 a_2 \ldots a_p},{}^{\alpha} (C_\lambda)_{\alpha},{}_{c_p \ldots c_2 c_1}^{d_q \ldots d_1} = \delta_{c_1}^{a_1} \delta_{c_2}^{a_2} \ldots \delta_{b_q}^{d_q}$$



(9.28)

$$C^\lambda \mathbf{P}_\mu = \delta_\lambda^\mu C^\lambda\,,$$
$$\mathbf{P}_\lambda C^\mu = \delta_\lambda^\mu C^\mu\,, \qquad (\text{no sum on } \lambda, \mu)\,, \qquad (9.29)$$

follows immediately from (9.7) and (9.27).

9.7 Irrelevancy of clebsches

(copied from *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [5])

As was emphasized in sect. 9.6, an explicit choice of clebsches is highly arbitrary; it corresponds to a particular coordinatization of the d_λ-dimensional subspace V_λ. For computational purposes clebsches are largely irrelevant. Nothing that a physicist wants to compute depends on an explicit coordinatization. For example, in QCD the physically interesting objects are color singlets, and all color indices are summed over: one needs only an expression for the projection operators, not for the C_λ's separately.

Again, a nice example is the Lie algebra generators T_i. Explicit matrices are often constructed (Gell-Mann λ_i matrices, Cartan's canonical weights); however, in any singlet they always appear summed over the adjoint rep indices. The summed combination of clebsches is just the adjoint rep projection operator, a very simple object compared with explicit T_i matrices ($\mathbf{P}_A$ is typically a combination of a few Kronecker deltas), and much simpler to use in explicit evaluations. As we shall show by many examples, all rep dimensions, casimirs, etc.. are computable once the projection operators for the reps involved are known. Explicit clebsches are superfluous from the computational point of view; we use them chiefly to state general theorems without recourse to any explicit realizations.

However, if one has to compute noninvariant quantities, such as subgroup embeddings, explicit clebsches might be very useful. Gell-Mann [8] invented λ_i matrices in order to embed $SU(2)$ of isospin into $SU(3)$ of the eightfold way. Cartan's canonical form for generators, summarized by Dynkin labels of a rep is a very powerful tool in the study of symmetry-breaking chains [7, 12]. The same can be achieved with decomposition by invariant matrices (a nonvanishing expectation value for a direction in the defining space defines the little group of transformations in the remaining directions), but the tensorial technology in this context is underdeveloped compared to the canonical methods. And, as Stedman [13] rightly points out, if you need to check your calculations against the existing literature, keeping track of phase conventions is a necessity.

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Exercises

9.1. **Irreps of SO(2).** Matrix

$$T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{9.30}$$

is the generator of rotations in a plane.

(a) Use the method of projection operators to show that for rotations in the kth Fourier mode plane, the irreducible $1D$ subspaces orthonormal basis vectors are

$$\mathbf{e}^{(\pm k)} = \frac{1}{\sqrt{2}} \left(\pm \mathbf{e}_1^{(k)} - i\, \mathbf{e}_2^{(k)} \right) .$$

How does T act on $\mathbf{e}^{(\pm k)}$?

(b) What is the action of the $[2\times2]$ rotation matrix

$$D^{(k)}(\theta) = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} , \qquad k = 1, 2, \cdots$$

on the $(\pm k)$th subspace $\mathbf{e}^{(\pm k)}$?

(c) What are the irreducible representations characters of SO(2)?

9.2. **Reduction of a product of two SO(2) irreps.** Determine the Clebsch-Gordan series for SO(2). Hint: Abelian group has 1-dimensional characters. Or, you are just multiplying terms in Fourier series.

9.3. **Irreps of O(2).** O(2) is a group, but not a Lie group, as in addition to continuous transformations generated by (9.30) it has, as a group element, a parity operation

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which cannot be reached by continuous transformations.

(a) Is this group Abelian, i.e., does T commute with $R(k\theta)$? Hint: evaluate first the $[T, \sigma]$ commutator and/or show that $\sigma D^{(k)}(\theta)\sigma^{-1} = D^{(k)}(-\theta)$.

(b) What are the equivalence (i.e., conjugacy) classes of this group?

(c) What are irreps of O(2)? What are their dimensions?
Hint: O(2) is the $n \to \infty$ limit of D_n, worked out in exercise 4.4 *Irreducible representations of dihedral group D_n*. Parity σ maps an SO(2) eigenvector into another eigenvector, rendering eigenvalues of any O(2) commuting operator degenerate. Or, if you really want to do it right, apply Schur's first lemma to improper rotations

$$R^{'}(\theta) = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} \sigma = \begin{pmatrix} \cos k\theta & \sin k\theta \\ \sin k\theta & -\cos k\theta \end{pmatrix}$$

to prove irreducibility for $k \neq 0$.

(d) What are irreducible characters of O(2)?

(e) Sketch a fundamental domain for O(2).

9.4. **Reduction of a product of two O(2) irreps.** Determine the Clebsch-Gordan series for O(2), i.e., reduce the Kronecker product $D^{(k)} \bigotimes D^{(\ell)}$.

9.5. **A fluttering flame front.**

(a) Consider a linear partial differential equation for a real-valued field $u = u(x,t)$ defined on a periodic domain $u(x,t) = u(x+L,t)$:

$$u_t + u_{xx} + \nu u_{xxxx} = 0\,, \qquad x \in [0,L]\,. \tag{9.31}$$

In this equation $t \geq 0$ is the time and x is the spatial coordinate. The subscripts x and t denote partial derivatives with respect to x and t: $u_t = \partial u/d\partial$, u_{xxxx} stands for the 4th spatial derivative of $u = u(x,t)$ at position x and time t. Consider the form of equations under coordinate shifts $x \to x + \ell$ and reflection $x \to -x$. What is the symmetry group of (9.31)?

(b) Expand $u(x,t)$ in terms of its SO(2) irreducible components (hint: Fourier expansion) and rewrite (9.31) as a set of linear ODEs for the expansion coefficients. What are the eigenvalues of the time evolution operator? What is their degeneracy?

(c) Expand $u(x,t)$ in terms of its O(2) irreducible components (hint: Fourier expansion) and rewrite (9.31) as a set of linear ODEs. What are the eigenvalues of the time evolution operator? What is their degeneracy?

(d) Interpret $u = u(x,t)$ as a 'flame front velocity' and add a quadratic nonlinearity to (9.31),

$$u_t + \tfrac{1}{2}(u^2)_x + u_{xx} + \nu u_{xxxx} = 0\,, \qquad x \in [0,L]\,. \tag{9.32}$$

This nonlinear equation is known as the Kuramoto-Sivashinsky equation, a baby cousin of Navier-Stokes. What is the symmetry group of (9.32)?

(e) Expand $u(x,t)$ in terms of its O(2) irreducible components (see exercise 9.3) and rewrite (9.32) as an infinite tower of coupled nonlinear ODEs.

(f) What are the degeneracies of the spectrum of the eigenvalues of the time evolution operator?

9.6. **O(2) fundamental domain for Kuramoto-Sivashinsky equation.** You have C_2 discrete symmetry generated by flip σ, which tiles the space by two tiles.

- Is there a subspace invariant under this C_2? What form does the tower of ODEs take in this subspace?
- How would you restrict the flow (the integration of the tower of coupled ODEs) to a fundamental domain?

This problem is indeed hard, a research level problem, at least for me and the grad students in our group. Unlike the beautiful full-reducibility, character-orthogonality representation theory of linear problems, in nonlinear problems symmetry reduction currently seems to require lots of clever steps and choices of particular coordinates, and we am not at all sure that our solution is the optimal one. Somebody looking at the problem with a fresh eye might hit upon a solution much simpler than ours. Has happened before :)

Burak Budanur's solution is written up in Budanur and Cvitanović [1] *Unstable manifolds of relative periodic orbits in the symmetry-reduced state space of the Kuramoto-Sivashinsky system* sect. 3.2 *O(2) symmetry reduction*, eq. (17) (get it here).

9.7. **Lie algebra from invariance.** Derive the Lie algebra commutator and the Jacobi identity as particular examples of the invariance condition, using both index and birdtracks notations. The invariant tensors in question are "the laws of motion," i.e., the generators of infinitesimal group transformations in the defining and the adjoint representations.

group theory - week 10

Lie groups, algebras

Homework HW10

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 10.1 *Conjugacy classes of* $SO(3)$	2 points (+ 2 bonus points, if complete)
Exercise 10.2 *The character of* $SO(3)$ *3-dimensional representation*	1 point
Exercise 10.3 *The orthonormality of* $SO(3)$ *characters*	2 point
Exercise 10.4 $U(1)$ *equivariance of two-modes system for finite angles*	3 points
Exercise 10.6 $SO(2)$ *or harmonic oscillator slice*	2 points

Bonus points

Exercise 10.5 *Integrate the two-modes system*	4 point
Exercise 10.7 *Invariant subspace of the two-modes system*	1 point
Exercise 10.8 *Slicing the two-modes system*	1 point
Exercise 10.9 *The symmetry reduced two-modes flow*	(difficult) 6 points

Total of 10 points = 100 % score.

2021-06-24 Predrag Lecture 19 Lie groups, algebras

The fastest way to watch any week's lecture videos is by letting YouTube run the

lecture playlist

There is way too much material in this week's notes. Watch the main sequence of video clips, that and recommended reading should suffice. The rest is optional.

- Bridging the step from discrete to continuous compact groups: invariant integration measures, characters, character orthonormality and completeness relations:

Rotations in 3 dimensions (30 min)

Lie algebra (21 min)

Birdtracks (6 min)

  - Sect. 10.7 *Lie groups for pedestrians* is advanced material, safely ignored, here only to whet your appetite for things not done in 19th century. It is a very condensed extract of chapters 3 *Invariants and reducibility* and 4 *Diagrammatic notation* from *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [11]. I am usually reluctant to use birdtrack notations in front of graduate students indoctrinated by their professors in the 1890's tensor notation, but now I'm emboldened by the very enjoyable article on *The new language of mathematics* by Dan Silver [22].
  - Ditto for sect. 10.10 *Birdtracks - updated history*. Your professor's notation is as convenient for actual calculations as -let's say- long division using roman numerals. So leave them wallowing in their early progressive rock of 1968, King Crimsons of their youth. You chill to beats younger than Windows 98, to grime, to trap, to hardvapour, to birdtracks.
  - Go to week 16 to learn more.

- OK, I see that formally $SU(2) \simeq SO(3)$, but who ordered "spin?"

Rotations in 2 complex dimensions (42 min)

  - Read sect. 10.3 *SU(2) Pauli matrices*

For overall clarity and pleasure of reading, I like Schwichtenberg [21] (click here) discussion best. If you read anything for this week's lectures, read Schwichtenberg sects. 3.4 to 3.6.

2021-06-29 Predrag Lecture 20 $O(2)$ symmetry sliced

- sect. 10.4 *Two-modes $SO(2)$-equivariant flow*

lecture playlist

In two real dimensions dynamics is boring (4:28 min)

Symmetries of solutions (18 min)

Symmetry reduction (1:44 min)

Moving frames: with freedom comes responsibility (8:35 min)

Phase of a relative periodic orbit, choice of moving frame (9:18 min)

Comoving frames (23:09 min)

Slice (4:00 min)

How to slice a continuous symmetry (14:16 min)

optional: *Low dimensional slices; 2D flat heart* (1:40 min)

Slices are not sections! (17 sec)

optional: *Cross-sections, orbitfolds* (1:18 min)

Symmetry reduced equations of motion (6:12 min)

Sections and slices are local, good up to a border (1:18 min)

A spatial Fourier expansion (5:11 min)

First Fourier mode slice (3:00 min)

In-slice time (1:54 min)

For the two-modes SO(2)-equivariant flow long version, see

ChaosBook example *Two-modes flow*.

ChaosBook chapt. *Slice & dice*, sect. 13.1 *Only dead fish go with the flow* to sect. 13.5 *First Fourier mode slice*.

This is difficult material, so it is OK if you do not get it this time around. None of this will be on the final - the main point is that once you face a nonlinear problem, nothing is easy - not even rotations on a circle.

10.1 Other sources (optional)

- You can glance through
  - sect. 10.5 *SO(3) character orthogonality*
  - sect. 10.6 *Linear algebra*

but I do not expect you to master this material.

- C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, works out in full detail the representations and Haar measures for 1-dimensional Lie groups, and explains the difference between rotations and translations.
- Chen, Ping and Wang [7] *Group Representation Theory for Physicists*, Sect 5.3 *Lie algebras* and Sect 5.4 *Finite transformations* work out several $SU(2)$ and $O(3)$ examples (click here). Sects 5.5, 5.6 and 5.7 also merit a quick read.
- In his group theory notes D. Vvedensky, chapter 8, sect. 8.3 *Axis–angle representation of proper rotations in three dimensions*, has a very nice discussion of the (10.7) parametrization of the $SO(3)$ 3-dimensional group manifold: the parameter space corresponds to the interior of a sphere of radius π, and the over the classes of $SO(3)$ is given by integral over spherical shells. In sect. 8.4 he derives the Haar measure (without calling it so).

  In sect. 8.5 Vvedensky says: "For $SO(2)$, we were able to determine the characters of the irreducible representations directly, i.e., without having to determine the basis functions of these representations. The structure of $SO(3)$, however, does not allow for such a simple procedure, so we must determine the basis functions from the outset." That I disagree with; in birdtracks.eu sect. 15.1 *Reps of $SU(2)$* I construct the irreps and label them by their Young tableaus with no recourse to spherical harmonics.
- Reading: Chen, Ping and Wang [7] *Group Representation Theory for Physicists*, Sect 5.2 *Definition of a Lie group, with examples* (click here).
- Dirac belt trick applet
- If still anxious, maybe this helps: Mark Staley, *Understanding quaternions and the Dirac belt trick* arXiv:1001.1778.
- ChaosBook Sect 26.1 *Compact groups*
- I have enjoyed reading Mathews and Walker [16] Chap. 16 *Introduction to groups* (click here). Goldbart writes that the book is "based on lectures by Richard Feynman at Cornell University." Very clever. In particular, work through the example of fig. 16.2: it is very cute, you get explicit eigenmodes from group theory alone. The main message is that if you think things through first, you never have to go through using explicit form of representation matrices - thinking in terms of invariants, like characters, will get you there much faster.
- Any book, of 100s available, like Cornwell [8] *Group Theory in Physics: An introduction* that covers group theory might be more to your taste.
- *Hamilton's quaternions* (3:18 min)
- Stone and Goldbart [24] (click here) Chapter 17 Sect 17.6 *Analytic functions and topology* (wherein stereographic projection is revealed to be the geometric origin of the spinor representations of the rotation group)
- This week's lectures are related to AWH Chapter 3 *Vector Analysis* (click here) and Chapter 16 *Angular Momentum* (click here).

10.2 Discussion (optional)

Question 10.1. Henriette Roux asks
Q Why is this complex 2-dimensional vector called a 'spinor'?
A Historical, as Arfken, Weber & Harris [4] explain: "It turns out that half-integral angular momentum states are needed to describe the intrinsic angular momentum of the electron and many other particles. Since these particles also have magnetic moments, an intuitive interpretation is that their charge distributions are spinning about some axis; hence the term spin. It is now understood that the spin phenomena cannot be explained consistently by describing these particles as ordinary charge distributions undergoing rotational motion, [...] "

Schwichtenberg [21]: "[...] spinors have properties that usual vectors do not have. For instance, the factor $1/2$ in the exponent. This factor shows us that a spinor1 is after a rotation by 2π not the same, but gets a minus sign. This is a pretty crazy property, because all objects we deal with in everyday life are exactly the same after a rotation by $360^o = 2\pi$.

Question 10.2. Henriette Roux asks
Q What' relation of Pauli exclusion principle to the spinor 2π rotation amounting to overall minus sign?
A I think of fermion/Grassmann statistics as Archimedes principle + linearity, see my Field Theory [10] *chap. 4 Fermions*. Basically, usually a constraint is imposed by eliminating a variable, for example, given the constraint is $x^2 + y^2 + z^2 = 1$, one gets rid of z by replacing it everywhere with $z \rightarrow \sqrt{1 - x^2 - y^2}$. This makes a fully symmetric theory asymmetric and ugly. In linear setting, another option is to keep all the variables and the symmetry, but add a new variable which by construction subtracts a degree of freedom, what I call [12] a "negative dimension". In quantum field theory such variable is called a 'ghost'; it needs to be anti-commuting or Grassmann.

Question 10.3. Henriette Roux asks
Q This course is all about eigenfunctions of symmetry operators. Why are you not teaching us Bessel functions?
A Blame Feynman: On May 2, 1985 my stay at Cornell was to end, and Vinnie of college town *Italian Kitchen* made a special dinner for three of us regulars. Das Wunderkind noticed Feynman ambling down Eddy Avenue, kidnapped him, and here we were, two wunderkinds, two humans.

Feynman was a very smart, forever driven wunderkind. He naturally bonded with our very smart, forever driven wunderkind, who suddenly lurched out of control, and got very competive about at what age who summed which kind of Bessel function series. Something like age twelve, do not remember which one did the Bessels first. At that age I read *Palle Alone in the World*, while my nonwunderkind friend, being from California, watched television 12 hours a day.

When Das Wunderkind taught graduate E&M, he spent hours crafting lectures about symmetry groups and their representations as various eigenfunctions. Students were not pleased.

So, fuggedaboutit! if you have not done your Bessels yet, they are eigenfunctions, just like your Fourier modes, but for a spherical symmetry rather than for a translation symmetry; wiggle like a cosine, but decay radially.

When you need them you'll figure them out. Or sue me.

Question 10.4. Predrag asks
Q You are the best of students now. Are you ready for The Talk?
A Henriette Roux: I'm ready!

10.3 SU(2) Pauli matrices

A lightning, bullet points review.

