Recycling Fourier Spectra of Chaotic Systems

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Abstract

We develop the periodic orbit theory of Fourier power spectra for chaotic dynamical systems. The theory is tested numerically on several 1-dimensional[3] mappings.

1 INTRODUCTION

We apply the transfer operator dynamical averaging techniques to the evaluation of power spectra of chaotic time series. The key idea is the realization that the periodic orbit description of diffusion introduced in refs. [3, 4, 5] can be interpreted as the zero frequency component of the power spectrum of a chaotic dynamical flow; in this paper we generalize the diffusion formalism to evaluation of the power spectrum at any rational multiple of periods of short unstable cycles of the flow.

discus here invariant averages (Lyapunov, dimensions, etc.) vs. non-invariant ones, such as $\langle x_i \rangle$.

2 DYNAMICAL AVERAGING AND TRANSFER OPERATORS

Let $\phi(\tau, x_\tau)$ be any "observable" evaluated on a trajectory

$$x_{\tau+1} = f(x_\tau) \qquad (1) \quad \{1\}$$

of a dynamical system. The simplest example of an such observable is the trajectory itself, $\phi(\tau, x_\tau) = x_\tau$. If $f(x)$ generates a chaotic time series x_τ, we can think of $\phi(\tau, x_\tau)$ as executing a random walk, driven by external "random number" sequence $x_0, x_1, x_2, \ldots, x_t$. The cumulative effect of the random walk for a given initial x is given by

$$\Phi^t(x) = \sum_{\tau=0}^{t-1} \phi(\tau, x_\tau)\,, \qquad x_0 = x\,. \qquad (2) \quad \{1c\}$$

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We cycle on! - **CONTINUATION OF 4TH DRAFT BY AP in ATLANTA, 20 mar 2002**

If ϕ is bounded, and $f(x)$ generates a bounded time series for all initial conditions $x \in M$, a compact region in the "phase space" of the dynamical system, $\Phi^t(x)$ cannot grow faster than t, and it makes sense to study the "drift velocity" of $\phi(\tau, x_\tau)$,

$$\overline{\phi(x)} = \lim_{t\to\infty} \frac{1}{t}\Phi^t(x)\,. \tag{3}$$

{tim_ave}

and its higher moments [7]. However, the time average $\overline{\phi(x)}$ is in general a wild function of x; for a nice hyperbolic system it takes the same value $\langle\phi\rangle$ for almost all initial x, but a different value on any periodic orbit, *ie.* on a dense set of initial points. Hence for chaotic dynamical systems robust averaging requires also averaging over the initial x. We shall denote average over the "phase space" $x \in M$ by $\langle \ldots \rangle$. The expectation value $\langle\phi\rangle$, the asymptotic time and space average is given by

$$\begin{aligned} \langle\phi\rangle &= \frac{1}{|M|}\int dx\,\overline{\phi(x)}\,, \qquad\qquad |M| = \int dx \\ &= \lim_{t\to\infty}\frac{1}{t}\sum_{\tau=0}^{t-1}\frac{1}{|M|}\int dx\,\phi(\tau, f^\tau(x))\,. \end{aligned} \tag{4}$$

Consider the expectation value

$$\begin{aligned} \left\langle e^{\beta\cdot\Phi^t}\right\rangle &= \frac{1}{|M|}\int dx\, e^{\beta\cdot\Phi^t(x)} \\ &= \frac{1}{|M|}\int dxdy\,\delta(y - f^t(x))\, e^{\beta\cdot\Phi^t(x)}\,. \end{aligned} \tag{5}$$

Here β is an auxilliary variable, useful for evaluation of the moments $\langle\phi^k\rangle$. The $t\to\infty$ limit of such averages can be related to the eigenvalues of transfer operators [1], in this case

$$\mathcal{L}^t(y,x) = e^{\beta\cdot\Phi^t(x)}\delta(y - f^t(x))\,. \tag{6}$$

{(8)}

If the limit (4) exists,

$$\lim_{t\to\infty}\left\langle e^{\beta\cdot\Phi^t}\right\rangle \to e^{tQ(\beta)} \tag{7}$$

{2}

also exists, with $Q(\beta)$ given by the leading eigenvalue of $\mathcal{L}^t$, $\lambda_0 = e^{tQ(\beta)}$. For $\beta = 0$, the operator (6) is the Perron-Frobenius operator. If the system is closed, probability conservation implies that the leading eigenvalue is exactly $\lambda_0 = 1$, and consquently $Q(0) = 0$. The averages such as (4) are recovered by evaluating derivatives of $Q(\beta)$ at $\beta = 0$:

$$\left.\frac{\partial Q}{\partial\beta}\right|_{\beta=0} = \langle\phi\rangle = \lim_{t\to\infty}\frac{1}{t}\left\langle\Phi^t\right\rangle\,, \tag{8}$$

$$\left.\frac{\partial^2 Q}{\partial\beta^2}\right|_{\beta=0} = \lim_{t\to\infty}\frac{\left\langle(\Phi^t)^2\right\rangle - \left\langle\Phi^t\right\rangle^2}{t}\,. \tag{9}$$

and so forth. For example, for $\phi_i(x) = x_i(t)$, $\langle x\rangle = 0$, the spatial diffusion constant in $i = 1, 2, \ldots, d$ spatial dimensions is given by [3, 4, 5]

$$D = \frac{1}{2d}\left.\frac{\partial^2 Q}{\partial\beta^2}\right|_{\beta=0} = \lim_{t\to\infty}\frac{1}{2dt}\left\langle\left(\sum_0^{t-1} x_i(t)\right)^2\right\rangle\,, \tag{10}$$

{3}

2.1 Trace formulas

Extraction of the spectrum of $\mathcal{L}$ commences with the evaluation of the trace

$$\mathrm{tr}\mathcal{L}^t = \sum_{\alpha=0}^{\infty} \lambda_\alpha^t = \frac{1}{|M|}\int dx\, e^{\beta\cdot\Phi^t(x)}\delta(x - f^t(x)). \qquad (11) \quad \{(10)\}$$

For discrete time and hyperbolic dynamics we obtain [1]

$$\mathrm{tr}\mathcal{L}^t = \sum_{x_i \in Fix\, f^t} \frac{e^{\beta\Phi_i^t}}{|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)|}$$

Using the property of additivity of observable Φ^t along a trajectory, we can reduce the sum to the sum over prime cycles of f

$$\mathrm{tr}\mathcal{L}^t = \sum_{p\in\mathcal{P}} \tau_p \sum_{r=1}^{\infty} \frac{\delta_{t,\tau_p r}}{|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)|} e^{r\beta\cdot\Phi_p} \qquad \Phi_p = \Phi^{\tau_p}(x_i),\ x_i \in p \qquad (12)$$

where the sum is over τ_p periodic points $x_{p,m}$ of all prime cycles p whose period τ_p divides t, $x_{p,m}$ is the m-th point on the cycle, and $\mathbf{J}_p$ is the cycle stability matrix. $\mathbf{J}_p = Df^{\tau_p}(x_{p,m})$ is, by the chain rule, independent of the starting point $x_{p,m}$. For continuous flows the trace formulas (sect. 6.1) are of the essentially same form.

The Fredholm determinant associated with the trace formula (12) follows by the usual resummations [2]:

$$\begin{aligned} F &= \det(1 - e^{-Q(\beta)}\mathcal{L}) = \exp[\mathrm{tr}\log(1 - e^{-Q}\mathcal{L}] = \exp\left[-\sum_{t=1}^{\infty}\frac{e^{-Qt}}{t}\mathrm{tr}\mathcal{L}\right] \\ &= \exp\left(-\sum_{p\in\mathcal{P}}\sum_{r=1}^{\infty}\frac{e^{-Q(\beta)\tau_p r + r\beta\Phi_p}}{r}\frac{1}{|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)|}\right). \qquad (13) \end{aligned}$$

The equation $F(\beta, Q) = 0$ implicitly defines the function $Q(\beta)$. Thus the derivatives (8),(9) we are looking for can be calculated as

$$\frac{\partial Q}{\partial \beta} = -\left.\frac{\frac{\partial F}{\partial \beta}}{\frac{\partial F}{\partial Q}}\right|_{\beta=Q=0}$$

etc.

