Periodic Orbit Expansions for Power Spectra of Chaotic Systems

Predrag Cvitanović[1], Mitchell J. Feigenbaum
Rockefeller University, Box 75,
1230 York Ave., New York, NY 10021-6399

Arkady S. Pikovsky[2]
Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen Ø

Abstract [3]

Periodic orbit formulas for evaluation of power spectra of chaotic dynamical systems are derived and tested numerically on several 1-dimensional mappings.

1 INTRODUCTION

We apply the transfer operator techniques to the evaluation of power spectra of chaotic time series. The key idea is the realization that the periodic orbit description of diffusion introduced in refs. [3, 4, 5] can be interpreted as the zero frequency component of the power spectrum of a chaotic dynamical flow; in this paper we generalize the formalism to evaluation of rational frequencies components of power spectra of maps whose dynamics is chaotic. [4]

2 TRANSFER OPERATORS

We seek to compute the generating function for the Fourier transform of the orbit of a dynamical system

$$x_t = f^t(x) \,, \qquad x_0 = x \tag{1}$$

[1]Permanent address: Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen

[2]Current address: Nichtlineare Dynamik, Universität Potsdam, Neuen Palais (Physik), D-O-1571 Potsdam, Germany

[3]PC: file articles/pikovsky/feig6.tex – **3rd MITCHELL DRAFT**, 8 oct 1993.

[4]PC: some blah blah needed here!

averaged over initial conditions, or

$$\left\langle e^{\beta t \hat{x}_t(\omega,x)} \right\rangle_x , \qquad \hat{x}_t(\omega, x) = \frac{1}{t} \sum_{n=0}^{t-1} x_n e^{in\omega} . \tag{2}$$

(Technically, we ultimately consider a slightly more complicated object, since averaging over all the points of an orbit will phase-wise wash out (2). We will fix up (2) after determining our formalism.) We anticipate that as $t \to \infty$, (2) will behave as

$$\left\langle e^{\beta t \hat{x}_t(\omega,x)} \right\rangle_x \to e^{tQ(\beta)} = \lambda^t , \tag{3}$$

so that appropriate β derivatives will determine the expectation value of moments of $\hat{x}(\omega)$. In particular, for $\omega \neq 0$ the power spectrum will be given by

$$\lim_{t\to\infty} t \left\langle |\hat{x}_t(\omega)|^2 \right\rangle_x = \left. \frac{\partial^2 Q}{\partial \beta^2} \right|_{\beta=0} . \tag{4}$$

The power spectrum consists of broad band noise $D(\omega)$ and discrete spectrum $\Delta(\omega)$

$$\left\langle |t\hat{x}_t(\omega)|^2 \right\rangle \sim t^2 \Delta(\omega) + 2tD(\omega) .$$

$D(\omega)$ is nothing else but the diffusion constant [3, 4] for quantity $t\hat{x}_t(\omega)$, and $\Delta(\omega)$ is the drift term (factor 2 is the conventional normalization for diffusion in one dimension).

We seek to identify λ with the leading eigenvalue of a transfer operator, and to perform its calculation upon the periodic orbits of the dynamical system (1). Let us proceed. An obvious operator to consider is $\mathcal{L}_\omega^{(\tau)}$:

$$(\mathcal{L}_\omega^{(\tau)} \psi)(x) = \int dx' \, \delta(x - f(x')) \, e^{\beta x' e^{i\omega\tau}} \psi(x') , \tag{5}$$

where $\mathcal{L}$ is parametrized by ω *and* an integer time value τ. It then follows that

$$(\mathcal{L}_\omega^{(\tau-1)} \cdots \mathcal{L}_\omega^{(0)} \psi)(x) = \int dx' \, \delta(x - f^t(x')) \, e^{\beta t \hat{x}_t(\omega,x')} \psi(x') , \tag{6}$$

so that $\int dx \mathcal{L}_\omega^{(\tau-1)} \cdots \mathcal{L}_\omega^{(0)} \psi$ is just (2) for ψ the chosen measure over phase space. The difficulty with (6) is that we do *not* have the t-th power of an operator, which would then yield λ^t, but rather t compositions of *different* operators $\mathcal{L}^{(\tau)}$. This is unacceptable to our goal and must immediately be remedied.

