got symmetry?
here is how you slice it

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dynamical description of turbulent flows

state space

a manifold $\mathcal{M} \in \mathbb{R}^d$: d numbers determine the state of the system

representative point

$x(\tau) \in \mathcal{M}$
a state of physical system at instant in time

deterministic dynamics

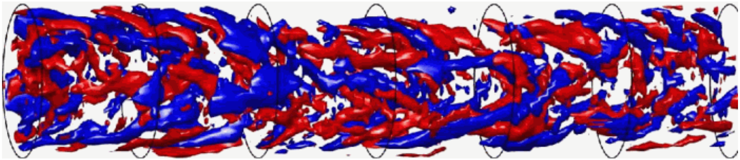
map $x(\tau) = f^\tau(x_0)$ = representative point time τ later

today's experiments

example of a representative point

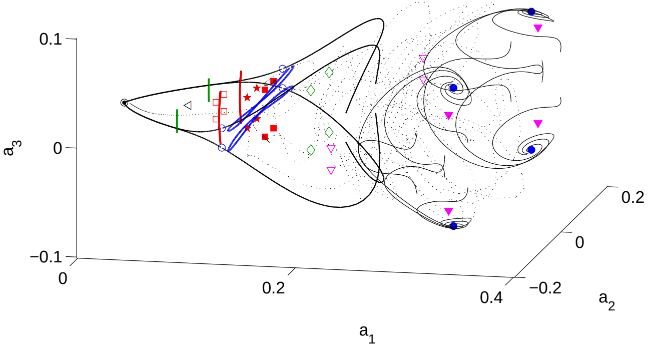
$x(\tau) \in \mathcal{M}$, $d = \infty$
a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry $\rightarrow$ $3D$ velocity field over the entire pipe[1]



[1] Casimir W.H. van Doorne (PhD thesis, Delft 2004)

can visualize 61,506 dimensional state space of turbulent flow



equilibria of turbulent plane Couette flow,
their unstable manifolds, and
myriad of turbulent videos mapped out as one happy family

for movies, please click through ChaosBook.org/tutorials today's talk's focus:

nature loves symmetry

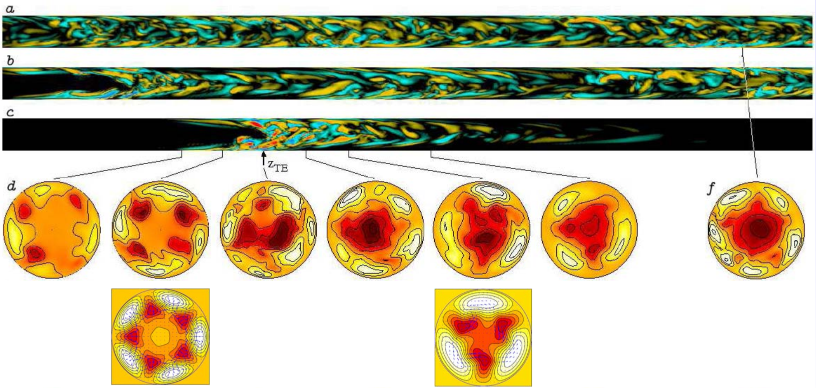
problem

physicists like symmetry more than Nature

Rich Kerswell

nature : turbulence in pipe flows

pipe flows : amazing data! amazing numerics!



Nature, she don't care : turbulence breaks all symmetries

symmetry of a dynamical system

a group G is a symmetry of the dynamics if

for every solution $f^{\tau}(x) \in \mathcal{M}$ and $g \in G$,

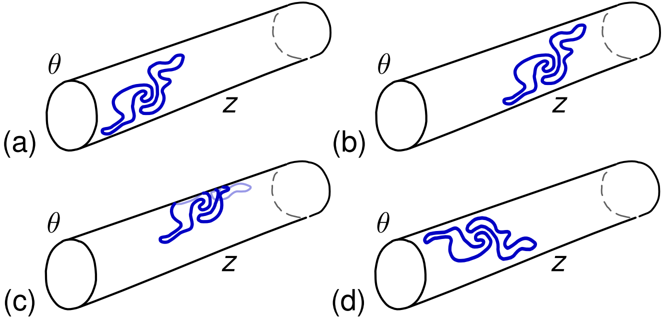
$gf^{\tau}(x) = f^{\tau}(gx)$ is also a solution