2.2 Cycle expansions

The associated with the Fredholm determinant Ruelle ζ function is obtained by replacement $|\det\left(\mathbf{1}-\mathbf{J}_p^r\right)| \to \Lambda_p = \prod_e \Lambda_{p,e}$, the product of the expanding eigenvalues of $\mathbf{J}_p$ (see ref. [2] for details):

$$
\begin{aligned}
1/\zeta(\beta,Q) &= \prod_{p\in\mathcal{P}} (1-t_p) \\
t_p &= \frac{1}{|\Lambda_p|} e^{\beta\Phi_p - Q\tau_p}\,, \qquad (14)
\end{aligned}
$$

Again, the function $Q(\beta)$ of (7) is the largest solution of the equation $1/\zeta(\beta,Q(\beta)) = 0$ which is equivalent to $F(\beta,Q(\beta))=0$. This replacement is also correct for continuous flow [10], trace formula (65).

The above infinite products can be rearranged as expansions with improved convergence properties [2]. The ζ function is expanded as a formal power series,

$$
\begin{aligned}
1/\zeta &= \prod_p (1-t_p) = 1 + \sum_{\{p_1p_2\dots p_k\}}{}^{'} t_{\{p_1p_2\dots p_k\}}, \\
t_{\{p_1p_2\dots p_k\}} &= (-1)^k t_{p_1}t_{p_2}\dots t_{p_k} \qquad (15)
\end{aligned}
$$

where the prime on the sum indicates that the sum is over all distinct non-repeating combinations of prime cycles. For $k>1$, $t_{\{p_1p_2\dots p_k\}}$ are "pseudo" cycles; they are sequences of shorter cycles that shadow a cycle with symbol sequence $p_1p_2\dots p_k$ along segments p_1, $p_2,\dots,$ p_k. For sufficiently small $z=e^{-Q}$ the sum makes sense as a power series in z.

2.3 Cycle formulas for dynamical averages

The implicit definition of $Q(\beta)$, $1/\zeta(\beta,Q(\beta)) = 0$, together with the expression for the variation in cycle weight (14) as function of β, Q

$$
\delta t_p = (\Phi_p(\omega)\delta\beta - \tau_p\delta Q)t_p\,,
$$

yields the cycle expansion for $\langle\phi\rangle$

$$
\begin{aligned}
\langle\phi\rangle = \frac{\partial Q}{\partial\beta} = -\frac{\frac{\partial\zeta^{-1}}{\partial\beta}}{\frac{\partial\zeta^{-1}}{\partial Q}} &= \frac{\sum' \Phi_{\{p_1p_2\dots p_k\}} t_{\{p_1p_2\dots p_k\}}}{\sum' \tau_{\{p_1p_2\dots p_k\}} t_{\{p_1p_2\dots p_k\}}}, \\
\Phi_{\{p_1p_2\dots p_k\}} &= \Phi_{p_1}+\Phi_{p_2}\dots+\Phi_{p_k}\,, \qquad \tau_{\{p_1p_2\dots p_k\}} = \tau_{p_1}+\tau_{p_2}\dots+\tau_{p_k}\,, \qquad (16)
\end{aligned}
$$

and similarly for the higher derivatives of $Q(\beta)$, such as for the diffusion constant (9)

$$
2D = \frac{\partial^2 Q}{\partial\beta^2} = -\frac{\frac{\partial^2\zeta^{-1}}{\partial\beta^2}}{\frac{\partial\zeta^{-1}}{\partial Q}} = \frac{\sum'(-1)^k(\Phi_{p_1}+\dots+\Phi_{p_k})^2/|\Lambda_{p_1}\cdots\Lambda_{p_k}|}{\sum'(-1)^k(\tau_{p_1}+\dots+\tau_{p_k})/|\Lambda_{p_1}\cdots\Lambda_{p_k}|} \qquad (17)
$$

{cexp}

(in writing (17) we have assumed that $\langle \phi \rangle = 0$, otherwise a more complex expression with the derivatives $\partial^2(\zeta^{-1})/\partial\beta\partial Q$ come in play), and with sums as in (15). Formally all such averages are of form $\langle \phi \rangle = \langle \Phi \rangle' / \langle \tau \rangle'$, but note that this average in not the one one would naively write down [15, 6] using trace formulas; this is not an approximate sum from partition of the phase space into neighborhoods of all periodic points of period t, but an exact sum over all prime cycles, with prefactors $(-1)^k$ ensuring curvature cancelations.

3 FOURIER SPECTRA

3.1 Averages over Fourier transforms

If we chose as the "observable" $\phi(\tau, x_\tau) = e^{i2\pi\omega\tau}\phi(x_\tau)$, the sum (2) becomes a Fourier transform

$$\Phi^t(\omega, x) = \sum_{\tau=0}^{t-1} e^{i2\pi\omega\tau}\phi(x_\tau)\,, \qquad x_0 = x\,, \tag{18}$$ {fou_tran}

and (4) becomes the space-averaged Fourier transform of the time series $\phi(x_0)$, $\phi(x_1)$, $\phi(x_2), \ldots$,

$$\langle \phi(\omega) \rangle = \lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \frac{1}{|M|} \int dx\, e^{i2\pi\omega\tau}\phi(f^\tau(x))\,. \tag{19}$$ {exp_fou_tr

In a chaotic system the space-averaged Fourier transform usually vanishes because

$$\frac{1}{|M|} \int dx\, \phi(f^\tau(x))$$

does not depend on τ. To obtain a non-vanishing quantity, we have first to take the absolute value of the Fourier transform, and after that to perform the averaging. In this way we immediately come to the power spectrum of the process:

$$\left\langle |\Phi^t(\omega)|^2 \right\rangle = t \sum_{\tau=-t+1}^{t-1} (1 - |\tau|/t) C(\tau) e^{i2\pi\omega\tau} \tag{20}$$ {pow_spec}

where $C(m) = \langle \phi(x_\tau)\phi(x_{\tau+m}) \rangle$ is the space-averaged time correlation function.

Generally, a power spectrum of a chaotic observable consists of a broad band noise $S(\omega)$ and of a discrete spectrum $\Delta(\omega)$ which correspond to two terms in the time growth rates of $\left\langle |\Phi^t(\omega)|^2 \right\rangle$:[4]

$$\left\langle |\Phi^t(\omega)|^2 \right\rangle \sim t^2 \Delta(\omega) + tS(\omega). \tag{21}$$ {11}

Comparing with (9), (10), we see that $\frac{1}{2}S(\omega) = D(\omega)$ is nothing else but the diffusion constant for quantity $\Phi^t(\omega)$, and $\Delta(\omega)$ is the drift term. With this interpretation of the observable Φ^t we can hope to obtain the power spectrum from the derivatives of the corresponding leading eigenvalue $Q(\beta)$ like in (9). However, in oder to be able to apply the trace formulas machinery, we need Φ^t to fulfill some requirements, such as additivity along the trajectories and periodic orbits. As we show below, these requirements are valid with restrictions.

[4] Strictly speaking, this time dependence can include powers of time different from one and two, in this situation one speaks on singular continuous spectra [].

3.2 Fourier cycle weights

The main difficulty in applying the trace formulas to the Fourier transformsis that in the former one needs additivity of an observable along the trajectory of a dynamical system. Let us look how the additivity can appear in the Fourier sum.