- U(n): unitary transformation $U = e^{iH}$
- Unitarity: $U^\dagger U = \mathbf{1} \Rightarrow H^\dagger = H$, the generator is hermitian.
- SU(n): special unitary transformation $\det U = 1$
- Must know: $\ln\det = \text{tr}\,\ln$ for any matrix, so the generator is traceless
  $\ln\det U = \text{tr}\,\ln U = \text{tr}\,H = 0$
- $\text{SU}(2) : H = \begin{pmatrix} a & c \\ e & b \end{pmatrix}$, $a, b, c, e \in \mathbb{C}$, eight real numbers in all.
- H is hermitian: $H = \begin{pmatrix} a & c+id \\ c-id & b \end{pmatrix}$, $a, b, c, d \in \mathbb{R}$,
- H is traceless: $0 = \text{tr}\,H \Rightarrow a + b = 0$, three real rotation parameters in all, so

$$\begin{aligned} H &= c\sigma_x + d\sigma_y + a\sigma_z \\ &= c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \quad (10.1)$$

where σ_j are known as Pauli matrices.

10.4 Two-modes SO(2)-equivariant flow

Consider the pair of U(1)-equivariant complex ODEs

$$\begin{aligned} \dot{z}_1 &= (\mu_1 - \mathrm{i}\, e_1)\, z_1 + a_1\, z_1 |z_1|^2 + b_1\, z_1 |z_2|^2 + c_1\, \overline{z}_1\, z_2 \\ \dot{z}_2 &= (\mu_2 - \mathrm{i}\, e_2)\, z_2 + a_2\, z_2 |z_1|^2 + b_2\, z_2 |z_2|^2 + c_2\, z_1^2 \,, \end{aligned} \quad (10.2)$$

with z_1, z_2 complex, and all parameters real valued.

This system is a generic example of a few-modes truncation of a Fourier representation of some physical flow, such as fluid dynamics convection flow, truncated in such a way that the model exhibits the same symmetries as the full original problem, while being drastically simpler to study. It is a merely a toy model with no physical interpretation, just like the iconic Lorenz flow. We use it to illustrate the effects of continuous symmetry on chaotic dynamics.

We refer to this toy model as the *two-modes* system. It belongs to the family of simplest ODE systems that we know that (a) have a continuous U$(1) \simeq$ SO(2), but no discrete symmetry (if at least one of $e_j \neq 0$). (b) models 'weather', in the same sense that Lorenz equation models 'weather', (c) exhibits chaotic dynamics, (d) can be easily visualized, in the dimensionally lowest possible setting required for chaotic dynamics, with the full state space of dimension $d = 4$, and the SO(2)-reduced dynamics taking place in 3 dimensions, and (e) for which the method of slices reduces the symmetry by a single global slice hyperplane.

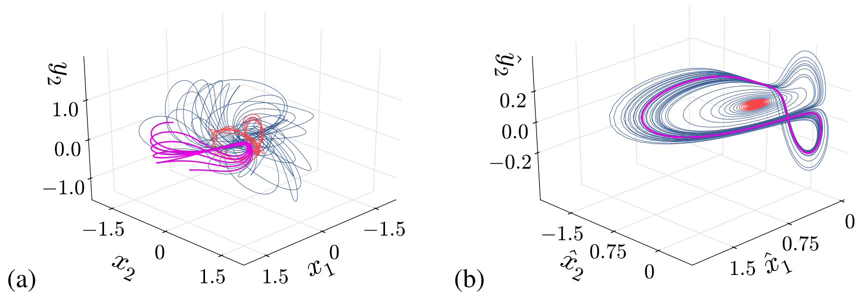


Figure 10.1: Two-modes flow before (a) and after (b) symmetry reduction by first Fourier mode slice. Here a long trajectory (red and blue) starting on the unstable manifold of the TW_1 (red), until it falls on to the strange attractor (blue) and the shortest relative periodic orbit $\overline{1}$ (magenta). Note that the relative equilibrium becomes an equilibrium, and the relative periodic orbit becomes a periodic orbit after the symmetry reduction.

The model has an unreasonably high number of parameters. After some experimentation we fix or set to zero various parameters, and in the numerical examples that follow, we settle for parameters set to

$$\begin{aligned} \mu_1 &= -2.8\,,\ \mu_2 = 1\,,\ e_1 = 0\,,\ e_2 = 1\,, \\ a_1 &= -1\,,\ a_2 = -2.66\,,\ b_1 = 0\,,\ b_2 = 0\,,\ c_1 = -7.75\,,\ c_2 = 1\,, \end{aligned} \tag{10.3}$$

unless explicitly stated otherwise. For these parameter values the system exhibits chaotic behavior. Experiment! If you find a more interesting behavior for some other parameter values, please let us know. The simplified system of equations can now be written as a 3-parameter $\{\mu_1, c_1, a_2\}$ two-modes system,

$$\begin{aligned} \dot{z}_1 &= \mu_1\, z_1 - z_1|z_1|^2 + c_1\, \overline{z}_1\, z_2 \\ \dot{z}_2 &= (1-\mathrm{i})\, z_2 + a_2\, z_2|z_1|^2 + \, z_1^2\,. \end{aligned} \tag{10.4}$$

In order to numerically integrate and visualize the flow, we recast the equations in real variables by substitution $z_1 = x_1 + i\, y_1$, $z_2 = x_2 + i\, y_2$. The two-modes system (10.2) is now a set of four coupled ODEs

exercise 10.5

$$\begin{aligned} \dot{x}_1 &= (\mu_1 - r^2)\, x_1 + c_1\, (x_1 x_2 + y_1 y_2)\,, \qquad r^2 = x_1^2 + y_1^2 \\ \dot{y}_1 &= (\mu_1 - r^2)\, y_1 + c_1\, (x_1 y_2 - x_2 y_1) \\ \dot{x}_2 &= x_2 + y_2 + x_1^2 - y_1^2 + a_2 x_2 r^2 \\ \dot{y}_2 &= -x_2 + y_2 + 2\, x_1 y_1 + a_2 y_2 r^2\,. \end{aligned} \tag{10.5}$$

Try integrating (10.5) with random initial conditions, for long times, times much beyond which the initial transients have died out. What is wrong with this picture? Figure 10.4 (a) is a mess. As we show here, the attractor is built up by a nice ‘stretch & fold’ action, hidden from the view by the continuous symmetry induced drifts. That is fixed by ‘quotienting’ model’s SO(2) symmetry, and reducing the dynamics to a 3-dimensional symmetry-reduced state space, figure 10.4 (b).

exercise 10.6
exercise 10.7
exercise 10.8

10.5 SO(3) character orthogonality (optional)

In 3 Euclidean dimensions, a rotation around z axis is given by the SO(2) matrix

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp\varphi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \qquad (10.6)$$

An arbitrary rotation in $\mathbb{R}^3$ can be represented by

$$R_{\boldsymbol{n}}(\varphi) = e^{-i\varphi\, \boldsymbol{n}\cdot\boldsymbol{L}} \qquad \boldsymbol{L} = (L_1, L_2, L_3)\,, \qquad (10.7)$$

where the unit vector $\boldsymbol{n}$ determines the plane and the direction of the rotation by angle φ. Here L_1, L_2, L_3 are the generators of rotations along x, y, z axes respectively,

$$L_1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad L_2 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (10.8)$$

with Lie algebra relations

$$[L_i, L_j] = i\varepsilon_{ijk} L_k\,. \qquad (10.9)$$

All SO(3) rotations by the same angle θ around different rotation axis $\boldsymbol{n}$ are conjugate to each other,

$$e^{i\phi\boldsymbol{n}_2\cdot\boldsymbol{L}} e^{i\theta\boldsymbol{n}_1\cdot\boldsymbol{L}} e^{-i\phi\boldsymbol{n}_2\cdot\boldsymbol{L}} = e^{i\theta\boldsymbol{n}_3\cdot\boldsymbol{L}}\,, \qquad (10.10)$$

with $e^{i\phi\boldsymbol{n}_2\cdot\boldsymbol{L}}$ and $e^{-i\theta\boldsymbol{n}_2\cdot\boldsymbol{L}}$ mapping the vector $\boldsymbol{n}_1$ to $\boldsymbol{n}_3$ and back, so that the rotation around axis $\boldsymbol{n}_1$ by angle θ is mapped to a rotation around axis $\boldsymbol{n}_3$ by the same θ. The conjugacy classes of SO(3) thus consist of rotations by the same angle about all distinct rotation axes, and are thus labelled the angle θ. As the conjugacy class depends only on θ, the characters can only be a function of θ. For the 3-dimensional special orthogonal representation, the character is

exercise 10.2

$$\chi = 2\cos(\theta) + 1\,. \qquad (10.11)$$

For an irrep labeled by j, the character of a conjugacy class labeled by θ is

$$\chi^{(j)}(\theta) = \frac{\sin(j+1/2)\theta}{\sin(\theta/2)} \qquad (10.12)$$

To check that these characters are orthogonal to each other, one needs to define the group integration over a parametrization of the SO(3) group manifold. A group element is parametrized by the rotation axis $\boldsymbol{n}$ and the rotation angle $\theta \in (-\pi, \pi]$, with $\boldsymbol{n}$ a unit vector which ranges over all points on the surface of a unit ball. Note however, that a π rotation is the same as a $-\pi$ rotation ($\boldsymbol{n}$ and $-\boldsymbol{n}$ point along the same direction), and the $\boldsymbol{n}$ parametrization of SO(3) is thus a 2-dimensional surface of a unit-radius ball with the opposite points identified.

The Haar measure for $\mathrm{SO}(3)$ requires a bit of work, here we just note that after the integration over the solid angle (characters do not depend on it), the Haar measure is

$$dg = d\mu(\theta) = \frac{d\theta}{2\pi}(1-\cos(\theta)) = \frac{d\theta}{\pi}\sin^2(\theta/2)\,. \tag{10.13}$$

exercise 10.3

With this measure the characters are orthogonal, and the character orthogonality theorems follow, of the same form as for the finite groups, but with the group averages replaced by the continuous, parameter dependant group integrals

$$\frac{1}{|G|}\sum_{g\in G} \to \int_G dg\,.$$

The good news is that, as explained in ChaosBook.org Chap. *Relativity for cyclists* (and in *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [11]), one never needs to actually explicitly construct a group manifold parametrizations and the corresponding Haar measure.

10.6 Linear algebra (optional)

In this section we collect a few basic definitions. A sophisticated reader might prefer skipping straight to the definition of the Lie product (10.21), the big difference between the group elements product used so far in discussions of finite groups, and what is needed to describe continuous groups.

Vector space. A set V of elements $\mathbf{x},\mathbf{y},\mathbf{z},\ldots$ is called a *vector* (or *linear*) *space* over a field $\mathbb{F}$ if

(a) *vector addition* "+" is defined in V such that V is an abelian group under addition, with identity element $\mathbf{0}$;

(b) the set is *closed* with respect to *scalar multiplication* and vector addition

$$\begin{aligned} a(\mathbf{x}+\mathbf{y}) &= a\mathbf{x}+a\mathbf{y}\,, \qquad a,b\in\mathbb{F}\,, \quad \mathbf{x},\mathbf{y}\in V \\ (a+b)\mathbf{x} &= a\mathbf{x}+b\mathbf{x} \\ a(b\mathbf{x}) &= (ab)\mathbf{x} \\ 1\,\mathbf{x} &= \mathbf{x}\,, \qquad 0\,\mathbf{x}=\mathbf{0}\,. \end{aligned} \tag{10.14}$$

Here the field $\mathbb{F}$ is either $\mathbb{R}$, the field of reals numbers, or $\mathbb{C}$, the field of complex numbers. Given a subset $V_0 \subset V$, the set of all linear combinations of elements of V_0, or the *span* of V_0, is also a vector space.

A basis. $\{\mathbf{e}^{(1)},\cdots,\mathbf{e}^{(d)}\}$ is any linearly independent subset of V whose span is V. The number of basis elements d is the *dimension* of the vector space V.

Dual space, dual basis. Under a general linear transformation $g \in GL(n, \mathbb{F})$, the row of basis vectors transforms by right multiplication as $\mathbf{e}^{(j)} = \sum_k (\mathbf{g}^{-1})^j{}_k \, \mathbf{e}^{(k)}$, and the column of x_a's transforms by left multiplication as $x' = \mathbf{g}x$. Under left multiplication the column (row transposed) of basis vectors $\mathbf{e}_{(k)}$ transforms as $\mathbf{e}_{(j)} = (\mathbf{g}^\dagger)_j{}^k \mathbf{e}_{(k)}$, where the *dual rep* $\mathbf{g}^\dagger = (\mathbf{g}^{-1})^\top$ is the transpose of the inverse of $\mathbf{g}$. This observation motivates introduction of a *dual* representation space $\bar{V}$, the space on which $GL(n, \mathbb{F})$ acts via the dual rep $\mathbf{g}^\dagger$.

Definition. If V is a vector representation space, then the *dual space* $\bar{V}$ is the set of all linear forms on V over the field $\mathbb{F}$.

If $\{\mathbf{e}^{(1)}, \cdots, \mathbf{e}^{(d)}\}$ is a basis of V, then $\bar{V}$ is spanned by the *dual basis* $\{\mathbf{e}_{(1)}, \cdots, \mathbf{e}_{(d)}\}$, the set of d linear forms $\mathbf{e}_{(k)}$ such that

$$\mathbf{e}_{(j)} \cdot \mathbf{e}^{(k)} = \delta_j^k \,,$$

where δ_j^k is the Kronecker symbol, $\delta_j^k = 1$ if $j = k$, and zero otherwise. The components of dual representation space vectors $\bar{y} \in \bar{V}$ will here be distinguished by upper indices

$$(y^1, y^2, \dots, y^n) \,. \tag{10.15}$$

They transform under $GL(n, \mathbb{F})$ as

$$y'^a = (\mathbf{g}^\dagger)^a{}_b y^b \,. \tag{10.16}$$

For $GL(n, \mathbb{F})$ no complex conjugation is implied by the † notation; that interpretation applies only to unitary subgroups $U(n) \subset GL(n, \mathbb{C})$. In the index notation, $\mathbf{g}$ can be distinguished from $\mathbf{g}^\dagger$ by keeping track of the relative ordering of the indices,

$$(\mathbf{g})_a^b \to g_a{}^b \,, \qquad (\mathbf{g}^\dagger)_a^b \to g^b{}_a \,. \tag{10.17}$$

Algebra. A set of r elements $\mathbf{t}_\alpha$ of a vector space $\mathcal{T}$ forms an algebra if, in addition to the vector addition and scalar multiplication,

(a) the set is *closed* with respect to multiplication $\mathcal{T} \cdot \mathcal{T} \to \mathcal{T}$, so that for any two elements $\mathbf{t}_\alpha, \mathbf{t}_\beta \in \mathcal{T}$, the product $\mathbf{t}_\alpha \cdot \mathbf{t}_\beta$ also belongs to $\mathcal{T}$:

$$\mathbf{t}_\alpha \cdot \mathbf{t}_\beta = \sum_{\gamma=0}^{r-1} \tau_{\alpha\beta}{}^\gamma \mathbf{t}_\gamma \,, \qquad \tau_{\alpha\beta}{}^\gamma \in \mathbb{C} \,; \tag{10.18}$$

(b) the multiplication operation is *distributive*:

$$\begin{aligned} (\mathbf{t}_\alpha + \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma &= \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma + \mathbf{t}_\beta \cdot \mathbf{t}_\gamma \\ \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta + \mathbf{t}_\gamma) &= \mathbf{t}_\alpha \cdot \mathbf{t}_\beta + \mathbf{t}_\alpha \cdot \mathbf{t}_\gamma \,. \end{aligned}$$

The set of numbers $\tau_{\alpha\beta}{}^\gamma$ are called the *structure constants*. They form a matrix rep of the algebra,

$$(\mathbf{t}_\alpha)_\beta{}^\gamma \equiv \tau_{\alpha\beta}{}^\gamma \,, \tag{10.19}$$

whose dimension is the dimension r of the algebra itself.

Depending on what further assumptions one makes on the multiplication, one obtains different types of algebras. For example, if the multiplication is associative

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta) \cdot \mathbf{t}_\gamma = \mathbf{t}_\alpha \cdot (\mathbf{t}_\beta \cdot \mathbf{t}_\gamma) \,,$$

the algebra is *associative*. Typical examples of products are the *matrix product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c \,, \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \,, \tag{10.20}$$

and the *Lie product*

$$(\mathbf{t}_\alpha \cdot \mathbf{t}_\beta)_a^c = (t_\alpha)_a^b (t_\beta)_b^c - (t_\alpha)_c^b (t_\beta)_b^a \,, \qquad \mathbf{t}_\alpha \in V \otimes \bar{V} \tag{10.21}$$

which defines a *Lie algebra*.

10.7 Lie groups for pedestrians (optional)

[...] which is an expression of consecration of angular momentum.

— Mason A. Porter's student

Definition: A Lie group is a topological group G such that (i) G has the structure of a smooth differential manifold, and (ii) the composition map $G \times G \to G : (g, h) \to gh^{-1}$ is smooth, i.e., $\mathbb{C}^\infty$ differentiable.

Do not be mystified by this definition. Mathematicians also have to make a living. The compact Lie groups that we will deploy here are a generalization of the theory of SO$(2) \simeq$ U(1) rotations, i.e., Fourier analysis. By a 'smooth differential manifold' one means objects like the circle of angles that parameterize continuous rotations in a plane, figure 10.2, or the manifold swept by the three Euler angles that parameterize SO(3) rotations.

By 'compact' one means that these parameters run over finite ranges, as opposed to parameters in hyperbolic geometries, such as Minkowsky SO$(3, 1)$. The groups we focus on here are compact by default, as their representations are linear, finite-dimensional matrix subgroups of the unitary matrix group U(d).