Now, $\hat{x}_t(\omega)$ is basically the average $\frac{1}{t}\sum a_n x_n$ where a_n is also the orbit of a dynamical system, although generally (and in our case of Fourier analysis) a different one from that of (1). Thus by considering a larger dynamical system, namely the direct product of that for a_t and that for x_t, we can construct a fixed operator $\mathcal{L}$ for which the time-ordered product on the left of (6) becomes precisely $\mathcal{L}^t$. In the case of Fourier analysis $a_t = e^{i\theta_t}$, and the extended dynamical system is

$$\begin{aligned} x_{t+1} &= f(x_t) \\ \theta_{t+1} &= g_\omega(\theta_t) \;=\; \omega + \theta_t \quad (\mathrm{mod}\, 2\pi) \end{aligned} \tag{7}$$

where the θ_t dynamics is the trivial dynamics on the circle. The operator

$$(\mathcal{L}_\omega\, \psi)(x,\theta) \;=\; \int dx'\, d\theta' \delta(x - f(x'))\, \delta(\theta - g_\omega(\theta'))\, e^{\beta x' e^{i\theta'}}\, \psi(x',\theta')\,, \tag{8}$$

(where the θ δ-function is taken mod 2π) satisfies the semi-group property $\mathcal{L}^s\mathcal{L}^t = \mathcal{L}^{s+t}$, so

$$(\mathcal{L}^t_\omega\, \psi)(x,\theta) \;=\; \int dx'\, d\theta' \delta(x - f^t(x'))\, \delta(\theta - g^t_\omega(\theta'))e^{\beta e^{i\theta'} t \hat{x}_t(\omega',x')}\psi(x',\theta')\,, \tag{9}$$

and $\mathcal{L}^t_\omega\, \psi$ integrated over x and θ is precisely (2) enhanced by a richer choice of measures: with $\psi(x,\theta) = \psi(x)\delta(\theta)$, we have (2) itself. Now, with λ the leading eigenvalue $\mathcal{L}^t_\omega\, \psi \to \lambda^t \psi_\lambda$ as $t \to \infty$ and so, unless $(\psi_\lambda, \psi) = 0$, the integral of (9) will behave as λ^t.

We intend to determine λ by evaluating $\det(1 - z\mathcal{L}_\omega) = 0$ for the solution $z_\lambda = 1/\lambda$ of smallest modulus, with the determinant evaluated through

$$\det(1 - z\mathcal{L}) = e^{tr\,\ln(1 - z\mathcal{L})} = e^{-\sum z^t\, tr \mathcal{L}^t/t} \tag{10}$$

so that we need the trace of $\mathcal{L}^t_\omega$ for each t. z_λ itself is just the radius of convergence of the power series in the exponent of (10) and so for it we merely require the exponential (in t) part of $\mathrm{tr}\, \mathcal{L}^t_\omega$ for t asymptotically large. Thus, we must determine these traces. This requires care.

3 TRACE OVER θ

Proceeding formally, we need $\sum_\mu(\psi_\mu, \mathcal{L}_\omega^t\psi_\mu)$ for ψ_μ an orthonormal basis in our $x-\theta$ product space. Clearly $e^{in\theta}/\sqrt{2\pi}$ serve on the circle, and by Fourier analysis, we can take $\psi_\mu(x)e^{in\theta}$ as the x-θ basis for each ψ_μ, the x-basis. Accordingly $\mathrm{tr}\,\mathcal{L}_\omega^t$ can be calculated by independently θ-tracing and then x-tracing. The traces of operators like (5) and (8) are generally of the form

$$\begin{aligned}\mathrm{tr}\,\mathcal{L} &= \sum_\mu(\psi_\mu, \mathcal{L}\psi_\mu) = \sum_\mu \int du\, \psi_\mu^\dagger(u) \int dv\, L(u,v)\psi_\mu(v) \\ &= \int dudv\, L(u,v)\sum_\mu \psi_\mu^\dagger(u)\psi_\mu(v) \;=\; \int du\, L(u,u)\,, \end{aligned} \tag{11}$$

where we used the completeness of the basis ψ_μ. By (11), when we trace $\mathcal{L}^t$, all we need do is drop the ψ in $\mathcal{L}^t\psi$ (as, for example, in (9)) and set the kernel L variables u, v equal to the variable of integration.