Evaluated on the rth repeat of a prime cycle p having period τ_p, the Fourier sum (18) factorizes into

$$\begin{aligned} \Phi^{r\tau_p}(\omega, x_{p,m}) &= r\,\Phi_p(\omega)\, e^{-i2\pi\omega m}\frac{1}{r}\sum_{k=0}^{r-1} e^{i2\pi\omega\tau_p k}\,, && (22) \\ \Phi_p(\omega) &= \sum_{\tau=0}^{p-1} e^{i2\pi\omega\tau}\phi(x_{p,\tau})\,. && (23) \end{aligned}$$

The sum in (22) takes values

$$\frac{1}{r}\sum_{k=0}^{r-1} e^{i2\pi\omega\tau_p k} = \left\{ \begin{array}{llrcl} 1 & \text{if} & \tau_p\omega & = & \text{integer}, \\ 0 & \text{if} & \tau_p\omega & \neq & \text{integer}, \;, \quad r\tau_p\omega = \text{integer}, \quad . \\ O(1/r) & \text{if} & r\tau_p\omega & \neq & \text{integer}. \end{array} \right. \qquad (24)$$

In the $t = r\tau_p \to \infty$ limit the $O(1/r)$ terms vanish, and the sum (18) evaluated on the rth repeat of a prime cycle p projects out all frequencies ω which are not harmonics of the prime cycle frequency $1/\tau_p$:

$$\Phi^{r\tau_p}(\omega, x_{p,m}) \sim \left\{ \begin{array}{llrcl} r\,\Phi_p\, e^{-i2\pi\omega m} & \text{if} & \tau_p\omega & = & \text{integer} \\ 0 & \text{if} & \tau_p\omega & \neq & \text{integer} \end{array} \right. . \qquad (25) \quad \{\text{Phi\_om}\}$$

This formula can be interpreted in two ways. Let us first fix the periodic orbit under consideration, i.e. fix τ_p. Then, this orbit according to (25) gives non-wanishing weights for frequencies $0, \frac{1}{\tau_p}, \frac{2}{\tau_p}, \ldots$. This means that the orbit "contributes" to the power spectrum at those frequencies only, which frequencies can be presented in the spectrum of this orbit. At the other hand, let us fix a frequency ω. If this frequency is irrational, no periodic orbit "contributes" to it. If the frequency is rational $\omega = \frac{l}{q}$, then all orbits with periods $q, 2q, 3q, \ldots$ contribute to the power spectrum at this frequency.

Therefore, we have to consider the rational frequencies only. From (24),(25) it follows, that for these frequencies also the additivity of the Fourier weight along the periodic orbit of the dynamical system holds, provided the periods of orbits satisfy the relation $\tau_p\omega$ = integer. To ensure that only such orbits come in play, we can do the following: given a rational frequency $\omega = \frac{l}{q}$, let us consider the q-th iteration of (1):

$$x_{\tau+q} = f^q(x_\tau) \qquad (26) \quad \{\text{1q}\}$$

and as an obversable let us take

$$\phi^{(q)}(x_\tau) = \sum_{t=0}^{q-1} e^{i2\pi\omega t}\phi(x_{\tau+t}) \qquad (27) \quad \{\text{phiq}\}$$

Then the Fourier transform (18) for this frequency reduces to the sum (2), and the trace formula (12) is valid. After finding the diffusion constant for (26),(27) one should not forget to divide it by q to get the diffusion constant in the original time scale.

A small modification of the trace formula is still needed, because our observable (27) is complex, and we want to average the square of its absolute value. To do this it is convenient to consider the auxillary variable β as a complex one $\beta = \beta_r + i\beta_i$, and to average in (5) the real part, so that instead of (7) we write

$$\lim_{t\to\infty} \left\langle e^{\mathrm{Re}(\beta\cdot\Phi^t)} \right\rangle = \lim_{t\to\infty} \left\langle e^{\beta_r\cdot\mathrm{Re}(\Phi^t)-\beta_i\cdot\mathrm{Im}(\Phi^t)} \right\rangle \to e^{tQ(\beta)} \quad (28) \quad \{2r\}$$

The diffusion constant can be then represented through the derivatives [5]

$$\left.\frac{\partial^2 Q}{\partial \beta_r^2}\right|_{\beta_r=\beta_i=0} + \left.\frac{\partial^2 Q}{\partial \beta_i^2}\right|_{\beta_r=\beta_i=0} = \lim_{t\to\infty} \frac{1}{t} \left\langle |\Phi^t(\omega)|^2 \right\rangle = S(\omega)\,. \quad (29) \quad \{4bb\text{-}reim\}$$

3.3 Cycle expansions for power spectrum

`AP: calculations in this section not rechecked`

In view of discussion above we have to apply the main cycle expansion formula to the q-th iteration of the map to find the power spectrum at frequency $\omega = \frac{l}{q}$. This means that index p in (13)-(17) counts all prime cycles of the map f^q, not of the map f. The main formula of cycle expansion of the power spectrum now reads

$$S(\omega) \;=\; \frac{\sum'(-1)^k |\Phi_{p_1} + \cdots + \Phi_{p_k}|^2/|\Lambda_{p_1}\cdots\Lambda_{p_k}|}{\sum'(-1)^k (\tau_{p_1} + \cdots + \tau_{p_k})/|\Lambda_{p_1}\cdots\Lambda_{p_k}|} \quad (30) \quad \{cexp1\}$$

Here ω is rational $\omega = \frac{l}{q}$; p are prime cycles of the q-th iteration of the map, and Φ_p are (complex) Fourier sums along the cycles calculated according to (27).

As all the prime cycles of the map f^q stem from cycles of the original map f, it is instructive to write the corresponding ζ-function. It is clear that we should divide all primary cycles of f can be divided in two classes:

1. To the set $\mathcal{P}_q$ belong all primary cycles of f having periods q, $2q$, $3q, \ldots$. They appear for the map f^q as primary cycles of periods $1, 2, 3, \ldots$ correspondingly.
2. All other primary cycles of f with periods $r \neq n \cdot q$ also appear as primary cycles of f^q, namely as period-s cycles where s is a minimal integer satisfying together with another integer m the relation $r \cdot m = s \cdot q$

[5] We assume $\langle \Phi^t \rangle = 0$, what means that there is no discrete spectrum at the chosen frequency. For zero frequency this can be ensured by chosing the observable $\phi(x)$ having zero mean, for other frequencies this is ensured by mixing properties of the system. For a special consideration of non-mixing case see section 4.3 below

Next, we note that to each primary cycle p of f of type 1 (having period τ_p) correspond q cycles of f^q lying on its trajectory. The factors Φ_p for these cycles differ by the phase shift $e^{i2\pi\frac{l}{q}j}$. Thus, to each such cycle corresponds a product

$$\prod_{j=0}^{\tau_p-1}(1-t_{pj}) \qquad t_{pj}=\frac{1}{|\Lambda_p|}e^{\beta\Phi_p(\omega)e^{i2\pi\omega j}-Q\tau_p}$$

For a cycle of type 2, the Fourier weight vanishes according to (25). It is equivalent to setting $\beta=0$ at the corresponding t_p. Due to remaining additivity of the periods and multiplicativity of the multipliers, we can write the contribution as

$$(1-(t_p(0))^m)$$

where the zero argument means that here effectively $\beta=0$. As a result, we can write ζ-function for the calculation of power spectrum at frequency $\omega=\frac{l}{q}$ as

$$\frac{1}{\zeta}=\prod_{p\in\mathcal{P}_q}\prod_{j=0}^{\tau_p-1}(1-t_{pj})\prod_{p\notin\mathcal{P}_q}(1-(t_p(0))^m) \tag{31}$$ {zetafreq}

4 APPLICATINS TO 1-d MAPS

There exist maps - typically 1-d piecewise-linear maps - for which the natural measure is available in closed form. As for such maps the power spectra are known analytically, we can use them as benchmarks for tests of cycle expansions. We start with two maps whose symbolic dynamics is described by the full (0,1) binary shift, *ie.* all sequences of "0" and "1" are realizable: the Bernoulli shift and the skew tent map. C. Beck has further results for the Ulam map, based on the Chebyshev polynomials method of ref. [13]. While the cycle expansions do not appear to be convenient for rederivation of the analytic results, they converge faster than exponentially in numerical evaluations, and are also applicable to generic flows, where the natural measure is not analytically available.