Example 1. Circle group. A circle with a direction, figure 10.2, is invariant under rotation by any angle $\theta \in [0, 2\pi)$, and the group multiplication corresponds to composition of two rotations $\theta_1 + \theta_2 \mod 2\pi$. The natural representation of the group action is by a complex numbers of absolute value 1, i.e., the exponential $e^{i\theta}$. The composition rule is then the complex multiplication $e^{i\theta_2} e^{i\theta_1} = e^{i(\theta_1+\theta_2)}$. The circle group is a *continuous group*, with infinite number of elements, parametrized by the continuous parameter $\theta \in [0, 2\pi)$. It can be thought of as the $n \to \infty$ limit of the cyclic group C$_n$. Note that the circle divided into n segments is *compact*, in distinction to the infinite lattice of integers $\mathbb{Z}$, whose limit is a *line* (noncompact, of infinite length).

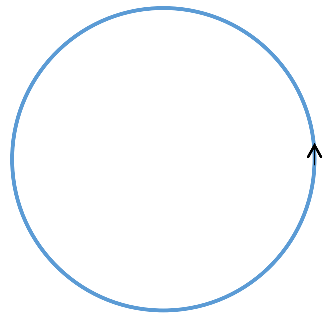


Figure 10.2: Circle group $S^1 = \mathrm{SO}(2)$, the symmetry group of a circle with directed rotations, is a compact group, as its natural parametrization is either the angle $\phi \in [0, 2\pi)$, or the perimeter $x \in [0, L)$.

An element of a $[d\times d]$ -dimensional matrix representation of a *Lie group* continuously connected to identity can be written as

$$g(\boldsymbol{\phi}) = e^{i\boldsymbol{\phi}\cdot T}\,, \qquad \boldsymbol{\phi}\cdot T = \sum_{a=1}^{N} \phi_a T_a\,, \tag{10.22}$$

where $\boldsymbol{\phi}\cdot T$ is a *Lie algebra* element, T_a are matrices called 'generators', and $\boldsymbol{\phi} = (\phi_1, \phi_2, \cdots, \phi_N)$ are the parameters of the transformation. Repeated indices are summed throughout, and the dot product refers to a sum over Lie algebra generators. Sometimes it is convenient to use the Dirac bra-ket notation for the Euclidean product of two real vectors $x, y \in \mathbb{R}^d$, or the product of two complex vectors $x, y \in \mathbb{C}^d$, i.e., indicate complex x-transpose times y by

$$\langle x|y\rangle = x^\dagger y = \sum_i^d x_i^* y_i\,. \tag{10.23}$$

Finite unitary transformations $\exp(i\boldsymbol{\phi}\cdot T)$ are generated by sequences of infinitesimal steps of form

$$g(\boldsymbol{\delta\phi}) \simeq 1 + i\boldsymbol{\delta\phi}\cdot T\,, \quad \boldsymbol{\delta\phi} \in \mathbb{R}^N\,, \quad |\boldsymbol{\delta\phi}| \ll 1\,, \tag{10.24}$$

where T_a, the *generators* of infinitesimal transformations, are a set of linearly independent $[d\times d]$ hermitian matrices (see figure 10.3 (b)).

The reason why one can piece a global transformation from infinitesimal steps is that the choice of the "origin" in coordinatization of the group manifold sketched in figure 10.3 (a) is arbitrary. The coordinatization of the tangent space at one point on the group manifold suffices to have it everywhere, by a coordinate transformation g, i.e., the new origin y is related to the old origin x by conjugation $y = g^{-1}xg$, so all tangent spaces belong the same class, they are geometrically equivalent.

Unitary and orthogonal groups are defined as groups that preserve 'length' norms, $\langle gx|gx\rangle = \langle x|x\rangle$, and infinitesimally their generators (10.24) induce no change in the

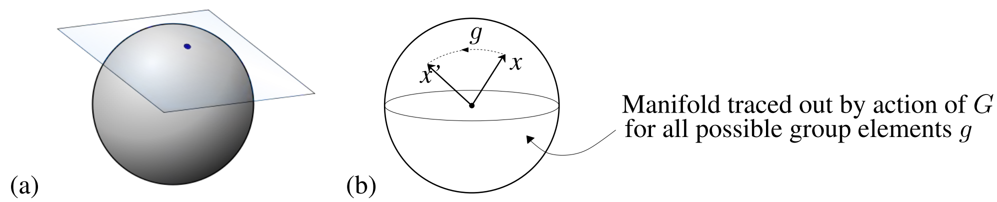


Figure 10.3: (a) Lie algebra fields $\{t_1, \cdots, t_N\}$ span the tangent space of the group orbit $\mathcal{M}_x$ at state space point x, see (10.26) (figure from WikiMedia.org). (b) A global group transformation $g : x \to x'$ can be pieced together from a series of infinitesimal steps along a continuous trajectory connecting the two points. The group orbit of state space point $x \in \mathbb{R}^d$ is the N-dimensional manifold of all actions of the elements of group G on x.

norm, $\langle T_a x | x \rangle + \langle x | T_a x \rangle = 0$, hence the Lie algebra generators T_a are hermitian for,

$$T_a^\dagger = T_a \,. \tag{10.25}$$

The flow field at the state space point x induced by the action of the group is given by the set of N tangent fields

$$t_a(x)_i = (T_a)_{ij} x_j \,, \tag{10.26}$$

which span the d-dimensional *group tangent space* at state space point x, parametrized by $\boldsymbol{\delta\phi}$.

For continuous groups the Lie algebra, i.e., the algebra spanned by the set of N generators T_a of infinitesimal transformations, takes the role that the $|G|$ group elements play in the theory of discrete groups (see figure 10.3).

10.7.1 Invariants

One constructs the irreps of finite groups by identifying matrices that commute with all group elements, and using their eigenvalues to decompose arbitrary representation of the group into a unique sum of irreps. The same strategy works for the compact Lie groups, (10.30), and is indeed the key idea that distinguishes the invariance groups classification developed in *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [11] from the 19th century Cartan-Killing classification of Lie algebras.

Definition. The vector $q \in V$ is an *invariant vector* if for any transformation $g \in \mathcal{G}$

$$q = Gq \,. \tag{10.27}$$

Definition. A tensor $x \in V^p \otimes \bar{V}^q$ is an *invariant tensor* if for any $g \in G$

$$x^{a_1 a_2 \dots a_p}_{b_1 \dots b_q} = G^{a_1}{}_{c_1} G^{a_2}{}_{c_2} \dots G_{b_1}{}^{d_1} \dots G_{b_q}{}^{d_q} x^{c_1 c_2 \dots c_p}_{d_1 \dots d_q} \,. \tag{10.28}$$

If a bilinear form $m(\bar{x}, y) = x^a M_a{}^b y_b$ is invariant for all $g \in \mathcal{G}$, the matrix

$$M_a{}^b = G_a{}^c G^b{}_d M_c{}^d \tag{10.29}$$

is an *invariant matrix*. Multiplying with ${G_b}^e$ and using the unitary, we find that the invariant matrices *commute* with all transformations $g \in \mathcal{G}$:

$$[G, \mathbf{M}] = 0 \,. \tag{10.30}$$

Definition. An *invariance group* $\mathcal{G}$ is the set of all linear transformations (10.28) that preserve the primitive invariant relations (and, by extension, *all* invariant relations)

$$\begin{aligned} p_1(x, \bar{y}) &= p_1(Gx, \bar{y}G^\dagger) \\ p_2(x, y, z, \ldots) &= p_2(Gx, Gy, Gz \ldots) \,, \qquad \ldots . \end{aligned} \tag{10.31}$$

Unitarity guarantees that all contractions of primitive invariant tensors, and hence all composed tensors $h \in H$, are also invariant under action of $\mathcal{G}$. As we assume unitary $\mathcal{G}$, it follows that the list of primitives must always include the Kronecker delta.

Example 2. If $p^a q_a$ is the only invariant of $\mathcal{G}$

$$p'^a q'_a = p^b (G^\dagger G)_b{}^c q_c = p^a q_a \,, \tag{10.32}$$

then $\mathcal{G}$ is the full *unitary group* $U(n)$ (invariance group of the complex norm $|x|^2 = x^b x_a \delta_b^a$), whose elements satisfy

$$G^\dagger G = 1 \,. \tag{10.33}$$

Example 3. If we wish the z-direction to be invariant in our 3-dimensional space, $q = (0, 0, 1)$ is an invariant vector (10.27), and the invariance group is $O(2)$, the group of all rotations in the x-y plane.

10.7.2 Infinitesimal transformations, Lie algebras

A unitary transformation G infinitesimally close to unity can be written as

$$G_a{}^b = \delta_a^b + i D_a^b \,, \tag{10.34}$$

where D is a hermitian matrix with small elements, $|D_a^b| \ll 1$. The action of $g \in \mathcal{G}$ on the conjugate space is given by

$$(G^\dagger)_b{}^a = G^a{}_b = \delta_b^a - i D_b^a \,. \tag{10.35}$$

D can be parametrized by $N \leq n^2$ real parameters. N, the maximal number of independent parameters, is called the *dimension* of the group (also the dimension of the Lie algebra, or the dimension of the adjoint rep).

Here we shall consider only infinitesimal transformations of form $G = 1 + iD$, $|D_b^a| \ll 1$. We do not study the entire group of invariant transformation, but only the transformations connected to the identity. For example, we shall not consider invariances under coordinate reflections.

The generators of infinitesimal transformations (10.34) are hermitian matrices and belong to the $D_b^a \in V \otimes \bar{V}$ space. However, not any element of $V \otimes \bar{V}$ generates an allowed transformation; indeed, one of the main objectives of group theory is to define the class of allowed transformations.

This subspace is called the *adjoint* space, and its special role warrants introduction of special notation. We shall refer to this vector space by letter A, in distinction to the defining space V. We shall denote its dimension by N, label its tensor indices by $i, j, k \ldots$, denote the corresponding Kronecker delta by a thin, straight line,

$$\delta_{ij} = i \;\text{———}\; j\,, \qquad i, j = 1, 2, \ldots, N\,, \tag{10.36}$$

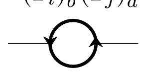
and the corresponding transformation generators by

$$(C_A)_{i,}{}_b^{\,a} = \frac{1}{\sqrt{a}}(T_i)_b^a =$$

$$a, b = 1, 2, \ldots, n \qquad i = 1, 2, \ldots, N\,.$$

Matrices T_i are called the *generators* of infinitesimal transformations. Here a is an (uninteresting) overall normalization fixed by the orthogonality condition

$$(T_i)_b^a (T_j)_a^b = \mathrm{tr}\,(T_i T_j) = a\,\delta_{ij}$$



$$= a \;\text{———}\,. \tag{10.37}$$

For every invariant tensor q, the infinitesimal transformations $G = 1 + iD$ must satisfy the invariance condition (10.27). Parametrizing D as a projection of an arbitrary hermitian matrix $H \in V \otimes \bar{V}$ into the adjoint space, $D = \mathbf{P}_A H \in V \otimes \bar{V}$,

$$D_b^a = \frac{1}{a}(T_i)_b^a \epsilon_i\,, \tag{10.38}$$

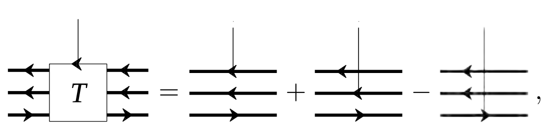
we obtain the *invariance condition* which the *generators* must satisfy: they *annihilate* invariant tensors:

$$T_i q = 0\,. \tag{10.39}$$

To state the invariance condition for an arbitrary invariant tensor, we need to define the action of generators on the tensor reps. By substituting $G = 1 + i\epsilon \cdot T + O(\epsilon^2)$ and keeping only the terms linear in ϵ, we find that the generators of infinitesimal transformations for tensor reps act by touching one index at a time:

$$\begin{aligned}(T_i)_{b_1\ldots b_q}^{a_1 a_2\ldots a_p}{}_{,}{}_{c_p\ldots c_2 c_1}^{d_q\ldots d_1} &= (T_i)_{c_1}^{a_1}\delta_{c_2}^{a_2}\ldots\delta_{c_p}^{a_p}\delta_{b_1}^{d_1}\ldots\delta_{b_q}^{d_q} \\ &+\delta_{c_1}^{a_1}(T_i)_{c_2}^{a_2}\ldots\delta_{c_p}^{a_p}\delta_{b_1}^{d_1}\ldots\delta_{b_q}^{d_q} + \ldots + \delta_{c_1}^{a_1}\delta_{c_2}^{a_2}\ldots(T_i)_{c_p}^{a_p}\delta_{b_1}^{d_1}\ldots\delta_{b_q}^{d_q} \\ &-\delta_{c_1}^{a_1}\delta_{c_2}^{a_2}\ldots\delta_{c_p}^{a_p}(T_i)_{b_1}^{d_1}\ldots\delta_{b_q}^{d_q} - \ldots - \delta_{c_1}^{a_1}\delta_{c_2}^{a_2}\ldots\delta_{c_p}^{a_p}\delta_{b_1}^{d_1}\ldots(T_i)_{b_q}^{d_q}\,.\end{aligned} \tag{10.40}$$

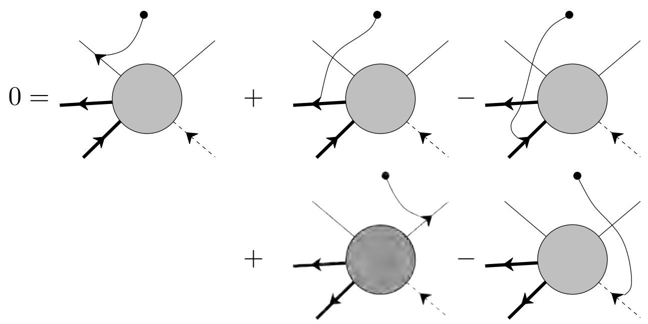
This forest of indices vanishes in the birdtrack notation, enabling us to visualize the formula for the generators of infinitesimal transformations for any tensor representation:



(10.41)

with a relative minus sign between lines flowing in opposite directions. The reader will recognize this as the Leibnitz rule.

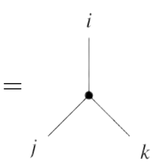
The invariance conditions take a particularly suggestive form in the birdtrack notation. Equation (10.39) amounts to the insertion of a generator into all external legs of the diagram corresponding to the invariant tensor q:



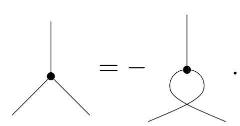
(10.42)

The insertions on the lines going into the diagram carry a minus sign relative to the insertions on the outgoing lines.

As the simplest example of computation of the generators of infinitesimal transformations acting on spaces other than the defining space, consider the adjoint rep. Where does the ugly word "adjoint" come from in this context is not obvious, but remember it this way: this is the one distinguished representation, which is intrinsic to the Lie algebra, with the explicit matrix elements $(T_i)_{jk}$ of the adjoint rep given by the fully antisymmetric structure constants iC_{ijk} of the algebra (i.e., its multiplication table under the commutator product). It's the continuous groups analogue of the multiplication table, or the regular representation for the finite groups. The factor i ensures their reality (in the case of hermitian generators T_i), and we keep track of the overall signs by always reading indices *counterclockwise* around a vertex:

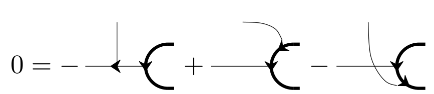
$-iC_{ijk} =$ 

(10.43)

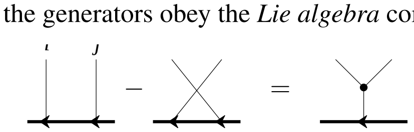


(10.44)

As all other invariant tensors, the generators must satisfy the invariance conditions (10.42):



Redrawing this a little and replacing the adjoint rep generators (10.43) by the structure constants, we find that the generators obey the *Lie algebra* commutation relation

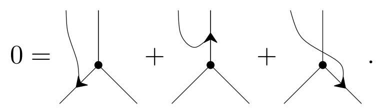


(10.45)

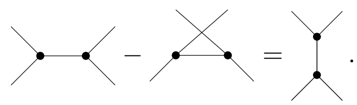
In other words, the Lie algebra commutator

$$T_iT_j \quad - \quad T_jT_i \quad = \quad iC_{ijk}T_k\,. \tag{10.46}$$

is simply a statement that T_i, the generators of invariance transformations, are themselves invariant tensors. Now, honestly, do you prefer the three-birdtracks equation (10.45), or the mathematician's page-long definition of the adjoint rep? It's a classic example of bad notation getting into way of understanding a relation of beautiful simplicity. The invariance condition for structure constants C_{ijk} is likewise



Rewriting this with the dot-vertex (10.43), we obtain



(10.47)

This is the Lie algebra commutator for the adjoint rep generators, known as the *Jacobi relation* for the structure constants

$$C_{ijm}C_{mkl} - C_{ljm}C_{mki} = C_{iml}C_{jkm}\,. \tag{10.48}$$

Hence, the Jacobi relation is also an invariance statement, this time the statement that the structure constants are invariant tensors.

10.8 Nobel Prize in Physics 2020 (optional)

Students –really, anybody who has learned some physics– often ask me: is space continuous or discrete?

We do not know, but this week's SO$(3) \approx$ SU(2) correspondence is one of the gateway drugs to speculations about quantum underpinnings of the observed spacetime. It start's with Hamilton's quaternions - the discovery that the building blocks of our apparent 3 Euclidian dimensions are 2-dimensional complex spin 1/2 'spinors', and it leads different people to different theories of quantum spacetime - one direction is the one taken by David Ritz Finkelstein, another one leads to Roger Penrose's description of Minkowski spacetime in terms of twistors.

In what follows, Erin Wells Bonning from Emory University and Predrag Cvitanović from the Georgia Tech explain the 2020 Nobel Prize in physics in terms accessible to all.

A half of the 2020 Nobel Prize in Physics was awarded to Roger Penrose, for the discovery that black hole formation is a robust prediction of the general theory of relativity. In 1957 Penrose, then a graduate student, met Georgia Tech's late David Ritz Finkelstein in a fateful meeting that changed both men's lives forever after. It was Finkelstein's extension of the Schwarzschild metric which provided Penrose with an opening into general relativity and set him on the path to his 1965 discovery celebrated by this year's prize.