Let us begin to evaluate $\mathrm{tr}\,\mathcal{L}_\omega^t$ by first θ-tracing and momentarily neglecting the x-parts:

$$\begin{aligned}\mathrm{tr}_\theta\mathcal{L}_\omega^t &= \int_0^{2\pi} d\theta\delta(\theta - g_\omega^t(\theta))u_\omega(\theta,x) \\ &= 2\pi\delta(\omega t \bmod 2\pi)\int_0^{2\pi}\frac{d\theta}{2\pi}u_\omega(\theta,x)\,. \end{aligned} \tag{12}$$

This is an immediate difficulty: $\mathrm{tr}_\theta\mathcal{L}_\omega^t$ is divergent (not trace class), so that our $\mathcal{L}_\omega$ at sharp frequency is not quite well defined. We treat this by considering a smeared $\mathcal{L}_\omega$:

$$\mathcal{L}_h = \int \frac{d\omega'}{2\pi}h(\omega'-\omega)\mathcal{L}_{\omega'}\,, \tag{13}$$

with $h(\omega)$ a test function centered about 0, which we ultimately let tend to $2\pi\,\delta(\omega)$. $\mathcal{L}_h^t$ is similar to (13) with h replaced by h convolved with itself t times. We now argue that as $t\to\infty$, $\mathrm{tr}\,\mathcal{L}_h^t$ is well-behaved. To see this, define $\nu = \omega/2\pi$. Since we mod 2π, ν need to be taken only in $[0,1)$. Then,

$$I_t(\nu) \;=\; 2\pi\delta(\omega t \bmod 2\pi) = \delta(\nu t \bmod 1) = \sum_{n=0}^{t-1}\delta(\nu t - n)$$

$$= \frac{1}{t}\sum_{n=0}^{t-1}\delta(\nu - \frac{n}{t}),$$

and

$$\int_0^1 d\nu I_t(\nu)h(\nu) = \frac{1}{t}\sum_{n=0}^{t-1} h(\frac{n}{t}) \to \int_0^1 d\nu h(\nu),$$

that is, $I_t(\nu)$ becomes the unit distribution as $t \to \infty$. [5] Thus the $2\pi\delta(\omega t)$ becomes harmless in computing $\mathrm{tr}\,\mathcal{L}_h^t$ which as $h \to 2\pi\delta$ is the meaning we assign to $\mathrm{tr}\,\mathcal{L}_\omega^t$, so that *so far as the divergence* of $\mathrm{tr}\,\mathcal{L}_h^t$ is concerned, we can simply legislate $2\pi\delta(\omega t \bmod 2\pi)$ to 1. *However*, prior to limits being taken, the $\delta(\omega t \bmod 2\pi)$ *has* an impact on u_ω within the θ integral of (12). In particular, as we shall now see, coupled with the t-periodicity $x_t = x_0$ following from the x-trace over $\delta(x - f^t(x))$, the Fourier transform $\hat{x}_t$ of (2) will approach a definite limit as $t \to \infty$, so that $t\hat{x}_t \approx t\hat{x}$ in the exponent in (9) correctly reflects the asymptotic behavior.

Let us now attempt to evaluate the full trace of $\mathcal{L}_\omega^t$. Dropping the $2\pi\delta(\omega t \bmod 2\pi)$ we have

$$\mathrm{tr}\,\mathcal{L}_\omega^t = \int dx\delta(x - f^t(x)) \int \frac{d\theta}{2\pi} e^{\beta t e^{i\theta}\hat{x}_t(\omega,x)}.$$

However, by power series expanding the exponential, it is clear that the θ integral is precisely 1 independent of β: as we said at the beginning the phases have washed out the integral, so that (2) and (8) must now be modified to obtain a non-trivial result. It is obvious what we must do: we should be considering the operator

$$(\mathcal{L}_\omega\,\psi)(x,\theta) \;=\; \int dx'\frac{d\theta'}{2\pi}\delta(x - f(x'))\,\delta(\theta - g_\omega(\theta'))e^{\beta x'\frac{1}{2}(e^{i\theta'}+e^{-i\theta'})}\psi(x',\theta') \quad (14)$$

for which

$$(\mathcal{L}_\omega^t\,\psi)(x,\theta) \;=\; \int dx'\frac{d\theta'}{2\pi}\delta(x - f^t(x'))\,\delta(\theta - g_\omega^t(\theta'))e^{\beta t|\hat{x}_t|\cos(\theta'+\alpha_t)}\psi(x',\theta'),$$

with α_t the argument of $\hat{x}_t$. For the measure uniform in θ the θ' integration can now be shifted to absorb α_t, so that after $\mathcal{L}_\omega^t\,\psi$ is integrated on x and θ, we are

[5] PC: I think this formalism fails for $\omega = 0$, the uncoupled dynamics appropriate for the classical diffusion constant?

determining $\left\langle \int \frac{d\theta}{2\pi} e^{\beta t|\hat{x}_t|\cos\theta} \right\rangle_x$, which for $t \to \infty$, by steepest descent, is

$$\left\langle \frac{e^{\beta t|\hat{x}_t|}}{\sqrt{2\pi\beta t|\hat{x}_t|}} \right\rangle_x . \tag{15}$$

Apart from a prefactor, this is the desired expectation value (3), which in turn behaves as λ^t, with λ the leading eigenvalue of $\mathcal{L}_\omega$ of (14).