a flow $\dot{x} = v(x)$ is G-equivariant if

$$v(x) = g^{-1}\, v(g\, x)\,, \qquad \text{for all } g \in G\,.$$

equations of motion of the same form in all frames

example : $\mathrm{SO}(2)_z \times \mathrm{O}(2)_\theta$ symmetry of pipe flow



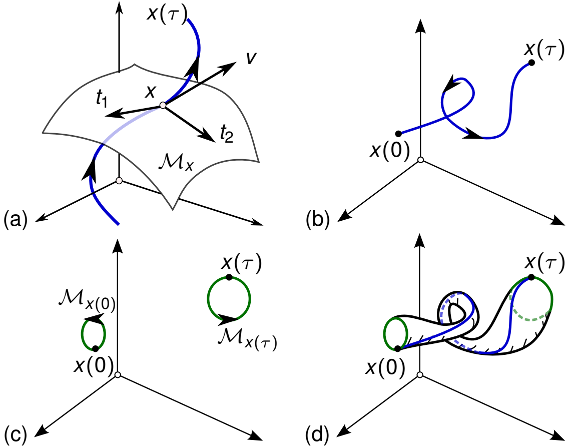
a fluid state, shifted by a stream-wise translation, azimuthal rotation g_p is a physically equivalent state

b) stream-wise

c) stream-wise, azimuthal

d) azimuthal flip

trajectories, orbits



(a) x tangent vectors:

$v(x)$ along time flow $x(\tau)$
$t_1(x), \cdots, t_N(x)$ group tangents

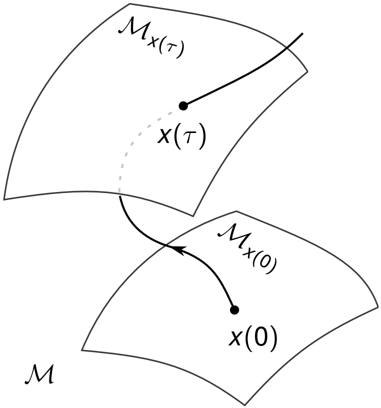
(b) trajectory $x(\tau)$

(c) group orbits $g\,x(\tau)$

(d) wurst $g\,x(\tau)$

foliation by group orbits

group orbits

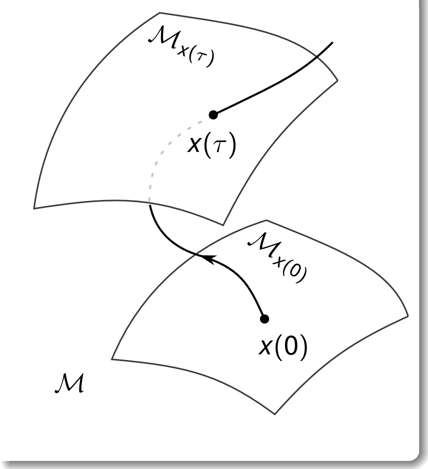


group orbit $\mathcal{M}_x$ of x is the set of all group actions

$$\mathcal{M}_x = \{g\,x \mid g \in G\}$$

foliation by group orbits

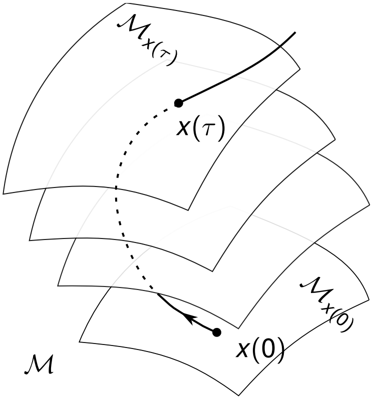
group orbits



any point on the manifold $\mathcal{M}_{x(\tau)}$ is equivalent to any other

foliation by group orbits

group orbits



action of a symmetry group foliates the state space $\mathcal{M}$ into a union of group orbits $\mathcal{M}_x$