A half of the 2020 Nobel Prize in Physics was awarded jointly to Reinhard Genzel and Andrea Ghez for the discovery of –in Ghez's words- "The Monster at the heart of the Milky Way," a black hole whose existence had been hypothesized since the early 1970s. In order to visually observe an object that famously does not emit any light, precise measurements of stars moving in the black hole's gravitational field had to be carried out. The independent work of Genzel and Ghez mapping the positions of these stars over many years has led to the clearest evidence yet that the center of our Milky Way galaxy contains "The Monster", that possibly every galaxy contains a black hole, and that the environment near it looks nothing like what was expected.

- *Nobel Lecture: Roger Penrose, Nobel Prize in Physics 2020 (34 min)*
- *Nobel Prize in Physics 2020 (56 min)*
  - Abstract
  - Penrose slides for Predrag's 1/2 of the presentation
  - 2020 Nobel Prizes in Chemistry and Physics, Explained
- *Roger Penrose gets Nobel Prize. How David Ritz Finkelstein and Roger Penrose met, and exchanged their lives' paths.*
- *Negative dimensions (6 min)*
- *Andrea Ghez: "The Monster at the Heart of our Galaxy"*
- *Veritasium: "The Infinite Pattern That Never Repeats"*

10.8.1 Quaternionic speculations

Predrag: putting this here for a further re-examination - safely ignored:)

Marek Danielewski (AGH), December 29, 2020, and L. Sapa: Foundations of the Quaternion Quantum Mechanics *Foundations of the Quaternion Quantum Mechanics*, Entropy, **2020**, 22, 1424:

"We show that quaternion quantum mechanics has well-founded mathematical roots and can be derived from the model of the elastic continuum by Cauchy, i.e., it can be regarded as representing the physical reality of elastic continuum. Starting from the Cauchy theory (classical balance equations for isotropic Cauchy-elastic material) and using the Hamilton quaternion algebra, we present a rigorous derivation of the quaternion form of the non- and relativistic wave equations. The family of the wave equations and the Poisson equation are a straightforward consequence of the quaternion representation of the Cauchy model of the elastic continuum. This is the most general kind of quantum mechanics possessing the same kind of calculus of assertions as conventional quantum mechanics. The problem of the Schrödinger equation, where imaginary 'i' should emerge, is solved. This interpretation is an attempt to describe the ontology of quantum mechanics, and demonstrates that, besides Bohmian mechanics, the complete ontological interpretations of quantum theory exists."

It has a quack feel to it, but should be easy to work through...

For a different approach, straightforward, no quackery, see Pavel A. Bolokhov *Quaternionic wave function* arXiv:1712.04795: " quaternions form a natural language for the description of quantum-mechanical wave functions with spin. We use the quaternionic spinor formalism which is in one-to-one correspondence with the usual spinor language. No unphysical degrees of freedom are admitted, in contrast to the majority of literature on quaternions. We build a Dirac Lagrangian in the quaternionic form, derive the Dirac equation and take the nonrelativistic limit to find the Schrödinger's equation. We show that the quaternionic formalism is a natural choice to start with, while in the transition to the noninteracting nonrelativistic limit, the quaternionic description effectively reduces to the regular complex wave function language. We provide an easy-to-use grammar for switching between the ordinary spinor language and the description in terms of quaternions. As an illustration of the broader range of the formalism, we also derive the Maxwell's equation from the quaternionic Lagrangian of Quantum Electrodynamics. In order to derive the equations of motion, we develop the variational calculus appropriate for this formalism. "

Commentary:

Manfried Faber, Richard Gill Quaternions were invented by Benjamin Olinde Rodrigues, before Hamilton. (He is also known for Rodrigues formula for Legendre polynomials.) In 1840 he published a result on transformation groups which applied Leonhard Euler's four squares formula, a precursor to the quaternions of William Rowan Hamilton, to the problem of representing rotations in space. In 1846 Arthur Cayley acknowledged Euler's and Rodrigues' priority describing orthogonal transformations.

Manfried Faber MathsHistory.st-andrews: In 1840 Rodrigues published a mathematical paper which contains the second result for which he is known today, namely his work on transformation groups where he derived the formula for the composition of successive finite rotations by an entirely geometric method. Rodrigues' composition of rotations is basically the composition of unit quaternions.

Predrag I teach my students that SU(2) is double cover of SO(3) and do not do more with quaternions. Octonions is another story...

Richard Gill According to Stigler's law of eponomy, everything worth remembering is associated with the name of someone we want to remember, who did something else.

10.9 What *really* happened (optional)

They do not make Norwegians as they used to. In his brief biographical sketch of Sophus Lie, Burkman writes: "I feel that I would be remiss in my duties if I failed to mention avery interesting event that took place in Lie's life. Klein (a German) and Lie had moved to Paris in the spring of 1870 (they had earlier been working in Berlin). However, in July 1870, the Franco-Prussian war broke out. Being a German alien in

France, Klein decided that it would be safer to return to Germany; Lie also decided to go home to Norway. However (in a move that I think questions his geometric abilities), Lie decided that to go from Paris to Norway, he would walk to Italy (and then presumably take a ship to Norway). The trip did not go as Lie had planned. On the way, Lie ran into some trouble–first some rain, and he had a habit of taking off his clothes and putting them in his backpack when he walked in the rain (so he was walking to Italy in the nude). Second, he ran into the French military (quite possibly while walking in the nude) and they discovered in his sack (in addition to his hopefully dry clothing) letters written to Klein in German containing the words 'lines' and 'spheres' (which the French interpreted as meaning 'infantry' and 'artillery'). Lie was arrested as a (insane) German spy. However, due to intervention by Gaston Darboux, he was released four weeks later and returned to Norway to finish his doctoral dissertation."

10.10 Birdtracks - updated history

Predrag Cvitanović November 7, 2018

Young tableaux and (non-Hermitian) Young projection operators were introduced by Young [26] in 1933 (Tung monograph [25] is a standard exposition). In 1937 R. Brauer [5] introduced diagrammatic notation for δ_{ij} in order to represent "Brauer algebra" permutations, index contractions, and matrix multiplication diagrammatically. R. Penrose's papers were the first to cast the Young projection operators into a diagrammatic form. In 1971 monograph [18] Penrose introduced diagrammatic notation for symmetrization operators, Levi-Civita tensors [20], and "strand networks" [17]. Penrose credits Aitken [2] with introducing this notation in 1939, but inspection of Aitken's book reveals a few Brauer diagrams for permutations, and no (anti)symmetrizers. Penrose's [19] 1952 initial ways of drawing symmetrizers and antisymmetrizers are very aesthetical, but the subsequent developments gave them a distinctly ostrich flavor [19]. In 1974 G. 't Hooft introduced a double-line notation for U(n) gluon group-theory weights [1]. In 1976 Cvitanović [9] introduced analogous notation for SU(N), SO(n) and Sp(n). For several specific, few-index tensor examples, diagrammatic Young projection operators were constructed by Canning [6], Mandula [15], and Stedman [23].

The 1975–2008 Cvitanović diagrammatic formulation of the theory of all semisimple Lie groups [11] as a way to compute group theoretic wights without any recourse to symbols goes conceptually and profoundly beyond the Penrose notation (indeed, Cvitanović "birdtracks" bear no resemblance to Penrose's "fornicating ostriches" [19]).

A chapter in Cvitanović 2008 monograph [11] sketches how birdtrack (diagrammatic) Young projection operators for arbitrary irreducible representation of SU(N) could be constructed (this text is augmented by a 2005 appendix by Elvang, Cvitanović and Kennedy [13] which, however, contains a significant error). Keppeler and Sjödahl [14] systematized the construction by offering a simple method to construct Hermitian Young projection operators in the birdtrack formalism. Their iteration is easy to understand, and the proofs of Hermiticity are simple. However, in practice, the algorithm is inefficient - the expression balloon quickly, the Young projection operators soon become unwieldy and impractical, if not impossible to implement.

The Alcock-Zeilinger algorithm, based on the simplification rules of ref. [3], leads to explicitly Hermitian and drastically more compact expressions for the projection operators than the Keppeler-Sjödahl algorithm [14]. Alcock-Zeilinger fully supersedes Cvitanović's formulation, and any future full exposition of reduction of SU(N) tensor products into irreducible representations should be based on the Alcock-Zeilinger algorithm.

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Exercises

10.1. **Conjugacy classes of SO(3):** Show that all SO(3) rotations (10.7) by the same angle θ around any rotation axis $\boldsymbol{n}$ are conjugate to each other:

$$e^{i\phi \boldsymbol{n}_2\cdot\boldsymbol{L}} e^{i\theta \boldsymbol{n}_1\cdot\boldsymbol{L}} e^{-i\phi \boldsymbol{n}_2\cdot\boldsymbol{L}} = e^{i\theta \boldsymbol{n}_3\cdot\boldsymbol{L}} \tag{10.49}$$

Check this for infinitesimal ϕ, and argue that from that it follows that it is also true for finite ϕ. Hint: use the Lie algebra commutators (10.9).

10.2. **The character of SO(3) 3-dimensional representation:** Show that for the 3-dimensional special orthogonal representation (10.7), the character is

$$\chi = 2\cos(\theta) + 1\,. \tag{10.50}$$

Hint: evaluate the character explicitly for $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$, then explain what is the intuitive meaning of 'class' for rotations.

10.3. **The orthonormality of SO(3) characters:** Verify that given the Haar measure (10.13), the characters (10.12) are orthogonal:

$$\langle\chi(j)|\chi(j')\rangle = \int_G dg\, \chi^{(j)}(g^{-1})\, \chi^{(j')}(g) = \delta_{jj'}\,. \tag{10.51}$$

10.4. **U(1) equivariance of two-modes system for finite angles:** Show that the vector field in two-modes system (10.2) is equivariant under (10.22), the unitary group U(1) acting on $\mathbb{R}^4 \cong \mathbb{C}^2$ as the $k = 1$ and 2 modes:

$$g(\theta)(z_1, z_2) = (e^{i\theta} z_1, e^{i2\theta} z_2), \quad \theta \in [0, 2\pi). \tag{10.52}$$

10.5. **Integrate the two-modes system:** Integrate (10.5) and plot a long trajectory of two-modes in the 4d state space, (x_1, y_1, y_2) projection, as in figure 10.4 (a). To save you time (typing in (10.5) is tedious), we have prepared for you python code, and online graded problem set here. If you do this exercise, please get started early, in order to make sure that the autograder is working, and forward to us the grades that you receive from the autograder.

10.6. **SO(2) or harmonic oscillator slice:** Construct a moving frame slice for action of SO(2) on $\mathbb{R}^2$

$$(x, y) \mapsto (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$$

by, for instance, the positive y axis: $x = 0$, $y > 0$. Write out explicitly the group transformation that brings any point back to the slice. What invariant is preserved by this construction?

10.7. **Invariant subspace of the two-modes system:** Show that $(0, 0, x_2, y_2)$ is a flow invariant subspace of the two-modes system (10.5), i.e., show that a trajectory with the initial point within this subspace remains within it forever.

10.8. **Slicing the two-modes system:** Choose the simplest slice template point that fixes the 1. Fourier mode,

$$\hat{x}' = (1, 0, 0, 0). \tag{10.53}$$

(a) Show for the two-modes system (10.5), that the velocity within the slice, and the phase velocity along the group orbit are

$$\hat{v}(\hat{x}) = v(\hat{x}) - \dot{\phi}(\hat{x})t(\hat{x}) \tag{10.54}$$

$$\dot{\phi}(\hat{x}) = -v_2(\hat{x})/\hat{x}_1 \tag{10.55}$$

(b) Determine the chart border (the locus of point where the group tangent is either not transverse to the slice or vanishes).

(c) What is its dimension?

(d) What is its relation to the invariant subspace of exercise 10.7?

(e) Can a symmetry-reduced trajectory cross the chart border?

10.9. **The symmetry reduced two-modes flow:** Pick an initial point $\hat{x}(0)$ that satisfies the slice condition for the template choice (10.53) and integrate (10.54) & (10.55). Plot the three dimensional slice hyperplane spanned by (x_1, x_2, y_2) to visualize the symmetry reduced dynamics. Does it look like figure 10.4 (b)?

group theory - week 11

SU(2) and SO(3)

Homework HW11

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 11.1 *The characters of SO(3) representations* 4 points

Bonus points
Exercise 11.2 *Real and pseudo-real representations of SO(3)* 4 points
Exercise 11.3 *Total spin of N particles* 5 points

Total of 4 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2021-07-01 Predrag Lecture 21 SU(2) and SO(3)

The fastest way to watch any week's lecture videos is by letting YouTube run the

- *lecture playlist*
- Gutkin notes, Lect. 9 *SU(2), SO(3) and their representations*, Sects. 1-3.2.
- *14.1 Recap: Irreps of SO(2)* are not what you would have expected (24:37 min)
- *14.2 Defining reps of SO(3) and SU(2)* (5:18 min)
- *14.3 Cartan root lattices; irreps of SO(3)* (22:50 min)
- (optional) Anthony Zee [3] *Group Theory in a Nutshell for Physicists*: Cartan classification of Lie algebras and Dynkin diagrams (5 lecture course)
  - Read sect. 11.2 SU(2) $\simeq$ SO(3)
  - For overall clarity and pleasure of reading, I like Schwichtenberg [2] (click here) discussion best. If you read anything for this week's lectures, read Schwichtenberg sects. 3.4 to 3.6

2021-07-01 Predrag Lecture 22 SO(3) in QM

- *Andrew Scherbakov: Eigenvalues of J^2 and J_z operators* (10:56 min)
- *Andrew Scherbakov: Raising, lowering operators J_+, J_-* (10:40 min)
  - (optional) *14.3A Who ordered J+, J-?* (7:37 min)
- *Andrew Scherbakov: J_x, J_y, and J_z operators for spin-1 particle* (3:58 min)

11.1 Discussion (optional)

- (Still to be uploaded) *14.4 Discussion: How is SU(2) a double cover of SO(3) and what are the physical consequences? For dimensions higher than four, SO(n) and SU(n) get a divorce, and so far quaternions, octonion ideas have not panned out; instead, particle physics has followed ideas of internal SU(2), SU(3) etc symmetries, that currently culminate in the Standard Model. Negative dimensions.* (X:XX min)

11.1.1 Recap of the course, so far (optional)

Predrag This course is all about *class* (physically distinct symmetry operations) and *character* (mining numbers from symmetries).

Here are some question by the dream student Henriette Roux (pseudonym) that I have answered in part in class discussions, but still have to write up:

Henriette Roux Why is it that the Fourier transformation works? The presence of a discrete but infinite translational symmetry in a system calls for its use of it to diagonalize the matrix and thus make calculations easier, but exactly *why* is the Fourier transform able to do this?

Henriette Roux How is this Fourier transform as we have studied in the space/point groups section related to that which we have derived from the projection operators?

Henriette Roux As an extension of the Fourier transform, are there any equivalent of Fourier transforms for rotations or other infinite but discrete symmetries as well? So for example, if there is a system with a discrete but infinite rotational symmetry, is there a "rotational" transform where the representing matrix is diagonalized? Are there whole classes of such transformations?

Henriette Roux You say that position and momentum are "dual" to each other, and so is the real space and reciprocal space (I guess it's the same thing as position and momentum but just for argument sake). The commonality between these are the fact that they can be Fourier transformed from one space to another. Does this mean that unitary operations, e^{iHt}, suggest a Fourier transform from the "energy" or "frequency" space to "time" space as well?

Henriette Roux This seems very closely related to Noether's theorem as well, is there a way to explain this similarity?

Henriette Roux The special thing about Lie groups is that there exist analytic functions which link $g(a)$ and $c = f(b, a)$ for $g(c) = g(a)g(b)$. Does this need for analytic functions come from the fact that to construct a group manifold, the maps relating different "local" Euclidean spaces need to be C^∞, or smooth? If so, is there a reference we can refer to which explains how the Lie groups satisfy all the other conditions of a manifold (establishing an open ball, building an atlas and so on) as well? Just as an extension, how do you even study groups which do not fall under the realm of a manifold? Don't common functions like differentials and integrations not apply in spaces outside a manifold?

Henriette Roux Why is that we Taylor expand the group in the first place? How is this connected to the shift to left/right group operators?

The next few questions are about General Relativity, and how is what is covered in this course applicable to GR:

Henriette Roux We keep to the first order in the expansion for $g(\theta)$ as we are considering the tangent space to the manifold. In the context of the GR, the tangent space was defined as the space of directional derivatives at a point. In our case, we are studying groups, which are not, in general, vectors (well I guess they can be [1×1] vectors/matrices but that's only specific irreps, so how do we understand the concept of tangent space as you have define it?

- Or does it work out since Lie groups are always Abelian and thus have an infinite number of 1D irreps?

- What happens if we keep the expansion to the 2nd order? Does the mathematics change in any way? Is there a good reason to ignore the 2nd and higher order expansions, not just in the physics sense (keeping to largest order of significance) but in the mathematical way of understanding things?

Question 11.1. Henriette Roux asks
Q Why is this complex 2-dimensional vector called a 'spinor'?
A Historical, as Arfken, Weber & Harris [1] explain: "It turns out that half-integral angular momentum states are needed to describe the intrinsic angular momentum of the electron and many other particles. Since these particles also have magnetic moments, an intuitive interpretation is that their charge distributions are spinning about some axis; hence the term spin. It is now understood that the spin phenomena cannot be explained consistently by describing these particles as ordinary charge distributions undergoing rotational motion, [...] "

Schwichtenberg [2]: "[...] spinors have properties that usual vectors do not have. For instance, the factor $1/2$ in the exponent. This factor shows us that a spinor1 is after a rotation by 2π not the same, but gets a minus sign. This is a pretty crazy property, because all objects we deal with in everyday life are exactly the same after a rotation by $360^o = 2\pi$.