4.1 Bernoulli doubling map

For the Bernoulli map $f(x)=2x \mod (1)$ the natural measure is $\mu(x)=1$. Taking the observable $\phi=x-\langle x\rangle=x-1/2$ we obtain the variance

$$C(0)=\left\langle(x_t-\langle x\rangle)^2\right\rangle=\int_0^1 dx(x-\frac{1}{2})^2=\frac{1}{12}\,, \tag{32}$$ {correl_0_B

and the correlation function is

$$C(m)=\langle(x_t-\langle x\rangle)(x_{t+m}-\langle x\rangle)\rangle=\frac{1}{12}2^{-m}\,. \tag{33}$$ {correl_B}

The power spectrum follows from (20),(33)

$$S(\omega) \;=\; \frac{1}{4}\frac{1}{5-4\cos 2\pi\omega} \tag{34}$$

We shall compare this exact solution with the values obtained from cycle expansion.

The periodic points are easily computed according to the general formula:

$$x_{\epsilon_1\epsilon_2\ldots\epsilon_k} = .\overline{\epsilon_1\epsilon_2\ldots\epsilon_k} = \frac{2^{k-1}\epsilon_1 + 2^{k-2}\epsilon_2 + \cdots + \epsilon_k}{2^k - 1} \qquad (35) \quad \{per\_B\}$$

and are presented in the table.

Symbols	x	Φ_p for $\omega = 0$	Φ_p for $\omega = 1/2$	Φ_p for $\omega = 1/3$
0	0	-1/2		0
1	1	1/2		0
01	1/3	0	-1/3	
10	2/3		1/3	
001	1/7			$(1+2\xi+4\xi^2)/7$
010	2/7	-1/2		$(2+4\xi+\xi^2)/7$
100	4/7			$(4+\xi+2\xi^2)/7$
011	3/7			$(3+6\xi+5\xi^2)/7$
110	6/7	1/2		$(6+5\xi+3\xi^2)/7$
101	5/7			$(5+3\xi+6\xi^2)/7$
0001	1/15		-1/3	
0010	2/15		1/3	
0100	4/15	-1		
1000	8/15			
0011	3/15	0	0	
0110	6/15		0	
1100	12/15	0		
1001	9/15			
0111	7/15		-1/3	
1110	14/15			
1101	13/15	1		
1011	11/15		1/3	

4.1.1 Spectrum at zero frequency

Here we have a trival calculation of the diffusion coefficient, based on the prime cycles of the Bernoulli map. It is convenient to rewrite the general expression

$$S(0) = \frac{\partial^2 Q}{\partial \beta^2} = \frac{\sum'(-1)^k(\Phi_{p_1} + \Phi_{p_2} + \cdots + \Phi_{p_k})^2/(|\Lambda_{p1}\cdots\Lambda_{p_k}|)}{\sum'(-1)^k(n_{p_1} + n_{p_2} + \cdots + n_{p_k})^2/(|\Lambda_{p1}\cdots\Lambda_{p_k}|)} \qquad (36) \quad \{rat1\}$$

in the form making compensation of contributions of different cycles evident. This form follows immediately from the product (15)

$$
\begin{aligned}
0 &= \prod_p (1-t_p) \\
&= (1-t_0)(1-t_1)(1-t_{10})(1-t_{100})(1-t_{101})(1-t_{1000})(1-t_{1001})(1-t_{1011})\dots \\
&= 1-t_0-t_1 \\
&- (t_{10}-t_1 t_0) \\
&- (t_{100}-t_{10}t_0) - (t_{101}-t_{10}t_1) \\
&- (t_{1000}-t_{100}t_0) - (t_{1001}-t_{100}t_1) - (t_{1011}-t_{101}t_1) + t_{101}t_0 - t_{10}tt_1t_0 \\
&\dots
\end{aligned}
$$

Noting that the denominator in (36) is a frequency independent normalization constant

$$
\sum{}'(-1)^k \frac{(n_{p_1}+\dots+n_{p_k})}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|} = [-\frac{1}{2}-\frac{1}{2}] + [-\frac{1}{4}(2-1-1)] + [-\frac{1}{8}(3-3)-\frac{1}{8}(3-3)] + \dots = -1\,,
$$

we can rewrite the power spectrum at zero frequency as

$$
\begin{aligned}
&\frac{1}{2}(\Phi_0^2+\Phi_1^2) \\
&\frac{1}{4}(\Phi_{10}^2-(\Phi_0+\Phi_1)^2) \\
&\frac{1}{8}[(\Phi_{100}^2-(\Phi_{10}+\Phi_0)^2)+(\Phi_{101}^2-(\Phi_{10}+\Phi_1)^2)] \\
&\frac{1}{16}[(\Phi_{1000}^2-(\Phi_{100}+\Phi_0)^2)+(\Phi_{1001}^2-(\Phi_{100}+\Phi_1)^2] \\
&-\frac{1}{16}[(\Phi_{1011}^2-(\Phi_{101}+\Phi_1)^2)-((\Phi_{101}+\Phi_0)^2-(\Phi_{10}+\Phi_1+\Phi_0)^2)] \\
&\dots
\end{aligned}
\tag{37}
$$

{expsimp}

Here we should insert the values of Φ_p presented in the table. We this obtain for the numerator

$$
\begin{aligned}
&-\frac{1}{2}(\frac{1}{4}+\frac{1}{4}) \\
&-\frac{1}{4}(0-0) \\
&-\frac{1}{8}[(\frac{1}{4}-\frac{1}{4})+(\frac{1}{4}-\frac{1}{4})] \\
&-\frac{1}{16}[(1-1)+(0-0)] \\
&-\frac{1}{16}[(1-1)-(0-0)]+\cdots \\
&= -\frac{1}{4}
\end{aligned}
\tag{38}
$$

Here we have a perfect cancellation. The resulting value of spectrum is

$$
S(0) = \frac{1}{4}
$$

in accordance to (34).

4.1.2 Spectrum at frequency 1/2

Let us consider the second iteration of the Bernoulli map. For this iteration the Fourier sum according to (27) is

$$\phi^{(2)}(x_\tau) = \phi(x_\tau) - \phi(x_{\tau+1})$$

is a function of the state space which is additive on a trajectory. Thus, one can apply the cycle expansion for diffusion to this map. There are four basic symbols

$$a = 00\,, \qquad b = 01\,, \qquad c = 10\,, \qquad d = 11\,.$$

We write the product (15) as

$$\begin{aligned}
0 &= \prod(1-t_p) \\
&= (1-t_a)(1-t_b)(1-t_c)(1-t_d)(1-t_{ab})(1-t_{ac})(1-t_{ad})(1-t_{bc})(1-t_{bd})(1-t_{cd})\ldots \\
&= 1 - t_a - t_b - t_c - t_d \\
&- (t_{ab} - t_a t_b) - (t_{ac} - t_a t_c) - (t_{ad} - t_a t_d) \\
&- (t_{bc} - t_b t_c) - (t_{bd} - t_b t_d) - (t_{cd} - t_c t_d) \\
&\cdots
\end{aligned} \tag{39}$$

where

$$t_p = \frac{e^{\beta\Phi_p - Q n_p}}{\Lambda_p}\,, \qquad \Phi_p = x_p - f(x_p)\,.$$

The diffusion constant is

$$2S(1/2) = \frac{\partial^2 Q}{\partial \beta^2} = \frac{\sum'(-1)^k (A_{p_1} + A_{p_2} + \cdots + A_{p_k})^2/(|\Lambda_{p1}\cdots\Lambda_{p_k}|)}{\sum'(-1)^k (n_{p_1} + n_{p_2} + \cdots + n_{p_k})^2/(|\Lambda_{p1}\cdots\Lambda_{p_k}|)}\,, \tag{40}$$ {rat2}

where the factor 2 on the left hand side appears because the diffusion is calculated in the new doubled time.