each group orbit $\mathcal{M}_x$ is an equivalence class

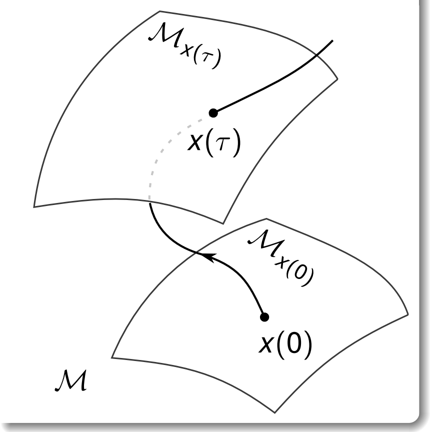
the goal

replace each group orbit by a unique point in a lower-dimensional

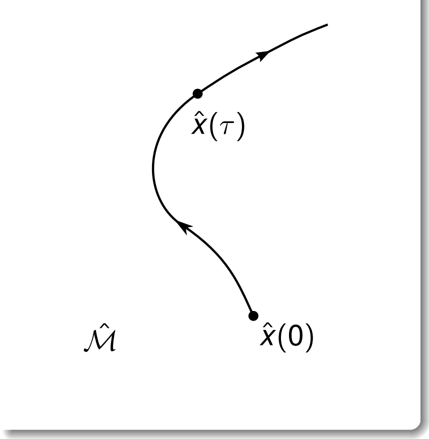
symmetry reduced state space $\mathcal{M}/G$

symmetry reduction

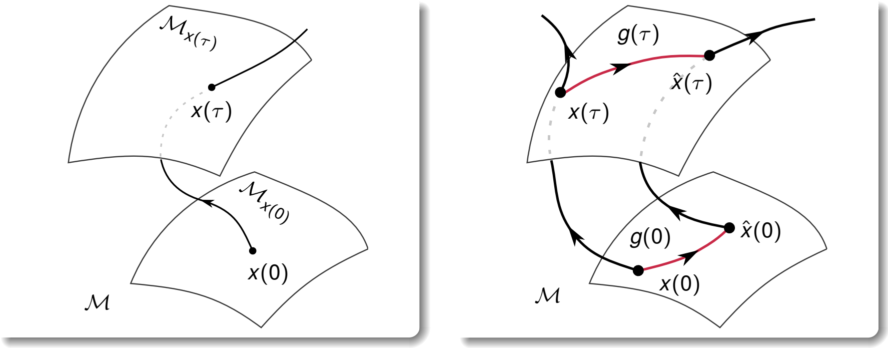
full state space



reduced state space



Cartan moving frame



free to redefine the flow any time instant by transformation to a frame moving along symmetry directions

method of slices

cut group orbits by a hypersurface (not a Poincaré section), each group orbit of symmetry-equivalent points represented by the single point

cut how?

geometers'choice

chose the frames so that distances are minimized

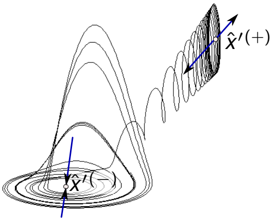
use a yardstick!

then cover the reduced manifold with a set of flat charts

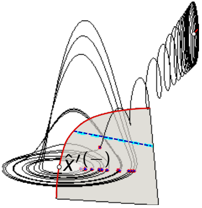
yes, we can do this with 10^6-dimensional flat sheets of ‘paper’

motivational : 2-chart sections atlas for Rössler flow

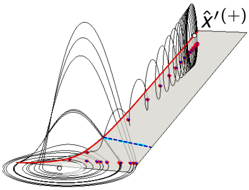
templates: 2 equilibria



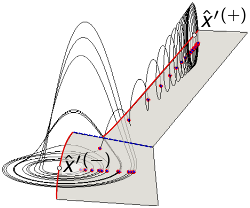
bottom chart



top chart



2-chart atlas



red : borders

blue : ridges

inspiration : pattern recognition

you are observing turbulence in a pipe flow, or your defibrillator has a mesh of sensors measuring electrical currents that cross your heart, and

you have a precomputed pattern, and are sifting through the data set of observed patterns for something like it

here you see a pattern, and there you see a pattern that seems much like the first one

how ‘much like the first one?’