11.2 SU(2) and SO(3)

K. Y. Short

An element of $\mathrm{SU}(2)$ can be written as

$$U_{\mathbf{n}}(\phi) = e^{i\phi\,\sigma\cdot\hat{\mathbf{n}}/2} \tag{11.1}$$

where σ_j is a Pauli matrix and ϕ is a real number. What is the significance of the $1/2$ factor in the argument of the exponential?

Consider a generic position vector $\boldsymbol{x} = (x, y, z)$ and construct a Hermitian matrix of the form

$$\begin{aligned}\sigma\cdot\boldsymbol{x} &= \sigma_x x + \sigma_y y + \sigma_z z \\ &= \begin{pmatrix} 0 & x \\ x & 0\end{pmatrix} + \begin{pmatrix} 0 & -iy \\ iy & 0\end{pmatrix} + \begin{pmatrix} z & 0 \\ 0 & -z\end{pmatrix} \\ &= \begin{pmatrix} z & x-iy \\ x+iy & -z\end{pmatrix}\end{aligned} \tag{11.2}$$

Its determinant

$$\det\begin{pmatrix} z & x-iy \\ x+iy & -z\end{pmatrix} = -(x^2+y^2+z^2) = -\boldsymbol{x}^2 \tag{11.3}$$

gives the length of a vector. Consider a SU(2) transformation (11.1) of this matrix, $U^\dagger(\sigma \cdot \boldsymbol{x})U$. Taking the determinant, we find the same expression as before:

$$\det U(\sigma \cdot \boldsymbol{x})U^\dagger = \det U \det(\sigma \cdot \boldsymbol{x}) \det U^\dagger = \det(\sigma \cdot \boldsymbol{x}) . \tag{11.4}$$

Just as SO(3), SU(2) preserves the lengths of vectors.

To make the correspondence between SO(3) and SU(2) more explicit, consider a SU(2) transformation on a complex two-component *spinor*

$$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{11.5}$$

related to $\boldsymbol{x}$ by

$$x = \frac{1}{2}(\beta^2 - \alpha^2), \quad y = -\frac{i}{2}(\alpha^2 + \beta^2), \quad z = \alpha\beta \tag{11.6}$$

Check that a SU(2) transformation of ψ is equivalent to a SO(3) transformation on $\boldsymbol{x}$. From this equivalence, one sees that a SU(2) transformation has three real parameters that correspond to the three rotation angles of SO(3). If we label the "angles" for the SU(2) transformation by α, β, and γ, we observe, for a "rotation" about $\hat{x}$

$$U_x(\alpha) = \begin{pmatrix} \cos\alpha/2 & i\sin\alpha/2 \\ i\sin\alpha/2 & \cos\alpha/2 \end{pmatrix} , \tag{11.7}$$

for a "rotation" about $\hat{y}$,

$$U_y(\beta) = \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix} , \tag{11.8}$$

and for "rotation" about $\hat{z}$,

$$U_z(\gamma) = \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} . \tag{11.9}$$

Compare these three matrices to the corresponding SO(3) rotation matrices:

$$R_x(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\zeta & \sin\zeta \\ 0 & -\sin\zeta & \cos\zeta \end{pmatrix} , \quad R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$
$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.10}$$

They're equivalent! Result: *Half the rotation angle generated by SU(2) corresponds to a rotation generated by SO(3).*

What does this mean? At this point, probably best to switch to Schwichtenberg [2] (click here) who explains clearly that SU(2) is a simply-connected group, and thus the "mother" or covering group, or the double cover of SO(3). This means there is a two-to-one map from SU(2) to SO(3); an SU(2) turn by 4π corresponds to an SO(3) turn by 2π. So, the building blocks of your 3-dimensional world are not 3-dimensional real vectors, but the 2-dimensional complex spinors! Quantum mechanics chose electrons to be spin 1/2, and there is nothing Fox Channel can do about it.

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Exercises

11.1. **The characters of SO(3) representations:** Show that for an irrep labeled by j, the character of a conjugacy class labeled by θ

$$\chi^{(j)}(\theta) = \frac{\sin(j+1/2)\theta}{\sin(\theta/2)} \tag{11.11}$$

can be obtained by taking the trace of $R_z^j(\theta)$. Verify that for $j = 1$ this character is the three dimensional special orthogonal representation character (10.11).

11.2. **Real and pseudo-real representations of SO(3).** Recall (Gutkin notes, sect. 4.5 *Representation Theory II*, Sect. 5 *5. Three types of representations*) that there are exist three types of representation which can be distinguished by the indicator (4.6):

$$\int_G d\mu(g)\chi^{(\ell)}(g^2) = \begin{cases} +1 & \text{real} \\ 0 & \text{complex} \\ -1 & \text{pseudo-real} \end{cases} . \tag{11.12}$$

Determine for which values of $\ell = 0, 1/2, 1, 3/2, 2\ldots$ the representation D_ℓ of SO(3) is real or pseudo-real.

Hint: The characters and Haar measure (10.13) of SO(3) are given by

$$\chi^{(\ell)}(g) = \frac{\sin\left(\left[\ell+\frac{1}{2}\right]\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}, \qquad d\mu(g) = \frac{d\theta}{\pi}\sin^2(\theta/2) \tag{11.13}$$

where θ is rotation angle for the group element g.

(B. Gutkin)

11.3. **Total spin of N particles.** Consider a system of four particles with spin $1/2$. Assuming that all (except spin) degrees of freedom are frozen the Hilbert space of the system is given by $V = V_{1/2} \otimes V_{1/2} \otimes V_{1/2} \otimes V_{1/2}$, with $V_{1/2}$ being two-dimensional space for each spin. $V = \oplus V_s$ can be decomposed then into different sectors V_s having the total spin s i.e., $\hat{S}^2 v = s(s+1)v$, for any $v \in V_s$. Here $\hat{S}^2 = (\sum_{i=1}^4 \hat{s}_i)^2$ and $\hat{s}_i = (\hat{s}_i^x, \hat{s}_i^y, \hat{s}_i^z)$ is spin operator for i-th particle.

(a) What are possible values s for the total spin of the system?

(b) Determine dimension of the subspace of V_0 with 0 total spin. In other words: how many times trivial representation enters into product:

$$D = D_{1/2} \otimes D_{1/2} \otimes D_{1/2} \otimes D_{1/2}\,? \tag{11.14}$$

(c) What is the answer to the above questions for N spins?

Hint: it is convenient to use (11.13) to decompose D into irreps.

(B. Gutkin)

11.4. **Splitting of degeneracies in a central potential.** Hamiltonian H_0 has rotational symmetry of SO(3).

(a) What are the possible energy level degeneracies of H_0?

A weak perturbation V with a symmetry T_d of full tetrahedron group is added (e.g., V is a potential created by lattice of atoms with a symmetry of T_d).

(b) What will be the degeneracies of new Hamiltonian $H_0 + V$?

(c) Assuming that the total angular momentum of the system before the perturbation is $l = 2$. How the degeneracies of the corresponding energy level will be split after the perturbation is applied?

(B. Gutkin)

11.5. **Quadrupole transitions.**

a) Write $Q_1 = xy$, $Q_2 = zy$, $Q_3 = x^2 - y^2$ and $Q_4 = 2z^2 - x^2 - y^2$ as components of spherical tensor of rank 2. *Hint:* use spherical harmonics $Y_l^m(\theta, \varphi)$.

b) The last quantity Q_4 is known as quadrupole moment. What are the selection rules for transitions induced by Q_4 in a system with SO(3) symmetry? In other words, for which m, l and k, j the transition rates:

$$P_{m,l\to k,j} \sim |\langle m\, l|Q_4|j\, k\rangle|^2$$

are non-zero?

c) By using Wigner-Eckart theorem write down the relationship between $|\langle m\, l|Q_4|j\, k\rangle|^2$ and $|\langle m\, l|Q_1|j\, k\rangle|^2$ in terms of Clebsch-Gordan coefficients.

(B. Gutkin)

group theory - week 12

Lorentz group; spin

Homework HW12

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 12.1 *Lie algebra of $SO(4)$ and $SU(2) \otimes SU(2)$*	6 points
Exercise 12.2 SO(n) Clebsch-Gordan series for $V \otimes V$.	3 points
Exercise 12.3 *Lorentz spinology*	5 points
Exercise 12.4 *Lorentz spin transformations*	5 points

Bonus points

Exercise 12.5 *The unbearable lightness of $SO(4)$ Lie algebra*	15 points

Total of 19 points = 100 % score.

2021-07-08 Predrag Lecture 23
$\mathrm{SO}(4) = \mathrm{SU}(2) \otimes \mathrm{SU}(2)$; Lorentz group

Lecture 15 (Unedited) $\mathrm{SU}(2)$ irreps. $\mathrm{SO}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$. More importantly: Minkowski metric, Lorentz group $\mathrm{SO}(1,3)$ irreps are also labeled by pairs of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ irrep labels. (2:29:20 h)

Gutkin notes sect. 9.2 *Representations of $SU(2)$ and $SO(3)$*.

Gutkin notes sects. 9.5-9.8 *Product representations of $SO(3)$*

∘ sect. 12.3 *Spinors and the Lorentz group*

For Lorentz group, read Schwichtenberg [3] Sect. 3.7 (click here).

2021-07-13 Predrag Lecture 24 $\mathrm{SO}(1,3)$; Spin

12.1 Other sources (optional)

∘ sect. 12.4 *Irreps of $SO(n)$*

14.3A Who ordered J+, J-? (7:37 min)

For $\mathrm{SO}(n)$ see also birdtracks.eu Chapt. 10 *Orthogonal groups*, pp. 121-123.

For $\mathrm{SO}(4) = \mathrm{SU}(2) \otimes \mathrm{SU}(2)$ see also birdtracks.eu sect. 20.3.1 *$SO(4)$ or Cartan $A_1 + A_1$ algebra*.

∘ sect. 12.5 *$SO(4)$ of the Kepler problem*

∘ sect. 12.5.1 *Central force problems*

Peter Voit [5] *Quantum Theory, Groups and Representations* (2017) has a nice calculation of spherical harmonics as $\mathrm{SO}(3)$ eigenvectors in polar coordinates. Used as problem set #6 for *Quantum Mechanics I Georgia Tech PHYS-6105*, November 17, 2022, should make it a section in these notes. S

John Wood's (click here) notes and exercise 12.5 *The unbearable lightness of $SO(4)$ Lie algebra.* The challenge: achieve some elegance in deriving the $\mathrm{SO}(4)$ commutator relations.

Ilya Kuprov [2] 'What exactly is spin?' (40 min). Starts out with very entertaining bits of physics history. Wlad Sobol writes: "Kuprov derives the Dirac equation from Wigner symmetry theorem, from a product of two Casimir operators of the Poincaré group." Revisit sect. 10.8.1 *Quaternionic speculations* for a different point of view.

12.2 Discussion (optional)

Henriette Roux In this course the Levi-Civita tensor appears to be the unique connection for SO(4); but in GR, I learnt that the choice of connection is actually arbitrary and there are theories of gravity which need not use the Levi-Civita tensor. Are these two different concepts which are not necessarily linked?

Predrag Sean Carroll answers your question in arXiv:9712019. He does not understand that the *invariant* tensors are good, as they are what *defines* a given symmetry group:

> *It is a remarkable property of the above tensors – the metric, the inverse metric, the Kronecker delta, and the Levi-Civita tensor – that, even though they all transform according to the tensor transformation law, their components remain unchanged in any Cartesian coordinate system in flat spacetime. In some sense this makes them bad examples of tensors, since most tensors do not have this property.*

However, he then goes on to explain that while in curved spacetime lengths and volumes are measured in the spacetime dependent way, we still need a notion of a volume of a hypercube as a skew product of its edges, ie, the determinant:

> *The Kronecker tensor can be thought of as the identity map from vectors to vectors (or from dual vectors to dual vectors), which clearly must have the same components regardless of coordinate system. The other tensors (the metric, its inverse, and the Levi-Civita tensor) characterize the structure of spacetime, and all depend on the metric. We shall therefore have to treat them more carefully when we drop our assumption of flat spacetime.*

What he then does in his eq. (2.39) is to promote Levi-Civita from 'tensor' to 'symbol' in order to be able to compute determinants, just like we do in flat space SO(n).

See also MathWorld discussion.

Are you happy now?

(A side, nomenclature remark: Levi-Civita is not a 'connection' in the sense the word 'connection' is used in GR.)

12.3 Spinors and the Lorentz group

A Lorentz transformation is any invertible real $[4 \times 4]$ matrix transformation Λ,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{12.1}$$

which preserves the Lorentz-invariant Minkowski bilinear form $\Lambda^T \eta \Lambda = \eta$,

$$x^{\mu} y_{\mu} = x^{\mu} \eta_{\mu\nu} y^{\nu} = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

with the metric tensor $\eta = diag(1,-1,-1,-1)$.

A contravariant four-vector $x^\mu = (x^0, x^1, x^2, x^3)$ can be arranged [4] into a Hermitian $[2\times2]$ matrix in $Herm(2,\mathbb{C})$ as

$$\underline{x} = \sigma_\mu x^\mu = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^3 \end{pmatrix} \tag{12.2}$$

in the hermitian matrix basis

$$\sigma_\mu = \bar\sigma^\mu = (\mathbb{1}_2, \boldsymbol{\sigma}) = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)\,, \quad \bar\sigma_\mu = \sigma^\mu = (\mathbb{1}_2, -\boldsymbol{\sigma})\,, \tag{12.3}$$

with $\boldsymbol{\sigma}$ given by the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{12.4}$$

With the trace formula for the metric

$$\frac{1}{2}\mathrm{tr}\,(\sigma_\mu \bar\sigma_\nu) = \eta_{\mu\nu}\,, \tag{12.5}$$

the covariant vector x^μ can be recovered by

$$\frac{1}{2}\mathrm{tr}\,(\underline{x}\bar\sigma^\mu) = \frac{1}{2}\mathrm{tr}\,(x^\nu \sigma_\nu \bar\sigma^\mu) = x^\nu \eta_\nu{}^\mu = x^\mu \tag{12.6}$$

The Minkowski norm squared is given by

$$\det \underline{x} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = x_\mu x^\mu\,, \tag{12.7}$$

and with (12.3)

$$\overline{x} = \begin{pmatrix} x^0-x^3 & -x^1+ix^2 \\ -x^1-ix^2 & x^0+x^3 \end{pmatrix} = \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix}\,, \tag{12.8}$$

the Minkowski scalar product is given by

$$x^\mu y_\mu = \frac{1}{2}\mathrm{tr}(\underline{x}\,\overline{y})\,. \tag{12.9}$$

The *special linear group* $SL(2,\mathbb{C})$ in two complex dimensions is given by the set of all matrices Λ such that

$$SL(2,\mathbb{C}) = \{\Lambda \in GL(2,\mathbb{C})\,|\,\det\Lambda = +1\}. \tag{12.10}$$

Let a matrix $\Lambda \in SL(2,\mathbb{C})$ act on $\underline{x} \in Herm(2,\mathbb{C})$ as

$$\underline{x} \mapsto \underline{x}' = \Lambda \underline{x} \Lambda^\dagger \tag{12.11}$$

where † denotes Hermitian conjugation. The Minkowski scalar product is preserved, $\det\,\underline{x}' = \det\,\underline{x}$. Thus $\underline{x}'$ can also be represented by a real linear combination of generalized Pauli matrices

$$\underline{x}' = \sigma_\mu x'^\mu \quad \text{with } x'_\mu x'^\mu = x_\mu x^\mu \tag{12.12}$$

and Λ explicitly acts as a Lorentz transformation (12.1), with $\Lambda^{\mu}{}_{\nu} = \frac{1}{2}\,\mathrm{tr}\,(\bar{\sigma}^{\mu}\Lambda\sigma_{\nu}\Lambda^{\dagger})$. The mapping is two-to-one, as two matrices $\pm\Lambda \in SL(2,\mathbb{C})$ generate the same Lorentz transformation $\Lambda\underline{x}\Lambda^{\dagger} = (-\Lambda)\underline{x}(-\Lambda)^{\dagger}$. This Λ belong to the proper orthochronous Lorentz group $SO^{+}(1,3)$, and it can be shown that $SL(2,\mathbb{C})$ is simply connected and is the double universal cover of the $SO^{+}(1,3)$.

Consider the fully antisymmetric Levi-Civita tensor $\varepsilon = -\varepsilon^{-1} = -\varepsilon^{\mathsf{T}}$ in two dimensions

$$\varepsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{12.13}$$

This defines a *symplectic* (i.e., *skew-symmetric*) bilinear form $\langle u, v\rangle = -\langle v, u\rangle$ on two *spinors* u and v, elements of the two-dimensional complex vector (or spinor) space $\mathbb{C}^2$

$$u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} , \quad v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} , \tag{12.14}$$

equipped with the symplectic form

$$\langle u, v\rangle = u^1 v^2 - u^2 v^1 = u^{\mathsf{T}}\varepsilon v\,. \tag{12.15}$$

This symplectic form is $SL(2,\mathbb{C})$-invariant

$$\langle u, v\rangle = u^{\mathsf{T}}\varepsilon v = \langle \Lambda u, \Lambda v\rangle = u^{\mathsf{T}}\Lambda^{\mathsf{T}}\varepsilon\Lambda v\,, \tag{12.16}$$

so one can interpret the group acting on spinors as $SL(2,\mathbb{C}) \cong \mathrm{Sp}(2,\mathbb{C})$, the complex symplectic group in two dimensions

$$\mathrm{Sp}(2,\mathbb{C}) = \{\Lambda \in GL(2,\mathbb{C})\,|\,\Lambda^{\mathsf{T}}\varepsilon\Lambda = \varepsilon\}\,. \tag{12.17}$$

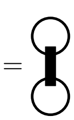
Summary. The group of Lorentz transformations of spinors is the group SL(2,C) of $[2\times 2]$ complex matrices with determinant 1, i.e., the invariant tensor is the 2-index Levi-Civita ε_{AB}. The SL(2,C) matrices are parametrized by three complex dimensions and therefore six real ones (the matrices have four complex numbers and one complex constraint on the determinant). This matches the 6 dimensions of the group manifold associated with the Lorentz group SO$(1,3)$.