The denominator is

$$\sum{}'(-1)^k \frac{(n_{p_1} + \cdots + n_{p_k})}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|} \tag{41}$$

$$\begin{aligned}
&= -\frac{1}{4} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} \\
&- \frac{1}{16}(2-2)\cdots \\
&+ \cdots = -1\,,
\end{aligned} \tag{42}$$ {time_aver2

The values of Φ_p are presented in the table We substitute this in the numerator:

$$
\begin{aligned}
&-\frac{1}{4}(\Phi_a^2+\Phi_b^2+\Phi_c^2+\Phi_d^2)\\
&-\frac{1}{16}[(\Phi_{ab}^2-(\Phi_a+\Phi_b)^2)+(\Phi_{ac}^2-(\Phi_a+\Phi_c)^2)+(\Phi_{ad}^2-(\Phi_a+\Phi_d)^2]\\
&-\frac{1}{16}[(\Phi_{bc}^2-(\Phi_b+\Phi_c)^2)+(\Phi_{bd}^2-(\Phi_b+\Phi_d)^2)+(\Phi_{cd}^2-(\Phi_c+\Phi_d)^2]\\
&=-\frac{1}{4}(0+(\frac{1}{3})^2+(\frac{1}{3})^2+0)\\
&-\frac{1}{16}[(\frac{1}{3})^2-(\frac{1}{3})^2)+(\frac{1}{3})^2-(\frac{1}{3})^2)+(0-0)]\\
&-\frac{1}{16}[(\frac{1}{3})^2-(\frac{1}{3})^2)+(\frac{1}{3})^2-(\frac{1}{3})^2)+(0-0)]\\
&=-\frac{1}{18}
\end{aligned}
\tag{43}
$$

As a result we obtain the correct value of $S(1/2) = 1/36$ and a perfect cancellation of higher-order contributions.

4.1.3 Representation of cycle formula for period 1/2 through the basic prime cycles

Because the cycles of the time-2 Bernoulli map are not the original ones, it is instructive to represent the cycle expansion in terms of basic prime cycles of the Bernoulli map.

Because the dependence on β in t_a and t_d disappears, we can write

$$t_a(Q,\beta)=t_a(Q,0)=t_{00}(Q,0)=t_0^2(Q,0) \qquad t_d(Q,\beta)=t_{11}(Q,0)=t_1^2(Q,0)$$

For the product of two remaining terms we obtain

$$
\begin{aligned}
(1-t_b)(1-t_c)&=1-t_{01}(Q,\beta)-t_{10}(Q,\beta)+t_{01}(Q,\beta)t_{10}(Q,\beta)\\
&=1-t_{01}(Q,\beta)-t_{10}(Q,\beta)+t_{01}^2(Q,0)
\end{aligned}
$$

Where we have used the property $\Phi_{01}+\Phi_{10}=0$ (which can be written also as $t_{01}(Q,\beta)=t_c(Q,-\beta)$)

Analogously, we can write

$$
\begin{aligned}
t_{ab}(Q,\beta)&=t_{0001}(Q,\beta)=t_{0010}(Q,-\beta)=t_{ac}(Q,-\beta)\\
t_{ad}(Q,\beta)&=t_{0011}(Q,\beta)=t_{0110}(Q,-\beta)=t_{bc}(Q,-\beta)\\
t_{bd}(Q,\beta)&=t_{0111}(Q,\beta)=t_{1011}(Q,-\beta)=t_{cd}(Q,-\beta)
\end{aligned}
$$

and the product of primary period-4 cycles appears as

$$
\begin{aligned}
&(1-t_{ab})(1-t_{ac})(1-t_{ad})(1-t_{bc})(1-t_{bd})(1-t_{cd})\\
=&(1-t_{0001}(Q,\beta)-t_{0010}(Q,\beta)+t_{0001}^2(Q,0))\\
&\times(1-t_{0011}(Q,\beta)-t_{0110}(Q,\beta)+t_{0011}^2(Q,0))\\
&\times(1-t_{0111}(Q,\beta)-t_{1101}(Q,\beta)+t_{0111}^2(Q,0))
\end{aligned}
$$

Finaly we obtain

$$\begin{aligned}
0 =&(1 - t_0^2(Q,0))(1 - t_1^2(Q,0))(1 - t_{01}(Q,\beta) - t_{10}(Q,\beta) + t_{01}^2(Q,0)) \\
&\times (1 - t_{0001}(Q,\beta) - t_{0010}(Q,\beta) + t_{0001}^2(Q,0))(1 - t_{0011}(Q,\beta) - t_{0110}(Q,\beta) + t_{0011}^2(Q,0)) \\
&\times (1 - t_{0111}(Q,\beta) - t_{1101}(Q,\beta) + t_{0111}^2(Q,0)) \quad \dots \\
=&1 - t_0^2(Q,0) - t_1^2(Q,0) \\
&- t_{01}(Q,\beta) - t_{10}(Q,\beta) \\
&- t_{0001}(Q,\beta) + t_0^2(Q,0)t_{01}(Q,\beta) - t_{0010}(Q,\beta) + t_0^2(Q,0)t_{10}(Q,\beta) \\
&- t_{0011}(Q,\beta) - t_{0110}(Q,\beta) + t_0^2(Q,0)t_1^2(Q,0) + t_{01}^2(Q,0) \\
&- t_{1110}(Q,\beta) + t_1^2(Q,0)t_{10}(Q,\beta) - t_{1101}(Q,\beta) + t_1^2(Q,0)t_{01}(Q,\beta) \quad \dots
\end{aligned} \tag{44}$$

One can see how the cancellating terms appear in the cycle expansion.

4.1.4 Spectrum at frequency 1/3

Here we have to coinsider the 3rd iteration of the Bernoulli map. It has 8 primary fixed points

$$a = 000 \quad b = 001 \quad c = 010 \quad d = 011 \quad e = 100 \quad f = 101 \quad g = 110 \quad h = 111$$

corresponding to the orbits in the table. Denoting $\xi = e^{i\frac{2\pi}{3}}$, we write also the corresponding values of Φ_p in the table.

Analogously, there are 28 primary cycles of period 2 (period-6 orbits for the Bernoulli map) labeled as

$$ab = 000001 \quad ac = 000010 \quad \cdots$$

Writing the zeta-function as

$$\begin{aligned}
0 =& \prod (1 - t_p) \\
&(1 - t_a)(1 - t_b)(1 - t_c)(1 - t_d)(1 - t_e)(1 - t_f)(1 - t_g)(1 - t_h) \\
&\times (1 - t_{ab})(1 - t_{ac})(1 - t_{ad})(1 - t_{ae})(1 - t_{af})(1 - t_{ag})(1 - t_{ah}) \\
&\times (1 - t_{bc})(1 - t_{bd})(1 - t_{cd}) \dots \\
=&1 - t_a - t_b - t_c - t_d - t_e - t_f - t_g - t_h \\
&- (t_{ab} - t_a t_b) - (t_{ac} - t_a t_c) - (t_{ad} - t_a t_d) \\
&- (t_{ae} - t_a t_e) - (t_{af} - t_a t_f) - (t_{ag} - t_a t_g) - (t_{ah} - t_a t_h) \\
&- (t_{bc} - t_b t_c) - (t_{bd} - t_b t_d) - (t_{cd} - t_c t_d) \\
&\dots
\end{aligned} \tag{45}$$

The cancellation is evident.

Because Φ_p are complex, we have to sum complex values of Φ_p along "pseudocycles", and to take the absolute value square of thisn sum:

$$3S(1/3) = \frac{\partial^2 Q}{\partial(\mathrm{Re}\beta)^2} + \frac{\partial^2 Q}{\partial(\mathrm{Im}\beta)^2} = \frac{\sum'(-1)^k |\Phi_{p_1} + \Phi_{p_2} + \dots + \Phi_{p_k}|^2/(|\Lambda_{p1} \cdots \Lambda_{p_k}|)}{\sum'(-1)^k (n_{p_1} + n_{p_2} + \dots + n_{p_k})^2/(|\Lambda_{p1} \cdots \Lambda_{p_k}|)} \tag{46}$$ {rat3}

where the factor 3 appears because the diffusion is calculated in the new tripled time.