Before proceeding we comment on two possible alternative formulations of the problem of evaluation of chaotic power spectra.

(1) Instead of extending dynamics as in (7), and deriving the trace formula constrained to rational frequencies, we could have considered from the start rational frequencies of form $\omega = 2\pi k/n$, and replaced the evolution operator (5) by one that advances the system n steps in time:

$$(\mathcal{L}_\omega^n \psi)(x) = \int dx'\, \delta(x - f^n(x')) \exp\left(\beta \sum_{k=0}^{n-1} x_k e^{2\pi i k m/n}\right) \psi(x') \,. \tag{16}$$

This operator satisfies the semi-group property $\mathcal{L}^{ns}\mathcal{L}^{nt} = \mathcal{L}^{n(s+t)}$, as the motion on the circle adjoined to the dynamics (7) is periodic with period n. The theorems that guarantee the discretness of the spectrum of $\mathcal{L}$ for the Axiom A flows [8] ensure faster than exponential convergence of the cycle expansions of the associated Fredholm determinants; we illustrate this with the numerical results of sect. 6. A disadvantage of using (16) instead of the 1-time step evolution operator (14) is that it requires a larger symbolic dynamics; for example, if the the 1-time step evolution is described by a 2-symbol alphabet, the n-time steps operator (16) requires an alphabet of 2^n symbols.

(2) An alternative formula for power spectrum is given by the Kinchine-Wiener relation

$$\left\langle |t\hat{x}_t(\omega, x)|^2 \right\rangle_x = t \sum_{n=-t+1}^{t-1} (1 - |n|/t) C(n) e^{i\omega n}$$

where $C(n) = \langle x_t x_{t+n} \rangle_x$ is the space-averaged time correlation function. We have also evaluated $C(n)$ by means of cycle expansions, but the convergence appears to be very slow.

4 TRACE FORMULA

Let us now determine

$$\mathrm{tr}\,\mathcal{L}_\omega^t = \int dx \delta(x - f^t(x)) \int \frac{d\theta}{2\pi} e^{\beta t |\hat{x}_t(\omega,x)| \cos\theta} \,. \tag{17}$$

The trace (17) picks up contributions from all periodic points $x = f^t(x)$. Every periodic point x belongs to some prime cycle p, $x \in \{x_{p,0}, x_{p,1}, \ldots, x_{p,n_p-1}\}$, where n_p is the minimal period of x under f, and $x_{p,m} = f^m(x_{p,0})$. Consider now $\hat{x}_t(\omega, x)$, given that x is periodic with period t. With $x = x_{p,0}$, n_p divides t (written $n_p|t$), or $t = rn_p\,, r \geq 1\,$. Define the Fourier transform of a single repeat of a prime cycle as

$$\hat{x}_p(\omega) = \frac{1}{n_p} \sum_{m=0}^{n_p-1} x_{p,m} e^{im\omega} \,.$$

Then the trajectory Fourier transform (2) is

$$\hat{x}_t(\omega, x) = \frac{1}{rn_p} \sum_{\ell=0}^{r-1} \sum_{m=0}^{n_p-1} x_{p,m} e^{i(m+n_p\ell)\omega} = \hat{x}_p(\omega) \frac{1}{r} \sum_{\ell=0}^{r-1} e^{in_p\ell\omega} \,. \tag{18}$$

As $\hat{x}_t(\omega, x_{p,m}) = e^{-i\omega m} \hat{x}_t(\omega, x_{p,0})$, $|\hat{x}_p(\omega)|$ is the same for all cycle points belonging to the prime cycle p. Although we have dropped the $2\pi\delta(\omega t \bmod 2\pi)$ in front of (17), ω *is* to be so constrained in evaluating $\hat{x}_t$. With $\omega = 2\pi n/t\,,\quad n = 0, \ldots, t-1\,$, the sum in (18) becomes

$$\frac{1}{r} \sum_{\ell=0}^{r-1} e^{2\pi i n\ell/r} = \delta_{n,kr}\,, \quad k = 0, \ldots, n_p - 1\,.$$