Andrew M. Steane writes "A spinor is the most basic mathematical object that can be Lorentz-transformed." His *An introduction to spinors*, arXiv:1312.3824, might help you develop intuition about spinors.

Andrzej Trautman tracks the origin of spinors to Euclid, and General Relativity to Clifford. He includes a letter from Hades saying, inter alia, "Unfortunately, it appears that there is now in your world a race of vampires, called referees, who clamp down mercilessly upon mathematicians unless they know the right passwords."

12.4 Irreps of SO(n) (optional)

The dimension of the defining representation of is SO(n) is given by the trace of the adjoint projection operator:

$$N = \mathrm{tr}\,\mathbf{P}_A =$$  $$= \frac{n(n-1)}{2}\,. \tag{12.18}$$

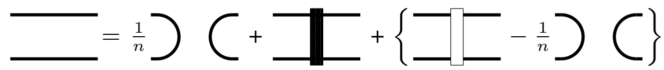
Young tableaux	$\square\times\square$	=	$\bullet$	+	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	+	$\square\square$
Dimensions	n^2	=	1	+	$\frac{n(n-1)}{2}$	+	$\frac{(n+2)(n-1)}{2}$
Projectors							

Table 12.1: SO(n) Clebsch-Gordan series for $V\otimes V$, worked out in detail in *Group Theory – Birdtracks, Lie's, and Exceptional Groups,* birdtracks.eu Chapt. 10 *Orthogonal groups*.

Dimensions of the other reps are listed in table 12.1.

12.5 SO(4) of the Kepler problem (optional)

One of "hidden" symmetries of quantum mechanics is the SO(4) of the Kepler problem.

John Baez discusses it in a fun read here: "if we take the angular momentum together with the Runge–Lenz vector, we get 6 conserved quantities—and these turn out to come from the group of rotations in 4 dimensions, SO(4), which is itself 6-dimensional. The obvious symmetries in this group just rotate a planet's elliptical orbit, while the unobvious ones can also squash or stretch it, changing the eccentricity of the orbit. [...] wavefunctions for bound states of hydrogen can be reinterpreted as functions on the 3-sphere, S^3. The sneaky SO(4) symmetry then becomes obvious: it just rotates this sphere! And the Hamiltonian of the hydrogen atom is closely connected to the Laplacian on the 3-sphere. The Laplacian has eigenspaces of dimensions n^2 where $n = 1, 2, 3, \ldots$, and these correspond to the eigenspaces of the hydrogen atom Hamiltonian. "

When the energy is fixed, the symmetry becomes Lie algebra SO$(3, 1)$ for positive-energy, scattering states, or SO(4) for negative-energy, bound states.

Michele Cini's lecture notes, p. 18 gives hydrogen as an example of why we don't believe in miracles such as "accidental" eigenvalue degeneracies, but assume that we must have missed a "hidden" symmetry. Cini writes: "Wolfgang Pauli in 1926 first solved [...] the H atom using the SO(4) symmetry." I didn't know that it was Pauli...

To dig deeper, skim through Baez *Mysteries of the gravitational 2-body problem.*

Bander and Itzykson [1] *Group theory and the hydrogen atom (I)* might be OK, but I have not read it.

12.5.1 Central force problems (optional)

For another way of looking at the H atom (and all solvable central force problems) download John Wood's chapter (click here) from *Quantum Mechanics for Nuclear Structure: I. A Primer,* IOP science series.

exercise 12.5

The $\mathrm{SO}(2,1)$ method can be extended to solve relativistic central force problems (one of my students did his Ph.D. thesis on this 20 years ago).

Q: Is the geometry associated with these algebraic structures, as applied to central force problems, explored?

References

[1] M. Bander and C. Itzykson, "Group theory and the hydrogen atom (I)", Rev. Mod. Phys. **38**, 330–345 (1966).

[2] I. Kuprov, *Spin: From Basic Symmetries to Quantum Optimal Control* (Springer, New York, 2022).

[3] J. Schwichtenberg, *Physics from Symmetry* (Springer, Berlin, 2015).

[4] E. Wigner, "On unitary representations of the inhomogeneous Lorentz group", Ann. Math. **40**, 149–204 (1939).

[5] P. Woit, *Quantum Theory, Groups and Representations* (Springer, New York, 2017).

Exercises

12.1. **Lie algebra of SO(4) and SU$(2) \otimes$ SU(2).** One particle Hamiltonian with a central potential has in general $\mathrm{SO}(3)$ symmetry group. It turns out, however, that for Coulomb potential the symmetry group is actually larger - $\mathrm{SO}(4)$, rather than $\mathrm{SO}(3)$. This explains why the energy level degeneracies in the hydrogen atom are anomalously large. So $\mathrm{SO}(4)$ and its representations are of a special importance in atomic physics.

(a) Show that the Lie algebra $\mathfrak{so}(4)$ of the group $\mathrm{SO}(4)$ is generated by real antisymmetric 4×4 matrices.

(b) What is the dimension of $\mathfrak{so}(4)$?

A natural basis in $\mathfrak{so}(4)$ is provided by antisymmetric matrices $M_{\mu\nu}, \quad \mu, \nu \in 1, 2, 3, 4$, $\mu \neq \nu$, generators of $\mathrm{SO}(4)$ rotations which leave invariant the $\mu\nu$-plane. The elements of these matrices are given by

$$(M_{\mu\nu})_{ij} = \delta_{i\mu}\delta_{j\nu} - \delta_{j\mu}\delta_{i\nu} \tag{12.19}$$

(c) Check that these matrices satisfy the commutation relationship

$$[M_{ab}, M_{cd}] = M_{ad}\delta_{bc} + M_{bc}\delta_{ad} - M_{ac}\delta_{bd} - M_{bd}\delta_{ac}. \tag{12.20}$$

(d) Show that Lie algebras of the groups $\mathrm{SO}(4)$ and $\mathrm{SU}(2) \times \mathrm{SU}(2)$ are isomorphic. **Path:**

(d.i) Define matrices

$$J_k = \frac{1}{2}\varepsilon_{kij} M_{i,j} , \qquad K_k = M_{k4} , \quad k = 1, 2, 3$$

and

$$\mathcal{A}_k = \frac{1}{2}\left(J_k + K_k\right) \quad \text{and} \quad \mathcal{B}_k = \frac{1}{2}\left(J_k - K_k\right) .$$

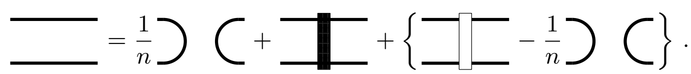
(d.ii) Show that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same commutation relations as two copies of $\mathfrak{su}(2)$.

(e) How does one construct irreps of $\mathfrak{so}(4)$ out of irreps of $\mathfrak{su}(2)$?

(f) Are groups $\mathrm{SO}(4)$ and $\mathrm{SU}(2) \otimes \mathrm{SU}(2)$ isomorphic to each other?

(B. Gutkin)

12.2. **SO(n) Clebsch-Gordan series for $V \otimes V$.**
(a) Show that the product of two n-dimensional reps of SO(n) decomposes into three irreps:


(12.21)

(b) Compute the dimensions of the three irreps.
(c) Which one is the adjoint one, and why? Hint: check the invariance condition. (This is worked out in detail in *Group Theory – Birdtracks, Lie's, and Exceptional Groups,* birdtracks.eu Chapt. 10 *Orthogonal groups*.)

12.3. **Lorentz spinology.**
Show that

(a)

$$x^2 = x_\mu x^\mu = \det \underline{x} \tag{12.22}$$

(b)

$$x_\mu y^\mu = \frac{1}{2}(\det(\underline{x} + \underline{y}) - \det(\underline{x}) - \det(\underline{y})) \tag{12.23}$$

(c)

$$x_\mu y^\mu = \frac{1}{2}\mathrm{tr}\,(\underline{x}\,\overline{y}) , \tag{12.24}$$

where $\overline{y} = \bar{\sigma}_\mu y^\mu$

12.4. **Lorentz spin transformations.**
Let a matrix $\Lambda \in SL(2,\mathbb{C})$ act on hermitian matrix $\underline{x}$ as

$$\underline{x} \mapsto \underline{x}' = \Lambda \underline{x} \Lambda^\dagger . \tag{12.25}$$

(a) Check that $\underline{x}'$ is Hermitian, and the Minkowski scalar product (12.23) is preserved.

(b) Show that Λ explicitly acts as a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

(c) Show that the mapping from a $\Lambda \in SL(2,\mathbb{C})$ to the Lorentz transformation in $\mathrm{SO}(1,3)$ is two-to-one.

(d) Consider the Levi-Civita tensor $\epsilon = -\epsilon^{-1} = -\epsilon^{\mathrm{T}}$ in two dimensions,

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \tag{12.26}$$

and the associated symplectic form

$$\langle u, v \rangle = u^{\mathrm{T}} \varepsilon v = u^1 v^2 - u^2 v^1 \,. \tag{12.27}$$

Show that this symplectic form is $SL(2, \mathbb{C})$-invariant

$$\langle u, v \rangle = u^{\mathrm{T}} \varepsilon v = \langle \Lambda u, \Lambda v \rangle = u^{\mathrm{T}} \Lambda^{\mathrm{T}} \varepsilon \Lambda v \,. \tag{12.28}$$

12.5. **The unbearable lightness of SO(4) Lie algebra.** Download John Wood's (click here) notes. The challenge: achieve some elegance in deriving the SO(4) commutator bracket relations, for example reduce the number of steps in the calculation by 30% or 50%.

The prize: a case of beer, details to be negotiated with John.

The challenges start on p. 9-8, following eq. (9.21), i.e., "(i)", "(iv)", and "(v)". For instance, on p. 9-11 John indicates all of the cancellations. These suggest that his solution is "calculating zero" unnecessarily. One could take linear combinations of the operators that possess these commutator bracket relations; but the combinations do not seem a priori warranted on the basis of the dynamics of the problem.

(J. Wood)

group theory - week 13

Simple Lie algebras; SU(3)

Homework HW13

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 13.1 *Root systems of simple Lie algebras*	5 points
Exercise 13.2 *Meson octet*	5 points

Bonus points

Exercise 13.3 *SU(3) symmetry in 3D harmonic oscillator*	5 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2021-07-13 Predrag Lecture 25 Reps of simple Lie algebras

In week 4 we learned that for finite groups there is one very special matrix rep, the *regular representation* constructed from the group multiplication table, that is intrinsic to the abstract group itself, and whose reduction yields all irreps of a given group.

In week 9 we saw that for continuous groups we need to study the Lie algebra of the finite number of generators T_j, rather than the infinity of group elements $g = \exp(i\phi \cdot T)$. Here the finite group multiplication table is replaced by the 'Lie product', i.e., the table of Lie commutators' fully antisymmetric structure constants iC_{ijk}.

So far we have chosen the hermitian basis T_j. But non-hermitian bases are also OK, as we know from raising / lowering operators of SU(2) of quantum mechanics irrep constructions.

- (Lost unedited 34:45 min file) *16.1 Adjoint representation and Killing form.*
- Gutkin notes, sect. 10.1 *Adjoint representation and Killing form.*
- (Lost unedited 15:13 min file) *16.2 Cartan sub-algebra and roots.*
- Gutkin notes, Sect. 10.2 *Cartan sub-algebra and roots.*
- (19:02 min) *16.3 Root systems.*
- Gutkin notes, Sect. 10.3 *Main properties of root systems.*
- (21:30 min) *16.4 Construction of representations.* SU(2) example; analogy to presentations of finite groups, such as D_n (4.1); 'grand circles' on Cartan lattices.
- Gutkin notes, Sect. 10.4 *Building up representations of* g.

2021-07-15 Predrag Lecture 26 Cartan's SU(3) irreps

- (unedited 2:03:14 h) *Representations of SU*(3).
- Gutkin notes, Sect. 10.5 *Representations of SU*(3).

13.1 Group theory news (optional)

Turns out, applications of group theory go way beyond what is covered in this course:

- Mathematicians map E_8, and it is bigger than the human genome.
- **Group theory of defamation:** The officers argued Sawant's statements impugned them individually even though she only spoke about the police department as a whole. The court says suing as individuals and advancing a group theory of defamation takes far more than the officers showed in their complaint.

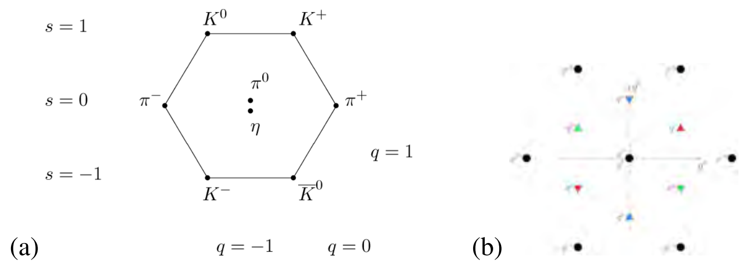


Figure 13.1: (a) The meson (pseudoscalars) octet. (b) The quark triplet, the anti-quark triplet and the gluon octet. (Wikipedia).

> [W]hether proceeding under an individual or group theory, Plaintiffs must plead that the statements "specifically" identified or singled them out, or was understood as "referring to [them] in particular." Sims, 20 Wn. App. at 236.

Exercises

13.1. **Root system of simple Lie algebras.**

a) Determine dimensions of Lie algebras $\mathfrak{so}(N), \mathfrak{su}(N)$ and dimensions of their Cartan subalgebras. What is the number of the positive roots for these Lie algebras?

b) Show that $N \times N$ diagonal matrices H_i with zero traces and uper/lower corner $N \times N$ matrices $E^{(a,b)}$ with the elements $E^{(a,b)}_{i,j} = \delta_{ia}\delta_{ib}$ provide Cartan-Weyl basis of $\mathfrak{su}(N)$. To put it differently, show that $E^{(a,b)}$ are eigenstates for adjoint representation of H_i's.

(B. Gutkin)

13.2. **Meson octet.** In Gutkin lecture notes, Lect. 11 *Strong interactions: flavor SU*(3), the meson octet, figure 13.1 (a)

$$
\begin{aligned}
\Phi &= \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K^0} & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ & 0 \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & K^+ \\ 0 & 0 & K^0 \\ K^- & \overline{K^0} & 0 \end{pmatrix} + \frac{\eta}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{aligned} \tag{13.1}
$$

is interpreted as arising from the adjoint representation of SU(3), i.e., the traceless part of the quark-antiquark $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$ outer product (see figure 13.1 (b)),

$$
\begin{pmatrix} u\overline{u} & u\overline{d} & u\overline{s} \\ d\overline{u} & d\overline{d} & d\overline{s} \\ s\overline{u} & s\overline{d} & s\overline{s} \end{pmatrix} . \tag{13.2}
$$

where we have replaced in (13.1) the constituent $q \otimes \overline{q}$ combinations by the names of the elementary particles they build.

Given the quark quantum numbers

	Q	I	I_3	Y	B
u	2/3	1/2	1/2	1/3	1/3
d	-1/3	1/2	-1/2	1/3	1/3
s	-1/3	0	0	-2/3	1/3

verify the strangeness and charge assignments of figure 13.1 (a).

13.3. **SU(3) symmetry in 3D harmonic oscillator.** The Hamiltonian of 3D isotropic harmonic oscillator is given by

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + \frac{m\omega^2}{2} x_i^2 = \hbar\omega \sum_{i=1}^{3} (a_i^\dagger a_i + 1/2),$$

where $a_i = \sqrt{\frac{m\omega}{2\hbar}} x_i + i\sqrt{\frac{1}{2m\omega\hbar}} p_i$ is creation ($a_i^\dagger$ resp. annihilation) operator satisfying $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = 0$.

a) Show that $a_i \to U_{i,j} a_j$, with $U \in U(3)$ is a symmetry of the Hamiltonian. In other words isotropic 3D harmonic oscillator has $U(3)$ rather than $O(3)$ symmetry!

b) Calculate degeneracy of the n-th level $E_n = \omega\hbar(n + 3/2)$ of the oscillator.

c) By comparison of dimensions find out which representations of SU(3) appear in the spectrum of harmonic oscillator.

(B. Gutkin)

group theory - week 14

Flavor SU(3)

Homework HW14

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Bonus points

Exercise 14.1 *Gell-Mann–Okubo mass formula*	8 points
Exercise 15.3 *Young tableaux for $SU(3)$*	3 points
Exercise 15.4 *Irrep projection operators for unitary groups*	5 points

All points are bonus points. Extra points accumulate, can help you if you had missed a few problems.

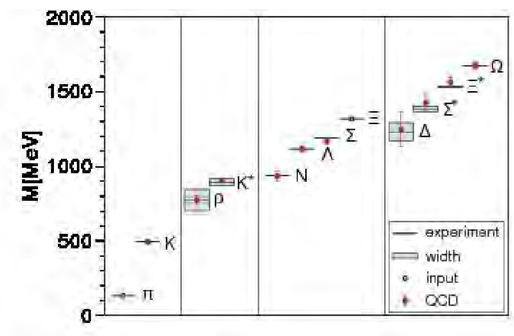


Figure 14.1: A lattice gauge theory calculation of the light QCD spectrum. Horizontal lines and bands are the experimental values with their decay widths. The π, K and and Ξ have no error bars because they are used to set the light and strange quark masses and the overall scale respectively. From Scholarpedia.

2021-07-22 Predrag Lecture 27 Flavor SU(3)

- (Unedited 1:26:11 h) *Gell-Mann–Okubo mass formula.* Pions and kaons fit into an $\mathrm{SU}(3)$ octet, but the strange mesons masses are much larger, breaking the symmetry $\mathrm{SU}(3) \to \mathrm{SU}(2) \times U(1)$. That leads to a Gell-Mann – Okubo parameter-free constraints on various masses, which were verified to high accuracy, and lead to predictions of masses for yet undiscovered particles.