The values of $|\Phi_p|^2$ are the same for all cycles $b \dots g$:

$$|\Phi_b|^2 = |\Phi_c|^2 = \dots = |\Phi_g|^2 = \frac{1}{7}$$

while for a and h $|\Phi_a| = |\Phi_h| = 0$. The denominator in (46) is -1, and we obtain

$$S(1/3) = \frac{1}{3} \cdot 6 \cdot \frac{1}{7} \cdot \frac{1}{8} = \frac{1}{28}$$

in accordance with (34).

Next we represent the zeta-function in terms of primary cycles of the Bernoulli map. From the cycles above the following are primary cycles of smaller periods:

$$\begin{aligned} t_a &= t_{000} = t_0^3(0) & t_h &= t_{111} = t_1^3(0) \\ t_{cf} &= t_{010101} = t_{01}^3(0) & t_{fc} &= t_{101010} = t_{10}^3(0) \end{aligned} \tag{47}$$

Here we have written the argument to show the absence of β-dependence in these terms.

The product (15) can be written in the terms of primary cycles, in the following we goup the terms according to their cancellation.

$$\begin{aligned} 0 =& 1 - t_0^3(0) - t_1^3(0) \\ & - t_{001} - t_{010} - t_{100} - t_{011} - t_{110} - t_{101} \\ & + (-t_{000001} + t_0^3 t_{001}) + (-t_{000010} + t_0^3 t_{010}) + (-t_{000100} + t_0^3 t_{100}) \\ & + (-t_{000011} + t_0^3 t_{011}) + (-t_{000110} + t_0^3 t_{110}) + (-t_{001100} + t_{001} t_{100}) \\ & + (-t_{000101} + t_0^3 t_{101}) + (-t_{001010} + t_{001} t_{010}) + (-t_{010100} + t_{010} t_{100}) \\ & + (-t_{000111} + t_0^3 t_1^3) + (-t_{001110} + t_{001} t_{110}) + (-t_{011100} + t_{011} t_{100}) \\ & + (-t_{001011} + t_{001} t_{011}) + (-t_{010110} + t_{010} t_{110}) + (-t_{101100} + t_{101} t_{100}) \\ & + (-t_{001101} + t_{001} t_{101}) + (-t_{011010} + t_{011} t_{010}) + (-t_{11010} + t_{110} t_{100}) \\ & + (t_{010} t_{101} - t_{01}^3) + (t_{101} t_{010} - t_{10}^3) \\ & + (-t_{011101} + t_{011} t_{101}) + (-t_{111010} + t_1^3 t_{010}) + (-t_{110101} + t_{110} t_{101}) \\ & + (-t_{001111} + t_1^3 t_{001}) + (-t_{011110} + t_{011} t_{110}) + (-t_{111100} + t_1^3 t_{100}) \\ & + (-t_{011111} + t_1^3 t_{011}) + (-t_{111110} + t_1^3 t_{110}) + (-t_{111101} + t_1^3 t_{101}) \\ & + \cdots \end{aligned} \tag{48}$$

4.1.5 Obtaining full spectrum from cycles

As we have seen, all higher contributions cancel, so it appears that the spectrum of the Bernoulli map can be fully represnted in the form of primary cycles.

Consider a frequency $\omega = p/q$. The primary cycles here are all cycles of period q, such cycles can be written as

$$x_{\epsilon_1 \epsilon_2 \dots \epsilon_q} = .\overline{\epsilon_1 \epsilon_2 \dots \epsilon_q} = \frac{2^{k-1}\epsilon_1 + 2^{k-2}\epsilon_2 + \cdots + \epsilon_q}{2^q - 1} \tag{49}$$ {qcyc}

The Fourier weight for this cycle is

$$\Phi_{\epsilon_1\epsilon_2\ldots\epsilon_q} = x_{\epsilon_1\epsilon_2\ldots\epsilon_q} + e^{i2\pi\frac{p}{q}} x_{\epsilon_2\epsilon_3\ldots\epsilon_q\epsilon_1} + \ldots e^{i2\pi\frac{p}{q}(q-1)} x_{\epsilon_q\epsilon_1\ldots\epsilon_{q-1}}$$

Substituting (49) and regrouping the terms we get

$$\begin{aligned}
\Phi_{\epsilon_1\epsilon_2\ldots\epsilon_q} =& \frac{\epsilon_q}{2^q-1}\left(1 + 2e^{i2\pi\frac{p}{q}} + 4e^{i2\pi\frac{p}{q}2} + \cdots + 2^{q-1}e^{i2\pi\frac{p}{q}(q-1)}\right) \\
&+ \frac{\epsilon_{q-1}}{2^q-1}\left(2 + 4e^{i2\pi\frac{p}{q}} + 8e^{i2\pi\frac{p}{q}2} + \cdots + e^{i2\pi\frac{p}{q}(q-1)}\right) \\
&\ldots \\
&= \sum_{s=q}^{1} \frac{\epsilon_s e^{-i2\pi\frac{p}{q}s}}{2^q-1} \sum_{k=0}^{q-1} 2^k e^{i2\pi\frac{p}{q}k} \\
&= \sum_{s=q}^{1} \frac{\epsilon_s e^{-i2\pi\frac{p}{q}s}}{1-2e^{i2\pi\frac{p}{q}}}
\end{aligned}$$

The power spectrum is the sum of squared absolute values of these terms:

$$S(\frac{p}{q}) = \frac{1}{q2^q}\frac{1}{5-4\cos 2\pi\frac{p}{q}} \sum_{\text{all cycles}} |\sum_{s=1}^{q} \epsilon_s e^{-i2\pi\frac{p}{q}s}|^2$$

where the sum is over all combinations of the symbols $0, 1$, i.e. the number of cycles is 2^q

`AP: It appears that`

$$\frac{1}{2^q}\sum_{\texttt{all } 2^q \texttt{ cycles}} |\frac{1}{\sqrt{q}}\sum_{s=1}^{q} \epsilon_s e^{-i2\pi\frac{p}{q}s}|^2 = \frac{1}{4}$$

`but I cannot proove it.`

4.2 Skew tent map

[6] The skew tent map

$$f(x) = \begin{cases} ax & \text{if } 0 \le x \le a^{-1}, \\ \frac{a}{a-1}(1-x) & \text{if } a^{-1} \le x \le 1. \end{cases} \qquad (50) \quad \{12.1\}$$

is another example where statistical properties can be obtained analytically. The correlation function was calculated by Grossmann and Thomae [11]:

$$C(m) = \langle (x_t - \langle x\rangle)(x_{t+m} - \langle x\rangle)\rangle = \frac{1}{12}\left(\frac{2-a}{a}\right)^m \qquad (51) \quad \{\texttt{correl}\}$$

The power spectrum is

$$\begin{aligned}
S(\omega) &= \frac{1}{6}\frac{a-1}{a(a-2)(1+\cos 2\pi\omega)+2} \\
S(0) &= \frac{1}{12(a-1)}, \qquad S(1/2) = \frac{a-1}{12}, \ldots \\
S(\omega) &= \frac{1}{12} \quad \text{for} \quad a = 2\,.
\end{aligned} \qquad (52)$$

[6] PC: this calculation should be done more analytically?