distance

assume that G is a subgroup of the group of orthogonal transformations $\mathrm{O}(d)$, and measure distance $|x|^2 = \langle x|x\rangle$ in terms of the Euclidean inner product

numerical fluids: PDE discretization independent L2 distance is

energy norm

$$\|\mathbf{u}-\mathbf{v}\|^2 = \langle \mathbf{u}-\mathbf{v}|\mathbf{u}-\mathbf{v}\rangle = \frac{1}{V}\int_\Omega d\mathbf{x}\,(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$$

experimental fluid:

image discretization independent distance

is pixel-to-pixel distance, or ???

take the first pattern

'template' or 'reference state'

a point $\hat{x}'$ in the state space $\mathcal{M}$

and use the symmetries of the flow to

slide and rotate the 'template'

act with elements of the symmetry group G on $\hat{x}' \to g(\phi)\,\hat{x}'$

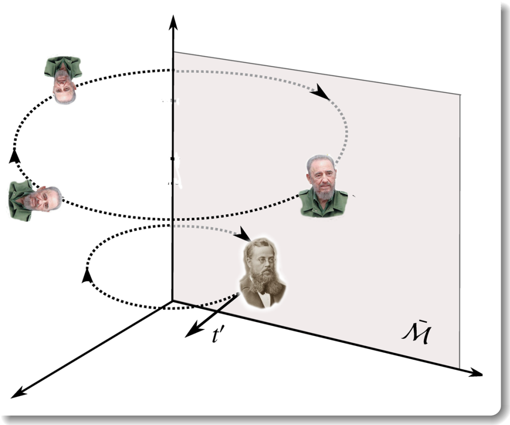
until it overlies the second pattern (a point x in the state space)

distance between the two patterns

$$|x - g(\phi)\,\hat{x}'| = |\hat{x} - \hat{x}'|$$

is minimized

idea: the closest match



template: Sophus Lie

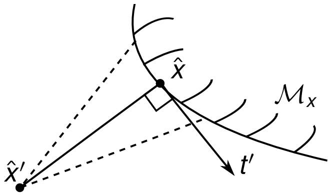
(1) rotate man with a beard x traces out the group orbit $\mathcal{M}_x$

(2) replace the group orbit by the closest match $\hat{x}$ to the template pattern $\hat{x}'$

the closest matches $\hat{x}$ lie in the $(d-N)$ symmetry reduced state space $\hat{\mathcal{M}}$

idea: the closest match

extremal condition for nearest distance from template $\hat{x}'$ to group orbit of x



minimal distance

is a solution to the extremum conditions

$$\frac{\partial}{\partial \phi_a} |x - g(\phi)\,\hat{x}'|^2$$

but what is

$$\frac{\partial}{\partial \phi_a} g(\phi)\,?$$

infinitesimal transformations

$$g \simeq 1 + \phi \cdot \mathbf{T}\,, \qquad |\delta\phi| \ll 1$$

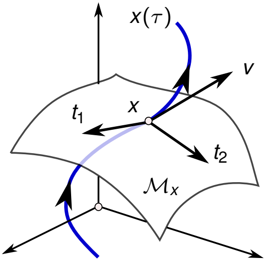
- T_a are generators of infinitesimal transformations
- here T_a are $[d \times d]$ antisymmetric matrices

now have the 'slice condition'

flow field at the state space point x induced by the action of the group is given by the set of *N tangent fields*

$$t_a(x)_i = (\mathbf{T}_a)_{ij} x_j$$

group tangent fields



slice condition

$$\frac{\partial}{\partial \phi_a} |x - g(\phi)\,\hat{x}'|^2 = 2\,\langle \hat{x} | t'_a \rangle = 0\,, \qquad t'_a = \mathbf{T}_a \hat{x}'$$