- Gutkin notes, Lect. 11 *Strong interactions: flavor SU*(3). Heisenberg isospin $\mathrm{SU}(2)$. Gell-Mann flavor $\mathrm{SU}(3)$. Gell-Mann-Okubo mass formula.

The Gell-Mann-Okubo mass sum rules [1, 2, 4] are an easy consequence of the approximate $\mathrm{SU}(3)$ flavor symmetry. Determination of quark masses is much harder - they are parameters of the standard model, determined by optimizing the spectrum of particle masses obtained by lattice QCD calculations as compared to the experimental baryon and meson masses. The best determination of the mass spectrum as of 2012 is given in figure 14.1. Up, down quarks are about 3 and 6 MeV, respectively, with strange quark mass about 100 MeV, all with large error brackets. As of 2021, I have not found an update to figure 14.1, but the latest on the subject can probably be traced in Georg von Hippel's latticeqcd.blogspot.com.

14.1 Isotropic quantum harmonic oscillator

One of the “hidden" quantum symmetries is the $\mathrm{SU}(3)$ of the 3D isotropic quantum harmonic oscillator. Murgan and Zender [3] *Energy eigenvalues of the three-dimensional quantum harmonic oscillator from SU(3) cubic Casimir operator* is a nice pedagogical intro to this $\mathrm{SU}(3)$. Would prefer no explicit irreps (see www.birdtracks.eu) but working this out is a good (but long) exercise.

References

[1] M. Gell-Mann, *The Eightfold Way: A Theory of Strong Interaction Symmetry* (CalTech, 1961).

[2] M. Gell-Mann, "Symmetries of baryons and mesons", Phys. Rev. **125**, 1067–1084 (1962).

[3] R. Murgan and A. Zender, "Energy eigenvalues of the three-dimensional quantum harmonic oscillator from SU(3) cubic Casimir operator", Eur. J. Phys. **40**, 015405 (2018).

[4] S. Okubo, "Note on unitary symmetry in strong interactions", Progr. Theor. Phys. **27**, 949–966 (1962).

Exercises

14.1. **Gell-Mann–Okubo mass formula.** The mass symmetry-breaking interaction for an isospin multiplet is proportional to the 3rd component of the isospin operator, I_3. Similarly, the symmetry-breaking interaction of SU(3) for the meson octet is given by the 8th component of the octet operator $Y = \lambda_8$. Derive the GMO mass formula for mesons

$$m_\eta^2 = \frac{4\,m_K^2 - m_\pi^2}{3}\,. \tag{14.1}$$

by eliminating the parameter for the strength of this interaction, as in Gutkin lecture notes, Lect. 11 *Strong interactions: flavor SU*(3) .

group theory - week 15

Many particle systems. Young tableaux

Homework HW15

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Bonus points

Exercise 15.1 *Representations of* $SU(3)$	5 points
Exercise 15.2 *Young tableaux for* S_5	3 points
Exercise 15.3 *Young tableaux for* $SU(3)$	3 points
Exercise 15.4 *Irrep projection operators for unitary groups*	5 points

All points are bonus.

(no videos) Predrag Lecture 28 Many particle systems. Young tableaux

Gutkin notes, Lect. 12 *Many particle systems*.

Excerpt from Predrag's monograph [5], fetch it here: Sect. 9.3 *Young tableaux*.

(no videos) Predrag Lecture 29 Young tableaux

Excerpts from Predrag's monograph [5], fetch them here:
Sect. 2.2 *First example: SU*(n) (skim over casimirs and beyond: this example gives you a flavor of birdtracks computations, you do not need to work it out in detail),
Sect. 6.1 *Symmetrization*,
Sect. 6.2 *Antisymmetrization*,
Sect. 9.1 *Two-index tensors*,
Sect. 9.2 *Three-index tensors*, and Table 9.1.

Reading for this week: Sect. 9.3 *Young tableaux*.

Young tableaux for SU(3) and SU(n) have not yet been covered in the lectures, but you can easily learn them yourself, from, for example, Gutkin notes, Lect. 12 *Young tableaux*. Boris Gutkin is a professor, beyond learning new stuff, so he follows old fashioned references such as Fulton and Harris [6]. The resulting simple recipe with 0 explanation can be found, for example, here: *Young diagrams* by C.G. Wohl.

A modern exposition is given in *Group Theory – Birdtracks, Lie's, and Exceptional Groups,* birdtracks.eu Chapt. 9 *Unitary groups*. Currently I am a fan of the Alcock-Zeilinger algorithm [1–3], based on the simplification rules of ref. [2], which leads to explicitly Hermitian and compact expressions for the projection operators.

Probably best to read Alcock-Zeilinger course *The Special Unitary Group, Birdtracks, and Applications in QCD* notes [4]. Alcock-Zeilinger fully supersedes Cvitanović's formulation, and any future full exposition of birdtracks reduction of SU(N) tensor products into irreducible representations should be based on the Alcock-Zeilinger algorithm.

15.1 Other sources (optional)

The clearest current exposition and the most powerful irrep reduction of SU(n) is given in the triptych of papers by Judith Alcock-Zeilinger and her thesis adviser H. Weigart, University of Cape Town:

Simplification rules for birdtrack operators [3],
Compact Hermitian Young projection operators [2], and
Transition operators [1].

Probably best to read Alcock-Zeilinger course *The Special Unitary Group, Birdtracks, and Applications in QCD* notes [4]. You want to study these in detail if your research leads you to study of multiparticle states.

References

[1] J. Alcock-Zeilinger and H. Weigert, "Transition operators", J. Math. Phys. **58**, 051702 (2016).

[2] J. Alcock-Zeilinger and H. Weigert, "Compact Hermitian Young projection operators", J. Math. Phys. **58**, 051702 (2017).

[3] J. Alcock-Zeilinger and H. Weigert, "Simplification rules for birdtrack operators", J. Math. Phys. **58**, 051701 (2017).

[4] J. M. Alcock-Zeilinger, The Special Unitary Group, Birdtracks, and Applications in QCD, tech. rep. (Univ. Tübingen, 2018).

[5] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2008).

[6] W. Fulton and J. Harris, *Representation Theory* (Springer, New York, 1991).

Exercises

15.1. **Representations of SU(3).** Any irrep of SU(3) can be labeled $D(p,q)$ by its highest weight $\lambda = p\lambda_1 + q\lambda_2$, where $\lambda_{1,2}$ are the two fundamental weights.

(a) Find all irreps $D(p,q)$ of SU(3) with the dimensions less then 20 (see lecture notes for the dimensions of $D(p,q)$).

(b) Draw the lattice Λ generated by $\lambda_{1,2}$ and mark there all the weights v (i.e., lattice nodes) which belong to irrep. $D(3,0)$. Is $D(3,0)$ a real irrep?

(c) Consider product (reducible) representation $3 \otimes 3$, where $3 = D(1,0)$ is the fundamental irrep. Mark all the weights v on Λ which belong to $3 \otimes 3$. Using this find out decomposition of $3 \otimes 3$ into irreps:

$$3 \otimes 3 = \square \oplus \triangle, \qquad \square = ?, \qquad \triangle = ?$$

Hint: see lecture notes for similar exercise on $3 \otimes \bar{3}$.

(d) Using previous results find decomposition of $3 \otimes 3 \otimes 3$ into irreps.

(B. Gutkin)

15.2. **Young tableaux for S_5.**

(a) Draw all Young diagrams for the symmetric group S_5. How many irreducible representations has it? Which of the diagrams correspond to one-dimensional irreps?

(b) Find Young diagram corresponding to the irrep of S_5 with the largest dimension? Draw Young tableaux corresponding to this irrep/Young diagram. What is the dimension of this irrep?

(c) What are the dimensions of the remaining irreps?

(B. Gutkin)

15.3. **Young tableaux for SU(3).** Solve exercise 15.1 (c,d) by using Young tableaux.

Remark: If Young tableaux for SU(3) are not covered in the lectures, learn them yourself from, for example, birdtracks.eu *Group Theory Birdtracks, Lie's, and Exceptional Groups.* The resulting simple recipe with 0 explanation can be found, for example, here *C.G. Wohl*.

(B. Gutkin)

15.4. **Irrep projection operators for unitary groups.** Derive projection operators and dimensions for irreps of the Kronecker product of the defining and the adjoint reps of SU(n) listed in *Group Theory Birdtracks, Lie's, and Exceptional Groups*, birdtracks.eu table 9.3. (Ignore "indices," we have not defined them here.)

group theory - week 16

Wigner 3- and 6-j coefficients

Homework HW16

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Bonus points

Exercise 16.1 *Gravity tensors*, part (a)	2 points
Exercise 16.1 *Gravity tensors*, part (b)	4 points
Exercise 16.1 *Gravity tensors*, part (c)	1 point
Exercise 16.1 *Gravity tensors*, part (d)	2 points
Exercise 16.1 *Gravity tensors*, part (e)	3 points
Exercise 16.1 *Gravity tensors*, part (f)	4 points
Exercise 16.1 *Gravity tensors*, part (g)	3 points
Exercise 16.1 *Gravity tensors*, part (h)	6 points
Exercise 16.1 *Gravity tensors*, part (i)	4 points
Exercise 16.1 *Gravity tensors*, part (j)	10 points

All points are bonus.

2021-07-27 Predrag Lecture 30 Wigner 3- and 6-j coefficients

(Unedited 2:04:59 h) *Sex, Lies and Videotape.*

What are we really trying to compute? Wigner 3n-j coefficients, birdtracks. How simple calculations lead to all Lie groups, including the exceptional ones.

The webbook for cyclists (medium level): *Tracks, Lie's, and Exceptional Magic*. Most of the webbook at a cyclist pace, in 50 overheads.

Birdtracks: Excerpts from Predrag's monograph [4], fetch them here:

Background reading on groups, vector spaces, tensors, invariant tensors, invariance groups (my advice is to start with Sect. 5.1 *Couplings and recouplings*, then backtrack to these introductory sections as needed):
Sect. 3.2 *Defining space, tensors, reps*,
Sect. 3.3 *Invariants*,
Sect. 4.1 *Birdtracks*,
Sect. 4.2 *Clebsch-Gordan coefficients*, and
Sect. 4.3 *Zero- and one-dimensional subspaces*.

The final result is invariant and highly elegant: any group-theoretical invariant quantity can be expressed in terms of Wigner 3- and 6-j coefficients:
Sect. 5.1 *Couplings and recouplings*,
Sect. 5.2 *Wigner $3n$-j coefficients*, and
Sect. 5.3 *Wigner-Eckart theorem.*

The rest is just bedside reading, nothing technical:
Sect. 4.8 *Irrelevancy of clebsches* and
Sect. 4.9 *A brief history of birdtracks*.

Course finale: *Indiana Jones vs. Fancy Footwork*. (1:30 min)
It's a matter of no small pride for a card-carrying dirt physics theorist to claim *full and total ignorance of group theory*.

16.1 Other sources (optional)

Predrag's monograph [4], *Group Theory Birdtracks, Lie's, and Exceptional Groups*, Sect. 2.1 *Basic concepts*; Sect. 2.1 *First example: $SU(n)$*; Chap. 1 explains pretty well what the monograph is about.

∘ A birdtracks refresher: sect. 2.7 *Permutations in birdtracks*

∘ Sect. 10.7 *Lie groups for pedestrians*

∘ sect. 10.10 *Birdtracks - updated history*

16.2 Gruppenpest (optional)

A practically-minded physicist always has been, and continues to be resistant to gruppenpest. Apparently already in 1910 James Jeans wrote, while discussing what should a physics syllabus contain: "We may as well cut out the group theory. That is a subject that will never be of any use in physics." In 1963 Eugene Wigner got the Nobel Prize in Physics, so by mid 60's gruppenpest was accepted in finer social circles.

ChaosBook Appendix A.6 Gruppenpest

Woit writes here about the "The Stormy Onset of Group Theory in the New Quantum Mechanics," citing Bonolis [2] *From the rise of the group concept to the stormy onset of group theory in the New Quantum Mechanics. A saga of the invariant characterization of physical objects, events and theories*.

Chayut [3] *From the periphery: the genesis of Eugene P. Wigner's application of group theory to quantum mechanics* traces the origins of Wigner's application of group theory to quantum physics to his early work as a chemical engineer, in chemistry and crystallography. "In the early 1920s, crystallography was the only discipline in which symmetry groups were routinely used. Wigner's early training in chemistry exposed him to conceptual tools which were absent from the pedagogy available to physicists for many years to come. This both enabled and pushed him to apply the group theoretic approach to quantum physics. It took many years for the approach first introduced by Wigner in the 1920s – and whose reception by the physicists was initially problematical – to assume the pivotal place it now holds." Another historical exposition is given by Scholz [6] *Introducing groups into quantum theory (1926–1930).*

So what is group theory good for? By identifying the symmetries, one can apply group theory to determine good quantum numbers which describe a physical state (i.e., the irreps). Group theory then says that many matrix elements vanish, or shows how are they related to others. While group theory does not determine the actual value of a matrix element of interest, it vastly simplifies its calculation.

The old fashioned atomic physics, fixated on $\mathrm{SO}(3)$ / $\mathrm{SU}(2)$, is too explicit, with too many bras and kets, too many square roots, too many deliriously complicated Clebsch-Gordan coefficients that you do not need, and way too many labels, way too explicit for you to notice that all of these are eventually summed over, resulting in a final answer much simpler than any of the intermediate steps.

I wrote my book [4] *Group Theory - Birdtracks, Lie's, and Exceptional Groups* to teach you how to compute everything you need to compute, without ever writing down a single explicit matrix element, or a single Clebsch-Gordan coefficient. There are two versions. There is a particle-physics / Feynman diagrams version that is index free, graphical and easy to use (at least for the low-dimensional irreps). The key insights are already in Wigner's book [8]: the content of symmetry is a set of invariant numbers that he calls $3n$-j's. Then there are various mathematical flavors (Weyl group on Cartan lattice, etc.), elegant, but perhaps too elegant to be computationally practical.

But it is nearly impossible to deprogram people from years of indoctrination in QM and EM classes. The professors have no time to learn new stuff, and students love manipulating their mu's and nu's.

Nothing to be done... (2:18:01 h)

Bonus: While waiting, exhale. (2:18:01 h)

References

[1] S. L. Adler, J. Lieberman, and Y. J. Ng, "Regularization of the stress-energy tensor for vector and scalar particles propagating in a general background metric", Ann. Phys. **106**, 279–321 (1977).

[2] L. Bonolis, "From the Rise of the Group Concept to the Stormy Onset of Group Theory in the New Quantum Mechanics. A saga of the invariant characterization of physical objects, events and theories", Rivista Nuovo Cim. **27**, 1–110 (2005).

[3] M. Chayut, "From the periphery: the genesis of Eugene P. Wigner's application of group theory to quantum mechanics", Found. Chem. **3**, 55–78 (2001).

[4] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2008).

[5] R. Penrose, "Applications of negative dimensional tensors", in *Combinatorial mathematics and its applications*, edited by D. J. A. Welsh (Academic, New York, 1971), pp. 221–244.

[6] E. Scholz, "Introducing groups into quantum theory (1926-1930)", Hist. Math. **33**, 440–490 (2006).

[7] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[8] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1931).

Exercises

16.1. **Gravity tensors.** In this problem we will apply diagrammatic methods ("birdtracks") to construct and count the numbers of independent components of the "irreducible rank-four gravity curvature tensors." However, any notation that works for you is OK, as long as you obtain the same irreps and their dimensions. The goal of this exercise (longish, as much of it is the recapitulation of the material covered in the book) is to give you basic understanding for how Young tableaux work for groups other than $\mathrm{U}(n)$. We start with

Part 1 : **$\mathbf{U}(n)$ Young tableaux decomposition.**

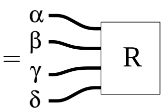
(a) The Riemann-Christoffel curvature tensor of general relativity has the following symmetries (see, for example, Weinberg [7] or the Riemann curvature tensor wiki):

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \tag{16.1}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{16.2}$$

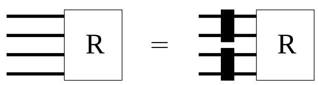
$$R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0\,. \tag{16.3}$$

Introducing a birdtrack notation for the Riemann tensor

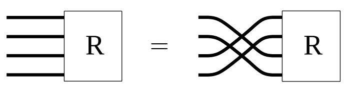
$R_{\alpha\beta\gamma\delta} =$  , (16.4)

check that we can state the above symmetries as

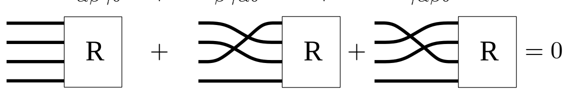
$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$

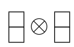
 , (16.5)

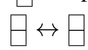
$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$

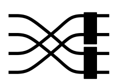
 , (16.6)

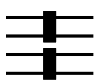
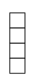
$R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0$

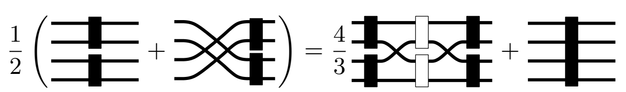
 (16.7)

The first condition says that R lies in the  subspace.

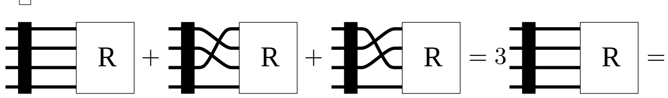
(b) The second condition says that R lies in the  interchange-symmetric subspace.