Consider first the zero frequency case, $\omega = 0$. In this case the Ruelle zeta function

$$1/\zeta = (1-t_0)(1-t_1)(1-t_{01})(1-t_{001})... = 1 - t_0 - t_1 - [t_{01} - t_0 t_1]...$$

consists of the fundamental part and curvature corrections, and calculation of diffusion constant is straightforward. Stability of a cycle with $n_{p,1}$ repeats of symbol 1 is $\Lambda_p = a^{n_p}/(1-a)^{n_{p,1}}$. The probability conservation follows trivially from $1/|\Lambda_0| + 1/|\Lambda_1| = 1/a + (a-1)/a = 1$ (all curvatures vanish for this peacewise-linear map), and the denominator in (17) is simply -1. For $\beta = 0$ the Fredholm determinant is given by

$$F = \left(1 - \frac{z}{|\Lambda_0|} - \frac{z}{|\Lambda_1|}\right)\left(1 - \frac{z}{|\Lambda_0|^2} + \frac{z}{|\Lambda_1|^2}\right)\cdots = (1-z)\left(1 - z\frac{2-a}{a}\right)\ldots . \tag{53}$$

The leading eigenvalue is 1, by probability conservation; the second eigenvalue controls the exponential falloff of the 2-point correlations, $\lambda_1 = a/(2-a)$, in agreement with (51).

The periodic points are easily computed:

$$\begin{aligned}
&x_0 = 0, \quad x_1 = \frac{a}{2a-1}, \quad x_{10} = \frac{a^2}{a^2+a-1}, \quad x_{01} = \frac{a}{a^2+a-1},\\
&x_{001} = \frac{a}{a^3+a-1} \quad x_{010} = \frac{a^2}{a^3+a-1} \quad x_{100} = \frac{a^3}{a^3+a-1}\\
&x_{011} = \frac{a}{a^3-a^2+2a-1} \quad x_{110} = \frac{a^2}{a^3-a^2+2a-1} \quad x_{110} = \frac{a^3-a^2+a}{a^3-a^2+2a-1}
\end{aligned} \tag{54}$$

For $a = 2$ we have:

$$\begin{aligned}
&x_0 = 0, \quad x_1 = 2/3, \quad x_{01} = 2/5, \quad x_{10} = 4/5, \quad x_{001} = 2/9, \quad x_{010} = 4/9, \quad x_{100} = 8/9,\\
&x_{011} = 2/7, \quad x_{110} = 4/7, \quad x_{101} = 6/7, \quad \ldots
\end{aligned} \tag{55}$$

(the general formula is given in the appendix). Even though the power spectrum is given explicitely by (52), the cycle expansion does not reproduce this expression in any obvious form; (17) yields [7] [8]

$$\begin{aligned}
S(0) =&(0-\frac{1}{2})^2\frac{1}{a} + \left(\frac{a}{2a-1} - \frac{1}{2}\right)^2 \frac{a-1}{a}\\
&+\left(\frac{a}{a^2+a-1} + \frac{a^2}{a^2+a-1} - 1\right)^2 \frac{a-1}{a^2} - \left(\frac{a}{2a-1} - 1\right)^2 \frac{a-1}{a^2} + \ldots\\
=&\frac{1}{4a} + \frac{a-1}{4(2a-1)^2 a}\\
&+\frac{a-1}{a^2(a^2+a-1)^2} - \frac{(a-1)^3}{(2a-1)^2 a^2}
\end{aligned} \tag{56}$$

$$\begin{aligned}
\Phi_{01}(1/2) &= x_{01} - x_{10} = \frac{a(a-1)}{a^2+a-1}, \quad \ldots\\
S(1/2) &= \frac{(a-1)^3}{(a^2+a-1)^2} + \text{4-cycle contributions}
\end{aligned} \tag{57}$$

[7]PC: these calculations not rechecked.
[8]AP: not recheked either

We see that there appears no evident cancellation of the cycles of different periods. Even in the case $a = 2$ there is no such cancellation, as one can see from the series for the $S(0)$:

$$S(0) = [\frac{1}{8} + \frac{1}{8 \cdot 9}] + [\frac{1}{4 \cdot 25} - \frac{1}{9 \cdot 4}] + \dots$$

Nevertheless, numerical summation of the series shows up nice convbergence to the analytic formula (52).

The result is given in fig. 1, with fig. 2 illustrating the convergence with maximal cycle length truncations. We have also checked that for the Ruelle zeta functions convergence is exponential, and that for the Fredholm determinant the convergence is faster than exponential. [9]

4.3 Pruned symbolic dynamics

{sec:psd}

If the symbolic dynamics is described by a subshift of a finite type, cycle expansions converge well [2]. An example is given by the tent map $f(x) = 1 - a|x|$, $a = (1 + \sqrt{5})/2$. This value of a corresponds to the $\overline{001}$, $\overline{011}$ 3-cycles bifurcation value, with the symbolic dynamics given by a simple pruning rule; the repeat $\_00\_$ is forbidden. Fig. 3 shows the numerical power spectrum evaluated by means of (17).

Difficulties can arise if the system is not sufficiently mixing. For example, the tent map $x_{t+1} = 1 - a|x_t|$ for $a = \sqrt{2}$ has two nonoverlaping bands. The system is not mixing, and the power spectrum contains a delta-function peak at $\omega = 1/2$. In the symbolic dynamics only sequences having "1" at all odd or at all even places are allowed. Let us focus our attention on the zeta function $1/\zeta$ at the frequency $\omega = 1/2$. The set of cycles with odd periods is empty (except for the fixed point "1", and it also disappears for $a < \sqrt{2}$) and

$$1/\zeta = \prod_{p \in \mathcal{P}_e} (1 - t_p(Q, \beta)) \qquad (58)$$ {r2e}

The straightforward differentiation of $1/\zeta$ does not give correct result: the drift term vanishes and the diffusion constant diverges. The reason is that the mapping f^2 is not mixing and has two symmetric invariant sets, so the zeta function (58) is a product of two zeta functions for these sets. The probability distribution function for Φ^t does not tend to a Gaussian hump as $t \to \infty$, but instead to two symmetric humps, drifting away from the origin in opposite directions (that is why the drift term for $1/\zeta$ vanishes). In order to describe the power spectrum for $\omega = 1/2$ correctly, we must restrict the averaging to one hump. This corresponds to considering one of the symmetric attractors of the map f^2. In terms of zeta function this means that we must consider square root of zeta function (58):

$$1/\zeta' = \prod_{p \in \mathcal{P}_e}{}' (1 - t_p(Q, \beta))$$

From this zeta function correct values of discrete and continuous components of the power spectrum at $\omega = 1/2$ are obtained as the drift term and the diffusion constant.

[9] PC: this must be reworked so that for finite Markov graphs, long cycles are re-expressed as the repeats of the fundamental cycles; work out in detail for 2-branch complete binary tent map?.

5 CONCLUSIONS

Fourier analysis for chaotic sets can be recast into the transfer operator formalism, but restricted to rational frequencies l/q. This is unsatisfactory in the sense that exponential numer of prime cycles is required for increasing q. [10]

6 APPENDIX A: Tent map cycles

{APPE}

[11] For the symmetric tent map ($a = 2$ in (50)) it is convenient to compactify the binary sequences by replacing $s_i = \{0, 1\}$ by the n-ary alphabet

$$n_i = \{1, 2, 3, 4, \cdots\} = \{1, 10, 100, 1000, \dots\} \qquad (59)$$ {dike_alph}

In this notation the itinerary of a point x, $S = \{n_1 n_2 n_3 n_4 \cdots\}$, and its binary expansion are related by $x = .1^{n_1} 0^{n_2} 1^{n_3} 0^{n_4} \cdots$. For example:

$$\begin{array}{rcl} x &=& .1011010011110000\ldots &=& .1^1 0^1 1^2 0^1 1^1 0^2 1^3 0^4 \ldots \\ S &=& 111011101001000\ldots &=& 11211234\ldots \end{array}$$