flow within the slice

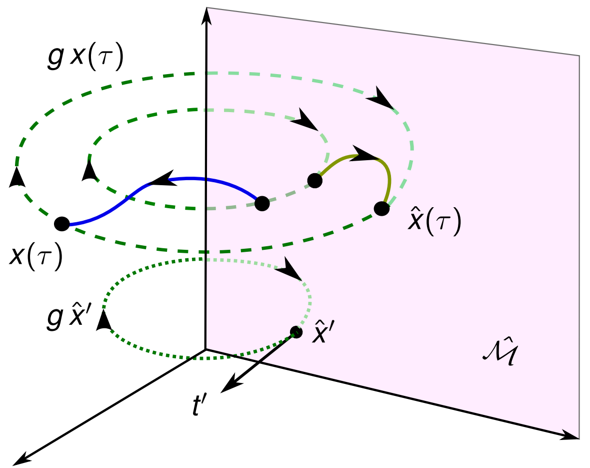
slice hyperplane : normal to template $\hat{x}'$ group tangent t'

reduced state space $\hat{\mathcal{M}}$ flow $\hat{v}(\hat{x})$

$$
\begin{aligned}
\hat{v}(\hat{x}) &= v(\hat{x}) - \dot{\phi}(\hat{x}) \cdot t(\hat{x}) \,, \qquad \hat{x} \in \hat{\mathcal{M}} \\
\dot{\phi}_a(\hat{x}) &= \langle v(\hat{x})^T | t'_a \rangle / \langle t(\hat{x})^T | t' \rangle \,.
\end{aligned}
$$

- v : velocity, full space
- $\hat{v}$: velocity component in slice
- $\dot{\phi} \cdot t$: velocity component normal to slice
- $\dot{\phi}$: reconstruction equation for the group phases

flow within the slice



full-space trajectory $x(\tau)$
rotated into the reduced state space $\hat{x}(\tau) = g(\phi)^{-1}x(\tau)$
by appropriate *moving frame* angles $\{\phi(\tau)\}$

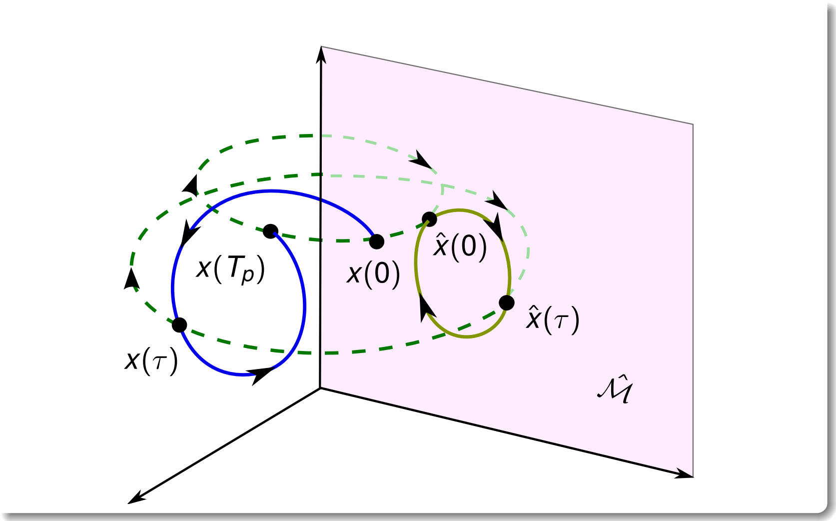
relative periodic orbit

a relative periodic orbit p is an orbit in state space $\mathcal{M}$ which exactly recurs

$$x_p(\tau) = g_p x_p(\tau + T_p)\,, \qquad x_p(\tau) \in \mathcal{M}_p$$

for a fixed relative period T_p and a fixed group action $g_p \in G$ that “rotates" the endpoint $x_p(T_p)$ back into the initial point $x_p(0)$.