Thus, $\hat{x}_p(\omega)$ contributes its own value to (17)

$$|\hat{x}_t(\omega, x_{p,m})| = \begin{cases} |\hat{x}_p(\omega)| & \text{if} \quad n_p|t \quad \text{and} \quad \omega = 2\pi k/n_p \\ 0 & \text{otherwise} \end{cases} \,. \tag{19}$$

Integrating (17) over x yields [1, 2] our final trace formula

$$\begin{aligned} \mathrm{tr}\,\mathcal{L}_\omega^t &= \sum_p n_p \sum_{r=1}^{\infty} \frac{\delta_{t,n_p r}}{|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)|} \left\langle e^{\beta t |\hat{x}_t(\omega)|} \right\rangle_p \\ \left\langle e^{\beta t |\hat{x}_t(\omega)|} \right\rangle_p &= \begin{cases} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{\beta r n_p |\hat{x}_p(\omega)| \cos\theta} = J_0(i\beta r n_p |\hat{x}_p(\omega)|) & \text{if } \omega = 2\pi \frac{k}{n_p} \\ 1 & \text{otherwise} \end{cases} \end{aligned} \tag{20}$$

Here $\mathbf{J}_p$ is the cycle stability matrix; for 1-d maps $\mathbf{J}_p = \Lambda_p = \frac{d}{dx} f^{n_p}(x)$, $x = x_{p,m}$.

Second derivative with respect to β yields a trace estimate of the power spectrum

$$\begin{aligned} D(\omega) &= \frac{1}{2} \frac{\partial^2 Q}{\partial \beta^2}\bigg|_{\beta=0} = \frac{1}{2t} \left\langle |t\hat{x}_t(\omega)|^2 \right\rangle_x + O\left(e^{tQ_1(0)}\right) \\ \frac{1}{t} \left\langle |t\hat{x}_t(\omega)|^2 \right\rangle_x &= \int dx \delta(x - f^t(x))\, t |\hat{x}_t(\omega, x)|^2 \\ &= \sum_p |n_p \hat{x}_p(\omega)|^2 \sum_{r=1}^{\infty} \frac{\delta_{t,n_p r}}{|1 - \Lambda_p^r|} \,. \end{aligned} \tag{21}$$

While this formula is already exponentially convergent with t, for nice hyperbolic systems the Fredholm determinant formulas of the next section are on general grounds [8] expected to be superexponentially convergent, and thus preferable to (21).

5 CYCLE EXPANSIONS

The Fredholm determinant corresponding to (20) follows from (10):

$$F(\beta, Q) = \det\left(1 - e^{-Q}\mathcal{L}\right) = \exp\left(- \sum_p \sum_{r=1}^{\infty} \frac{1}{r} \frac{e^{-r n_p Q} \left\langle e^{\beta r n_p |\hat{x}(\omega)|} \right\rangle_p}{|\det\left(\mathbf{1} - \mathbf{J}_p^r\right)|} \right) \,. \tag{22}$$

For given β the $\mathcal{L}_\omega$ eigenvalues $\lambda_\alpha = e^{Q_\alpha(\beta)}$ are given by the zeros of $F(\beta, Q(\beta)) = 0$. We shall need to evaluate only the derivatives of the leading eigenvalue λ_0 at $\beta = 0$. As for $\beta = 0$ the transfer operator (14) is simply the Perron-Frobenius operator, for bound dynamical systems probability conservation implies $Q(0) = 0$, and the leading eigenvalue is trivially $\lambda_0 = 1$. We use this condition as one of the checks of the quality of our cycle expansions.

The above formula can be simplified somewhat by observing that we only need the trace (17) for t asymptotically large,

$$\mathrm{tr}\, \mathcal{L}_\omega^t \approx \int dx \delta(x - f^t(x)) e^{\beta t |\hat{x}_t(\omega, x)|} \,, \tag{23}$$

to produce the *leading* eigenvalue λ_0. It is to be stressed that when we do so, although (15) and (23) are not exactly the outcome of an operator $\mathcal{L}$ with the semi-group property $\mathcal{L}^s \mathcal{L}^t = \mathcal{L}^{s+t}$, we have nevertheless demonstrated that they are the asymptotics of such an operator, namely $\mathcal{L}_\omega$ of (14), so that we can still expect fast convergence to λ of the cycle expansion technique which follows. In the saddlepoint approximation (15)

$$\left\langle e^{\beta t|\hat{x}_t(\omega)|}\right\rangle_p \rightarrow e^{\beta r n_p |\hat{x}_p(\omega)|}$$

the Fredholm determinant can be resummed into a product of the Selberg type [2]; for example, for 1-d maps (22) becomes

$$F_{sp}(\beta,Q) = \prod_p \prod_{k=0} \left(1 - \frac{t_p}{\Lambda_p^k}\right) , \qquad t_p = \frac{1}{|\Lambda_p|} e^{(\beta|\hat{x}_p(\omega)|-Q)n_p} . \tag{24}$$