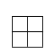
Use the characteristic equation for 

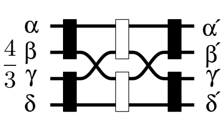
to split  into the  and  irreps:

 . (16.8)

(c) Show that the third condition (16.7) says that R has no components in the  irrep:

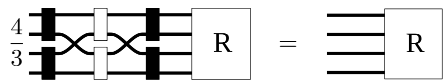
 (16.9)

Hence, the symmetries of the Riemann tensor are summarized by the  irrep projection operator [5]:

$(P_R)_{\alpha\beta\gamma\delta,}{}^{\delta'\gamma'\beta'\alpha'} = \frac{4}{3}$  (16.10)

(d) Verify that the Riemann tensor is in the  subspace

$$(P_R R)_{\alpha\beta\gamma\delta} \;=\; (P_R)_{\alpha\beta\gamma\delta,}{}^{\delta'\gamma'\beta'\alpha'}\, R_{\alpha'\beta'\gamma'\delta'} = R_{\alpha\beta\gamma\delta}$$



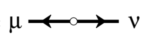
. (16.11)

(e) Compute the number of independent components of the Riemann tensor $\mathbf{R}_{\alpha\beta\gamma\delta}$ by taking the trace of the  irrep projection operator:

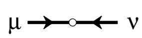
$$d_R = \operatorname{tr} \mathbf{P}_R = \frac{n^2(n^2-1)}{12} \,. \qquad (16.12)$$

Part 2 : **SO(n) Young tableaux decomposition**

The Riemann tensor has the symmetries of the  irrep of U(n). However, gravity is also characterized by the symmetric tensor $g_{\alpha\beta}$, that reduces the symmetry to a local SO(n) invariance (more precisely SO($1, n-1$), but compactness is not important here). The extra invariants built from $g_{\alpha\beta}$'s decompose U(n) reps into sums of SO(n) reps. Orthogonal group SO(n) is the group of transformations that leaves invariant a symmetric quadratic form $(q, q) = g_{\mu\nu} q^\mu q^\nu$, with a primitive invariant rank-2 tensor:

$g_{\mu\nu} = g_{\nu\mu} =$  $\qquad \mu, \nu = 1, 2, \ldots, n\,.$ (16.13)

If (q, q) is an invariant, so is its complex conjugate $(q, q)^* = g^{\mu\nu} q_\mu q_\nu$, and

$g^{\mu\nu} = g^{\nu\mu} =$  (16.14)

is also an invariant tensor. The matrix $A_\mu^\nu = g_{\mu\sigma} g^{\sigma\nu}$ must be proportional to unity, as otherwise its characteristic equation would decompose the defining n-dimensional rep. A convenient normalization is

$$g_{\mu\sigma} g^{\sigma\nu} \;=\; \delta_\mu^\nu$$

 . (16.15)

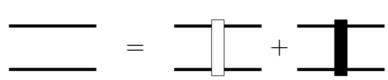
As the indices can be raised and lowered at will, nothing is gained by keeping the arrows. Our convention will be to perform all contractions with metric tensors with upper indices and omit the arrows and the open dots:

$g^{\mu\nu} \equiv$. (16.16)

The U(n) 2-index tensors can be decomposed into a sum of their symmetric and antisymmetric parts. Specializing to the subgroup SO(n), the rule is to lower all indices on all tensors, and the symmetrization projection operator is written as

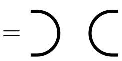
$$\begin{aligned} S_{\mu\nu,\rho\sigma} &= g_{\rho\rho'} g_{\sigma\sigma'} S_{\mu\nu,}{}^{\rho'\sigma'} \\ &= \frac{1}{2}\left(g_{\mu\sigma} g_{\nu\rho} + g_{\mu\rho} g_{\nu\sigma}\right) \end{aligned}$$

From now on, we drop all arrows and $g^{\mu\nu}$'s and write the decomposition into symmetric and antisymmetric parts as

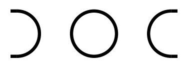


$$g_{\mu\sigma} g_{\nu\rho} \;=\; \frac{1}{2}(g_{\mu\sigma} g_{\nu\rho} + g_{\mu\rho} g_{\nu\sigma}) + \frac{1}{2}(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma})\,. \qquad (16.17)$$

The new invariant tensor, specific to SO(n), is the index contraction:

$$\mathbf{T}_{\mu\nu,\rho\sigma} = g_{\mu\nu} g_{\rho\sigma} \,, \qquad \mathbf{T} = $$  . (16.18)

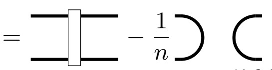
Its characteristic equation

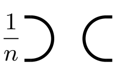
$$\mathbf{T}^2 = $$  $= n\mathbf{T}$ (16.19)

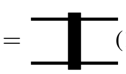
yields the trace and the traceless part projection operators. As $\mathbf{T}$ is symmetric, $S\mathbf{T} = \mathbf{T}$, only the symmetric subspace is reduced by this invariant.

(f) Show that SO(n) 2-index tensors decompose into three irreps:

traceless symmetric:

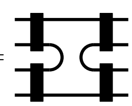
$$(P_2)_{\mu\nu,\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma} g_{\nu\rho} + g_{\mu\rho} g_{\nu\sigma}\right) - \frac{1}{n} g_{\mu\nu} g_{\rho\sigma} = $$  , (16.20)

singlet: $$(P_1)_{\mu\nu,\rho\sigma} = \frac{1}{n} g_{\mu\nu} g_{\rho\sigma} = \frac{1}{n}$$  , (16.21)

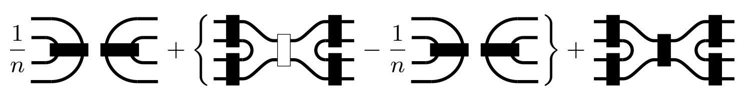
antisymmetric: $$(P_3)_{\mu\nu,\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}\right) = $$  (16.22)

What are the dimensions of the three irreps?

(g) In the same spirit, the U(n) irrep  is decomposed by the SO(n) intermediate 2-index state invariant matrix

$$\mathbf{Q} = $$  . (16.23)

Show that the intermediate 2-index subspace splits into three irreducible reps by (16.20) – (16.22):

$$\mathbf{Q} = $$ 

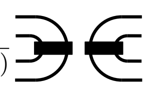
$$= \mathbf{Q}_0 + \mathbf{Q}_S + \mathbf{Q}_A \,. \qquad (16.24)$$

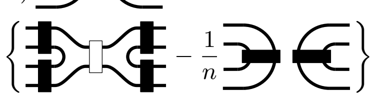
Show that the antisymmetric 2-index state does not contribute

$$\mathbf{P_R Q}_A = 0 \,. \qquad (16.25)$$

(Hint: The Riemann tensor is symmetric under the interchange of index pairs.)

(h) Fix the normalization of the remaining two projection operators by computing $\mathbf{Q}_S^2, \mathbf{Q}_0^2$:

$$\mathbf{P}_0 = \frac{2}{n(n-1)}$$  , (16.26)

$$\mathbf{P}_S = \frac{4}{n-2}$$  (16.27)

and compute their dimensions.

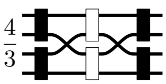
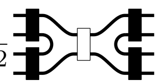
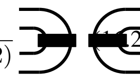
This completes the SO(n) reduction of the  U(n) irrep (16.11):

U(n)	$\to$	SO(n)				
	$\to$		+		+	$\circ$
$\mathbf{P}_R$	=	$\mathbf{P}_W$	+	$\mathbf{P}_S$	+	$\mathbf{P}_0$
$\frac{n^2(n^2-1)}{12}$	=	$\frac{(n+2)(n+1)n(n-3)}{12}$	+	$\frac{(n+2)(n-1)}{2}$	+	1

(16.28)

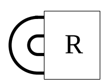
The projection operator for the SO(n) traceless  irrep is:

$$\mathbf{P}_W = \mathbf{P}_R - \mathbf{P}_S - \mathbf{P}_0$$

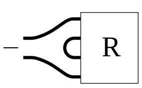
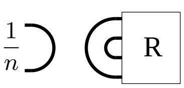
$\mathbf{P}_W = \frac{4}{3}$  $- \frac{4}{n-2}$  $+ \frac{2}{(n-1)(n-2)}$  (16.29)

(i) The above three projection operators project out the standard, SO(n)-irreducible general relativity tensors:

Curvature scalar:

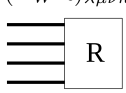
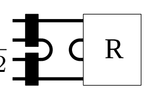
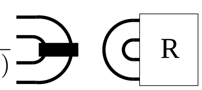
$R = -$  $= R^{\mu}{}_{\nu\mu}{}^{\nu}$ (16.30)

Traceless Ricci tensor:

$R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R = -$  $+ \frac{1}{n}$  (16.31)

Weyl tensor:

$C_{\lambda\mu\nu\kappa} = (\mathbf{P}_W R)_{\lambda\mu\nu\kappa}$

$=$  $- \frac{4}{n-2}$  $+ \frac{2}{(n-1)(n-2)}$ 

$$= R_{\lambda\mu\nu\kappa} + \frac{1}{n-2}\left(g_{\mu\nu}R_{\lambda\kappa} - g_{\lambda\nu}R_{\mu\kappa} - g_{\mu\kappa}R_{\lambda\nu} + g_{\lambda\kappa}R_{\mu\nu}\right)$$
$$- \frac{1}{(n-1)(n-2)}(g_{\lambda\kappa}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\kappa})R\,. \qquad (16.32)$$

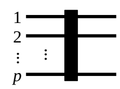
The numbers of independent components of these tensors are given by the dimensions of corresponding irreducible subspaces in (16.28).

What is the lowest dimension in which the Ricci tensor contributes? the Weyl tensor contributes? Show that in 2, respectively 3 dimensions, we have

$$\begin{aligned} n=2: \quad R_{\lambda\mu\nu\kappa} &= (P_0 R)_{\lambda\mu\nu\kappa} = \tfrac{1}{2}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})R \\ n=3: \quad &= g_{\lambda\nu}R_{\mu\kappa} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu} - g_{\lambda\kappa}R_{\mu\nu} \\ &\quad -\tfrac{1}{2}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})R\,. \end{aligned} \qquad (16.33)$$

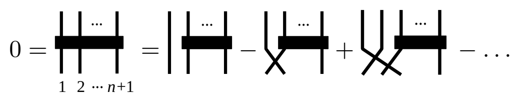
(j) The last example of this exercise is an application of birdtracks to general relativity index manipulations. The object is to find the characteristic equation for the Riemann tensor in *four dimensions*.

The antisymmetrization tensor $A_{a_1 a_2 \ldots,}{}^{b_p \ldots b_2 b_1}$ has nonvanishing components, only if all lower (or upper) indices differ from each other. If the defining dimension is smaller than the number of indices, the tensor A has no nonvanishing components:



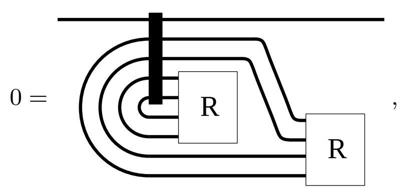
$= 0 \qquad \text{if } p > n\,.$ (16.34)

This identity implies that for $p > n$, not all combinations of p Kronecker deltas are linearly independent. A typical relation is the $p = n + 1$ case



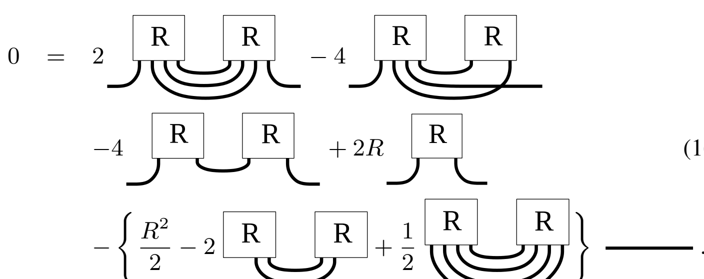
(16.35)

Contract (16.34) with two Riemann tensors:



(16.36)

and obtain the characteristic equation by expanding with (16.35):



(16.37)

This identity has been used by Adler *et al.*, eq. (E2) in ref. [1].

group theory - week 17

An overview, and the epilogue

If I had had more time, I would have written less
— Blaise Pascal, a remark made to a correspondent

This whole course has only one message:

If you have a symmetry, **use it!**

These notes in isolation do not make much sense - the essence of teaching is unveiling of concepts on a black/white board at human pace, interacting in live time. But nevertheless the notes might be useful to you, as they are hyperlinked to the literature that develops a given topic into depth. Here is a brief summary of the course, the ideas you want to take with you:

week 1 Linear algebra

The key idea: Projection operators (1.27) use eigenvalues of a matrix to split (reduce) a vector space into subspaces.

week 2 Finite groups

Groups, permutations, group multiplication tables, rearrangement theorem, subgroups, cosets, classes.

week 3 Group representations

Irreps, regular representation. So far, everything was intuitive: a representation of a group was bunch of 0's and 1's indicating how a group operation permutes physical objects. But now the first surprise:

Any representation of any finite group can be put into unitary form, and so complex-valued vector spaces and unitary representation matrices make their entrance.

week 4 Characters

Schur's Lemma. Unitary matrices can be diagonalized, and from that follows the Wonderful Orthogonality Theorem for Characters (coordinate independent, intrinsic numbers), and the *full reducibility* of any representation of any finite group.

week 5 Classes

The algebra of central or 'all-commuting' class operators, connects the reduction in terms of characters to the projection operators of week 1. The key idea:

Define a group by what objects (primitive invariant tensors) it leaves invariant.

week 6 Fundamental domain

Dynamical systems application: the Lorenz flow, its C_2 symmetry and its desymmetrization: if the system is nonlinear, its symmetry reduction is not easy.

week 7 Discrete Fourier representation

So far, everything was finite and compact. Next: two distinct ways of going infinite: (a) discrete translations, exemplified by deterministic diffusion and space groups of week 8, and (b) continuous Lie groups, exemplified by rotations of week 9.

week 8 Space groups

Translation group, Bravais lattice, wallpaper groups, reciprocal lattice, Brilluoin zone.

week 9 Continuous groups

Lie groups. Matrix representations. Invariant tensors. Lie algebra. Adjoint representation, Jacobi relation. Birdtracks.

Irreps of $\mathrm{SO}(2)$ and $\mathrm{O}(2)$ Clebsch-Gordan series (i.e., reduction of their products).

week 10 Lie groups, algebras; $\mathrm{O}(2)$ symmetry sliced

(a) Group integrals. $\mathrm{SO}(3)$ character orthogonality.

(b) Continuous symmetry reduction for a *nonlinear* system is much harder than discrete symmetry reduction of week 7. "Slicing" is a research level topic.

week 11 $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$

$\mathrm{SU}(2) \simeq \mathrm{SO}(3)$ correspondence leads to the next rude awakening; our 3-dimensional Euclidean space is not fundamental! All irreps of $\mathrm{SO}(3)$ are built from 2-dimensional complex vectors, or 1/2 spins.

week 12 Lorentz group; spin

(a) We now loose compactness: even though the $\mathrm{SO}(1,3)$ Lorentz invariance group of the Minkowski space symmetries is not compact, its Lie algebra still closes, as for the compact $\mathrm{SO}(4)$.

(b) $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \otimes \mathrm{SU}(2)$ correspondence leads to the Minkowski 4-dimensional space not being fundamental either - all irreps of the Lorentz group are built from combinations of 2-dimensional complex vectors, or spinors.

(c) together with general relativity, this leads to replacement of the Minkowski continuum by a 4-dimensional spacetime (or quantum) foam, a candidate theory of quantum gravity.

week 13 Simple Lie algebras; $\mathrm{SU}(3)$

The next profound shift:

So far all our group notions were based on tangible, spatial intuition: permutations, reflections, rotations. But now Lie groups take on a life of their own.

(a) The $\mathrm{SO}(3)$ theory of angular momenta generalizes to Killing-Cartan lattices, and a fully abstract enumeration of all possible semi-simple compact Lie groups.

(b) $\mathrm{SU}(2)$ is promoted to an *internal* isospin symmetry, decoupled from our Euclidean spatial intuition. Modern particle physics is born, with larger and larger internal symmetry groups, tacked onto higher and higher dimensional continuum spacetimes.

week 14 Flavor $\mathrm{SU}(3)$

Gell-Mann–Okubo formula. The next triumph of particle physics is yet another departure; observed baryons and mesons are built up from quarks, particles by assumption unobservable in isolation.

week 15 Young tableaux

We have come full circle now: as a much simpler alternative to the Cartan-Killing construction, irreps of the *finite* symmetric group S_n classify the irreps of the *continuous* $\mathrm{SU}(n)$ symmetry multi-particle states.

week 16 Wigner 3- and 6-j coefficients

The goal of group theory is to predict measurable numbers, numbers independent of any particular choice of coordinate. The full reducibility says that any such number is built from 3- and 6-j coefficients: they are the total content of group theory.

If you are reading this in preparation for a final exam, think of it as an opportunity to rethink the key ideas of this branch of mathematics, take with you the few essential insights that may serve you well in your career later on.

17.1 Student suggestions for improvements

Extracted from `studentEvals17.pdf`, `studentEvals19.pdf`.

improvements For this specific course, there are so many resources. It covers the best parts of many books most probably, but from the student's perspective every time they open the middle of a new book they can find themselves looking at the very first page at that book by following previous equations. Also the mathematical aspect of the course can screen the physics side of the picture. As an experimentalist I got lost in the second half of the course. I found it easier to follow when Dresselhaus or Tinkham book were being followed. Tinkham's book was hard for me at some point as well but these two books were easier to read for me than other resources.

The course could be dramatically improved with a stronger emphasis on fundamentals of group structure and constructive examples throughout, accompanied by a much more deliberate focus on a smaller set of topics. As an example of a fundamental topic which I think has been inadequately covered: I am still unsure what distinguishes irreducible representations from any representation of a group on an intuitive level. It is, as far as I can tell, something I should simply know by now, but has not been adequately explained to me.

comments This course has been bewildering for the latter half of the semester. On several occasions I have co-opted class time to have concepts from homework and previous lectures explained carefully, which has been consistently unhelpful. Overall, this course is by far the most stressful, confusing, and frustrating course I have taken at Georgia Tech and on more than one occasion has given me nightmares.

Instructor improvements The most needed improvement in the lectures is a stronger focus on examples (especially those reinforcing fundamental concepts) which reflect the Cvitanović expectations for the rest of the course.

to be CONTINUED