relative periodic orbit→ periodic orbit

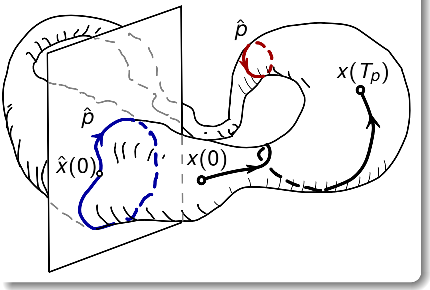


full state space relative periodic orbit $x(\tau)$
is rotated into the reduced state space periodic orbit

however : slice charts are local

a slice hyperplane cuts every group orbit at least twice

wurst, sliced



an $SO(2)$ relative periodic orbit is topologically a torus : the cuts are periodic orbit images of the same relative periodic orbit, the good close one, and the rest bad ones

nature couples many Fourier modes

group orbits of highly nonlinear states are highly contorted:
many extrema, multiple sections by a slice

example : group orbit of a pipe flow turbulent state

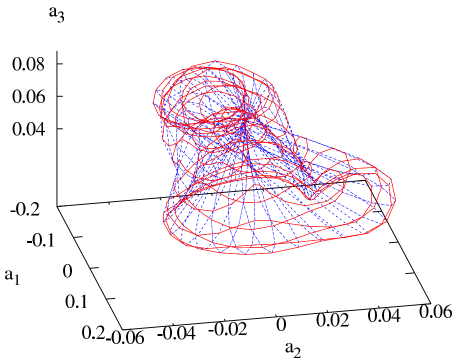
$\hat{x}'$ is Kerswell *et al* $N2\_M1$ relative equilibrium
($Re = 2400$, stubby $L = 2.5D$ pipe)

$\mathbf{SO}(2) \times \mathbf{SO}(2)$ symmetry
$\Rightarrow$ group orbit is 2-torus

a turbulent state

distance extremum condition

$$\frac{\partial}{\partial \phi_a} |x - g(\phi)\, \hat{x}'|^2 = 0$$



group orbits of highly nonlinear states are highly contorted:
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slice charts are local

reduced state space $\hat{\mathcal{M}}$ flow $\hat{v}(\hat{x})$

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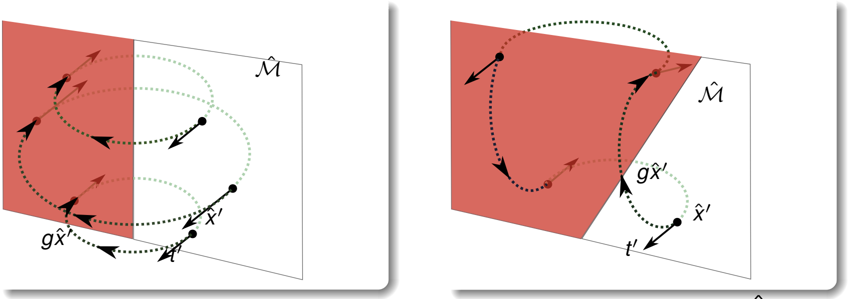
\begin{aligned}
\hat{v}(\hat{x}) &= v(\hat{x}) - \dot{\phi}(\hat{x}) \cdot t(\hat{x}) \,, \qquad \hat{x} \in \hat{\mathcal{M}} \\
\dot{\phi}_a(\hat{x}) &= (v(\hat{x})^T t'_a)/(t(\hat{x})^T \cdot t') \,.
\end{aligned}
$$

glitches!

group tangent of a generic trajectory orthogonal to the slice tangent at a sequence of instants τ_k

$$t(\tau_k)^T \cdot t' = 0$$

slice is good up to the chart border



SO(2) : two hyperplanes to a given template $\hat{x}'$; the slice $\hat{\mathcal{M}}$, and *chart border* $\hat{x}^* \in S$. Beyond :

group orbits pierce in the wrong direction

(a) a circle group orbit crosses the slice hyperplane twice.