As a further simplifying alternative, the leading eigenvalue can be extracted from the associated Ruelle ζ function [1], obtained by replacement $|\det\left(\mathbf{1}-\mathbf{J}_p^r\right)| \rightarrow |\Lambda_p|^r$. This amounts to keeping the $k = 0$ term in the above product (see ref. [2] for details):

$$1/\zeta(\beta,Q) = \prod_p (1 - t_p) . \tag{25}$$

The above infinite products can be rearranged as *cycle expansions* with improved convergence properties [2]. The ζ function is expanded as a formal power series,

$$\begin{aligned} 1/\zeta &= \prod_p (1-t_p) = 1 + \sum_{\{p_1 p_2 \ldots p_k\}}' t_{\{p_1 p_2 \ldots p_k\}}, \\ t_{\{p_1 p_2 \ldots p_k\}} &= (-1)^k t_{p_1} t_{p_2} \ldots t_{p_k} \end{aligned} \tag{26}$$

where the prime on the sum indicates that the sum is over all distinct non-repeating combinations of prime cycles. For $k > 1$, $t_{\{p_1 p_2 \ldots p_k\}}$ are “pseudo” cycles; they are sequences of shorter cycles that shadow a cycle with symbol sequence $p_1 p_2 \ldots p_k$ along segments p_1, $p_2, \ldots$, p_k. For sufficiently small z (we have absorbed z into the weights by $z^{n_p} t_p \rightarrow t_p$ substitution) the sum makes sense as a power series in z.

The implicit definition of $Q(\beta)$, $1/\zeta(\beta, Q(\beta)) = 0$, together with the expression for the variation in cycle weight (25) as function of β, Q

$$\delta t_p = (|\hat{x}_p(\omega)|\delta\beta - \delta Q)\, n_p t_p , \tag{27}$$

yields the cycle expansion [2] for

$$\begin{aligned}\lim_{t\to\infty}\langle t|\hat{x}_t(\omega,x)|\rangle_x &= \left.\frac{\partial Q}{\partial\beta}\right|_{\beta=0} = \frac{\sum'\hat{x}_{\{p_1p_2\ldots p_k\}}t_{\{p_1p_2\ldots p_k\}}}{\sum' n_{\{p_1p_2\ldots p_k\}}t_{\{p_1p_2\ldots p_k\}}},\\ \hat{x}_{\{p_1p_2\ldots p_k\}} &= n_{p_1}|\hat{x}_{p_1}(\omega)|+\cdots+n_{p_k}|\hat{x}_{p_k}(\omega)|\\ n_{\{p_1p_2\ldots p_k\}} &= n_{p_1}+n_{p_2}\cdots+n_{p_k}\,. \end{aligned}\tag{28}$$

Strictly speaking, this expression is not applicable in the present context, as the Bessel function in (20) is even function of β, so $\frac{\partial Q}{\partial\beta}=0$ prior to the saddlepoint approximation (15). However, the second derivative of $Q(\beta)$, eq. (4), does yield the power spectrum

$$D(\omega) = \frac{1}{2}\frac{\sum'(-1)^k\left(n_{p_1}|\hat{x}_{p_1}(\omega)|+\cdots+n_{p_k}|\hat{x}_{p_k}(\omega)|\right)^2/|\Lambda_{p_1}\cdots\Lambda_{p_k}|}{\sum'(-1)^k(n_{p_1}+\cdots+n_{p_k})/|\Lambda_{p_1}\cdots\Lambda_{p_k}|}\tag{29}$$

The sums are restricted to non-repeating combinations of prime cycles, as in (26). By (19) only those prime cycles p whose whose periods n_p are multiples of the frequency $\omega = 2\pi k/n$ denominator, $n_p|n$, contribute to the numerator of the cycle expansion (29). All cycles contribute to the denominator, but the denominator is a frequency-independent normalization factor which needs to be computed only once.