The periodic points correspond to the rational values of x, but we have to distinguish *even* and *odd* cycles, depending on whether $\sum_{k=1}^{n} s_k$ are even or odd. For the even cycles $t_k = t_{k+n}$, while for the odd cycles $t_k = t_{k+2n}$. The even (odd) cycles contain even (odd) number of n_i in the repeating block, and the periodic points are given by

$$\begin{aligned} x_{n_1 n_2 \cdots n_k} &= \frac{2^{n_p}}{2^{n_p} - 1} .1^{n_1} 0^{n_2} \cdots 0^{n_k}\,, \qquad n_p = \sum_{i=1}^{k} n_i \qquad k \text{ even} \\ &= \frac{4^{n_p}}{4^{n_p} - 1} .1^{n_1} 0^{n_2} \cdots 1^{n_k} 0^{n_1} 1^{n_2} \cdots 0^{n_k} \qquad k \text{ odd}\,. \end{aligned} \qquad (60)$$

For example:

$$\begin{array}{rcl} x_1 &=& x_1 &=& .10101\ldots &=& .\overline{10} &=& \frac{2}{3} \\ x_{10} &=& x_2 &=& .1^2 0^2 \ldots &=& .\overline{1100} &=& \frac{4}{5} \\ x_{100} &=& x_3 &=& .1^3 0^3 \ldots &=& .\overline{111000} &=& \frac{8}{9} \\ x_{101} &=& x_{21} &=& .\overline{110} = \frac{6}{7} \end{array}$$

6.1 APPENDIX B: Continuous flows

{Cont_flows

For continuous flow "observable" $\phi(\tau, x(\tau)) = e^{i2\pi\omega\tau}\phi(x(\tau))$, the sum (2) becomes a continuous time Fourier transform

$$\Phi^t(\omega, x) = \int_0^t d\tau e^{i2\pi\omega\tau}\phi(x(\tau))\,, \qquad x(0) = x\,, \qquad (61)$$ {fou_tran_c

[10]PC: seems not too smart - one has to go to long cycles to evaluate spectrum at frequencies with q large, but the spectrum itself is totally smooth. Besides, for continuous time flows this pulls out only one prime cycle at a time, as all periods τ_p are distinct. Need some smearing in ω?

[11]PC: this appendix will probably be thrown out again...

and (4) becomes the space-averaged Fourier transform of the observable along the flow $\phi(x(\tau))$

$$\langle\phi(\omega)\rangle = \lim_{t\to\infty}\frac{1}{|M|}\int dx\,\frac{1}{t}\int_0^t d\tau e^{i2\pi\omega\tau}\phi(f^\tau(x))\,. \qquad (62)$$ {exp_fou_tr

Evaluated on the rth repeat of a prime cycle p, the sum (61) factorizes into

$$\begin{aligned}\Phi^{r\tau_p}(\omega, x(\tau')) &= r\,\Phi_p(\omega)\,e^{-i2\pi\omega\tau'}\frac{1}{r}\sum_{k=0}^{r-1}e^{i2\pi\omega\tau_p k}\,,\\ \Phi_p(\omega) &= \int_0^{\tau_p} d\tau e^{i2\pi\omega\tau}\phi(x_p(\tau))\,,\end{aligned} \qquad (63)$$

As in (25), only the the harmonics of the prime cycle frequency $1/\tau_p$ survive the averaging:

$$\Phi^{r\tau_p}(\omega, x(\tau')) = \left\{\begin{array}{lccc} r\,\Phi_p(\omega)\,e^{-i2\pi\omega\tau'} & \text{if} \quad \tau_p\omega & = & \text{integer} \\ 0 & \text{if} \quad \tau_p\omega & \neq & \text{integer}\end{array}\right. . \qquad (64)$$

For continuous flows the trace (12) (with a similar to (**??**) separation of real and imaginary parts) takes form [10]

$$\begin{aligned}\mathrm{tr}\mathcal{L}^t(\omega) &= \sum_{p\in\mathcal{P}}\tau_p\sum_{r=1}^{\infty}\frac{\delta(t-r\tau_p)}{|\det(\mathbf{1}-\mathbf{J}_p^r)|}\left\langle e^{\beta_c\mathrm{Re}(\Phi^t(\omega))+\beta_s\mathrm{Im}(\Phi^t(\omega))}\right\rangle_p\\ \left\langle e^{\beta_c\mathrm{Re}(\Phi^t(\omega))+\beta_s\mathrm{Im}(\Phi^t(\omega))}\right\rangle_p &= \frac{1}{\tau_p}\int_0^{\tau_p} d\tau e^{r\beta_c\mathrm{Re}\Phi_p(\omega)\cos 2\pi\omega\tau + r\beta_s\mathrm{Im}\Phi_p(\omega)\sin 2\pi\omega\tau}\\ &= \int_0^{2\pi}\frac{d\theta}{2\pi}e^{r(\beta_c\mathrm{Re}\Phi_p(\omega)\cos\theta+\beta_s\mathrm{Im}\Phi_p(\omega)\sin\theta}\\ &= J_0\left(ir\sqrt{(\beta_c\mathrm{Re}\Phi_p(\omega))^2+(\beta_s\mathrm{Im}\Phi_p(\omega))^2}\right)\\ &= \sum_{k=0}^{\infty}\frac{r^{2k}(\beta_c^2(\mathrm{Re}\Phi_p(\omega))^2+\beta_s^2(\mathrm{Im}\Phi_p(\omega))^2)^k}{4^k(k!)^2}\,,\end{aligned} \qquad (65)$$

so only powers of $|\Phi_p(\omega)|^2$ survive the averaging. According to this formula, the sharp frequency Fourier transform picks out essentially a single prime cycle from the infinity of unstable cycles, the one resonant with $\tau_p\omega$ = integer. [12]

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Fig.1

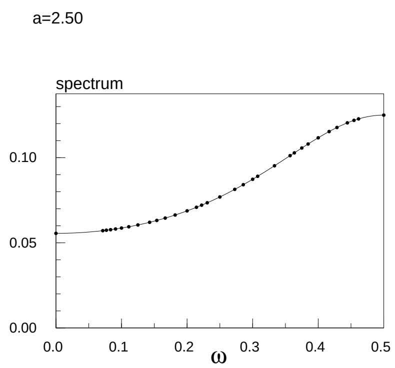

Figure 1: Power spectrum for skew tent map (50) with $a = 2.5$, evaluated from the Fredholm determinant (13) at all rational frequencies $\omega = l/q,\ \ q \leq 14$. The exact spectrum (52) is indicated by the solid line. {fig1}

Fig.2

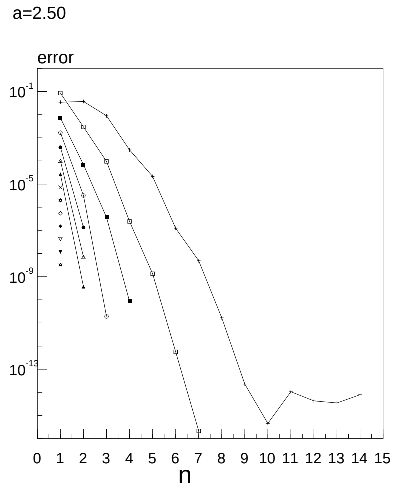

Figure 2: Error in determining the power spectrum of fig. 1 as function of the maximal cycle length used. plus: $\omega = 0$, square: $\omega = 1/2$; other symbols from top to bottom correspond to frequencies $1/3,\ 1/4, ..., 1/14$. {fig2}

Fig.3

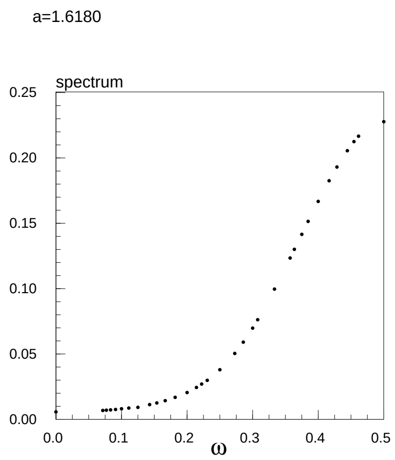

Figure 3: Power spectrum for the pruned symmetric tent map with $a = (1 + \sqrt{5})/2$. No analytic formula for spectrum is available for this case. {fig3}