The formalism, as it stands, is applicable only for $\omega\neq 0$. The $\omega = 0$ diffusion formulas [3, 4, 5] require substraction of the drift expectation value $\langle\hat{x}_t(0)\rangle$. [6]

6 NUMERICAL RESULTS

There exist maps - typically 1-d piecewise-linear maps - for which the natural measure is available in closed form. As for such maps the power spectra are known analytically, we can use them as a benchmarks for tests of cycle expansions. We start with two maps whose symbolic dynamics is described by the full (0,1) binary shift, *ie.* all sequences of “0” and “1” are realizable: the Bernoulli shift, and the skew tent map.

[6]PC: the remainder of the text has not been reworked yet.

6.1 Skew tent map

[7] For the skew tent map

$$f(x) = \begin{cases} ax & \text{if } 0 \le x \le a^{-1}, \\ \frac{a}{a-1}(1-x) & \text{if } a^{-1} \le x \le 1. \end{cases} \tag{30}$$

the correlation function was calculated by Grossmann and Thomae [9]:

$$C(m) = \langle (x_t - \langle x \rangle)(x_{t+m} - \langle x \rangle) \rangle = \frac{1}{12}\left(\frac{2-a}{a}\right)^m \tag{31}$$

The power spectrum is

$$S(\omega) \;=\; \frac{1}{6}\frac{a-1}{a(a-2)(1+\cos 2\pi\omega)+2}\,. \tag{32}$$

Consider first the zero frequency case, $\omega = 0$. In this case the Ruelle zeta function

$$1/\zeta = (1-t_0)(1-t_1)(1-t_{01})(1-t_{001})... = 1 - t_0 - t_1 - [t_{01} - t_0 t_1]...$$

consists of the fundamental part and curvature corrections, and calculation of diffusion constant is straightforward. Stability of a cycle with $n_{p,1}$ repeats of symbol 1 is $\Lambda_p = a^{n_p}/(1-a)^{n_{p,1}}$. The probability conservation follows trivially from $1/|\Lambda_0| + 1/|\Lambda_1| = 1/a + (a-1)/a = 1$ (all curvatures vanish for this piecewise-linear map), and the denominator in (29) is simply -1. For $\beta = 0$ the Fredholm determinant is given by

$$\begin{aligned} F &= \left(1 - \frac{z}{|\Lambda_0|} - \frac{z}{|\Lambda_1|}\right)\left(1 - \frac{z}{|\Lambda_0|^2} + \frac{z}{|\Lambda_1|^2}\right)\cdots \\ &= (1-z)\left(1 - z\frac{2-a}{a}\right)\cdots. \end{aligned}$$

The leading eigenvalue is 1, by probability conservation; the second eigenvalue controls the exponential falloff of the 2-point correlations, $\lambda_1 = a/(2-a)$, in agreement with (31).

[7] PC: this calculation should be done more analytically?

The periodic points are easily computed. However, even though the power spectrum is given explicitely by (32), the cycle expansion does not reproduce this expression in any obvious form, so we conclude this section with a numerical evaluation of the power spectrum for the skew tent map (30); the result is given in fig. 1, with fig. 2 illustrating the convergence with maximal cycle length truncations. We have also checked that for the Ruelle zeta functions convergence is exponential, and that for the Fredholm determinant the convergence is faster than exponential. [8]

6.2 Pruned symbolic dynamics

If the symbolic dynamics is described by a subshift of a finite type, cycle expansions converge well [2]. An example is given by the tent map $f(x) = 1 - a|x|$, $a = (1+\sqrt{5})/2$. This value of a corresponds to the $\overline{001}$, $\overline{011}$ 3-cycles bifurcation value, with the symbolic dynamics given by a simple pruning rule; the repeat _00_ is forbidden. Fig. 3 shows the numerical power spectrum evaluated by means of (29).

Difficulties can arise if the system is not sufficiently mixing. For example, the tent map $x_{t+1} = 1 - a|x_t|$ for $a = \sqrt{2}$ has two nonoverlaping bands. The system is not mixing, and the power spectrum contains a delta-function peak at $\omega = 1/2$. In symbolic dynamics only sequences having "1" at all odd or at all even places are allowed. Let us focus our attention on the zeta function $1/\zeta$ at the frequency $\omega = 1/2$. The set of cycles with odd periods is empty (except for the fixed point "1", and it also disappears for $a < \sqrt{2}$) and

$$1/\zeta = \prod_{p \in \mathcal{P}_e} (1 - t_p(Q, \beta)) \tag{33}$$

The straightforward differentiation of $1/\zeta$ does not give correct result: the drift term vanishes and the diffusion constant diverges. The reason is that the mapping f^2 is not mixing and has two symmetric attractors, so the zeta function (33) is a product of two zeta functions for these attractors. The probability distribution function for Φ_t does not tend to a Gaussian hump as $t \to \infty$, but instead to two symmetric humps, drifting away from the origin in opposite directions (that is why the drift term for $1/\zeta$ vanishes). In order to describe the power spectrum for $\omega = 1/2$ correctly, we must restrict the averaging to one hump. This corresponds to considering one of the symmetric attractors of the map f^2. In terms of zeta

[8] PC: this must be reworked so that for finite Markov graphs, long cycles are re-expressed as the repeats of the fundamental cycles; work out in detail for 2-branch complete binary tent map?.

function this means that we must consider square root of zeta function (33):

$$1/\zeta' = \prod_{p\in\mathcal{P}_e}{}' (1 - t_p(Q,\beta))$$

From this zeta function correct values of discrete and continuous components of the power spectrum at $\omega = 1/2$ are obtained as the drift term and the diffusion constant.

Mention Bernoulli hand calculations.

7 SUMMARY AND CONLUSIONS

We do not know how to deal with continuous flows.

Fourier analysis for chaotic sets can be recast into the transfer operator formalism, but restricted to rational frequencies $\omega = 2\pi k/n$.

While the cycle expansions do not appear to be convenient for rederivation of the analytic results, they converge faster than exponentially in numerical evaluations, and are also aapplicable to generic flows, where the natural measure is not analytically available

This is unsatisfactory in the sense that [9]

discus here invariant averages (Lyapunov, dimensions, etc.) vs. non-invariant ones, such as $\langle x_i \rangle$.

FIGURE CAPTIONS

1. Power spectrum for skew tent map (30) with $a = 2.5$, evaluated from cycle expansions of the Fredholm determinant (22) at all rational frequencies $\omega = 2\pi k/n$, $n \leq 14$. The exact spectrum (32) is indicated by the solid line.

[9]PC: seems not too smart - one has to go to long cycles to evaluate spectrum at frequencies with q large, but the spectrum itself is totally smooth. Besides, for continuous time flows this pulls out only one prime cycle at a time, as all periods T_p are distinct. Need some smearing in ω?

2. Error in determining the power spectrum of fig. 1 as function of the maximal cycle length used. (+) $\omega = 0$, (□) $\omega = 1/2$; other symbols from top to bottom correspond to frequencies $1/3,\ 1/4, ..., 1/14$.

3. Power spectrum for the pruned symmetric tent map with $a = (1 + \sqrt{5})/2$, evaluated from cycle expansions of the Fredholm determinant (22) at all rational frequencies $\omega = 2\pi k/n$, $n \leq 14$. No analytic formula for spectrum is available for this case.

References

[1] D. Ruelle, *Statistical Mechanics, Thermodynamic Formalism*, (Addison-Wesley, Reading MA, 1978).

[2] R. Artuso, E. Aurell and P. Cvitanović, "Recycling of strange sets I: Cycle expansions", *Nonlinearity* **3**, 325 (1990).

[3] P. Cvitanović, J.-P. Eckmann, and P. Gaspard, "Transport properties of the Lorentz gas in terms of periodic orbits", Niels Bohr Institute preprint (May 1991).

[4] P. Cvitanović, P. Gaspard, and T. Schreiber, "Investigation of the Lorentz Gas in terms of periodic orbits", *CHAOS* **2**, 85 (1992).

[5] R. Artuso, "Diffusive dynamics and periodic orbits of dynamical systems", *Phys. Lett.* **A 160**, 528 (1991).

[6] W.N. Vance, "Unstable periodic orbits and transport properties of nonequilibrium steady states", *Phys. Rev. Lett.* **96**, 1356 (1992).

[7] P. Cvitanović and B. Eckhardt, *J. Phys.* **A 24**, L237 (1991).

[8] For review of results on analyticity of Fredholm determinats of Axiom A flow evolution operators, see P. Cvitanović, P.E. Rosenqvist, H.H. Rugh, and G. Vattay, *A Fredholm determinant for semi-classical quantization*, *CHAOS* (1993), to appear.

[9] S. Grossmann and S. Thomae, *Z. Naturforsch.* **32 a**, 1353 (1977); reprinted in ref. [10].

[10] *Universality in Chaos, 2. edition*, P. Cvitanović, ed., (Adam Hilger, Bristol 1989).