(b) a group orbit for a combination of $m = 1$ and $m = 2$ Fourier modes resembles a baseball seam, and can be sliced 4 times, out of which only the point closest to the template is in the slice

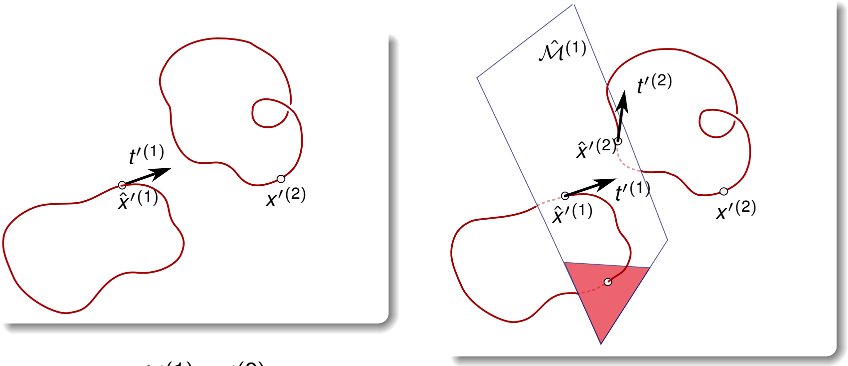
charting the state space

for turbulent/chaotic systems a set of charts is needed to capture the dynamics

templates should be representative of the dynamically dominant patterns seen in the solutions of nonlinear PDEs

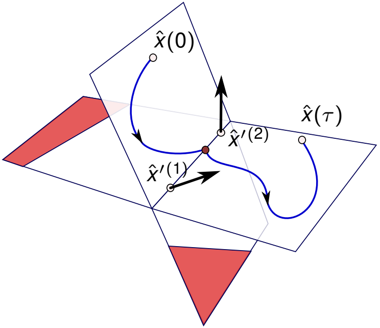
construct a global atlas of the dimensionally reduced state space $\hat{\mathcal{M}}$ by deploying linear slices $\hat{\mathcal{M}}^{(j)}$ across neighborhoods of the qualitatively most important patterns $\hat{x}'^{(j)}$

2-chart atlas

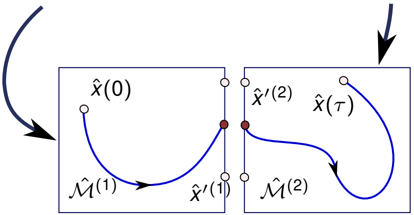


templates $\hat{x}'^{(1)}$, $x'^{(2)}$, with group orbits. Start with the template $\hat{x}'^{(1)}$. All group orbits traverse its $(d-1)$-dimensional slice hyperplane, including the group orbit of the second template $x'^{(2)}$. Replace the second template by its closest group-orbit point $\hat{x}'^{(2)}$, i.e., the point in slice $\hat{\mathcal{M}}^{(1)}$.

2-chart atlas

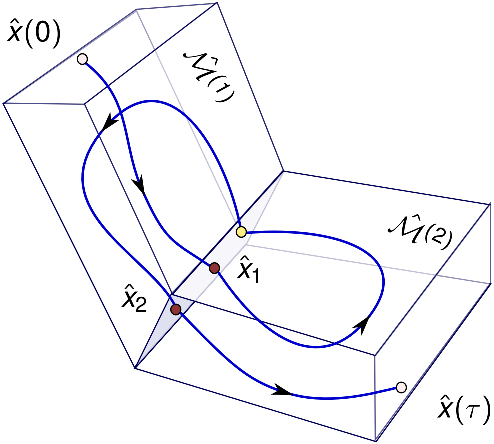


atlas : set of $(d-1)$-dimensional charts



2 templates reduced to the closest points viewed from either group orbit

tangent vectors have different orientations :
2 slice hyperplanes $\hat{\mathcal{M}}^{(1)}$, $\hat{\mathcal{M}}^{(2)}$
intersect in the *ridge*, a hyperplane of dimension $(d-2)$
each chart (page of the atlas) extends only as far as this ridge
if the templates are sufficiently close, the chart border of each slice (red region) is beyond this ridge



the two charts drawn as two $(d-1)$-dimensional slabs

shaded plane : the ridge, their $(d-2)$-dimensional intersection

rotation into a slice is not an average
over 3D pipe azimuthal angle

it is the full snapshot of the flow embedded in the

∞-dimensional state space

NO information is lost by symmetry reduction

- not modeling by a few degrees of freedom
- no dimensional reduction

today's talk's focus :

if you have a symmetry, reduce it!

your quandry

mhm - seems this would require extra thinking

what's the payoff?

example : dynamics simplified

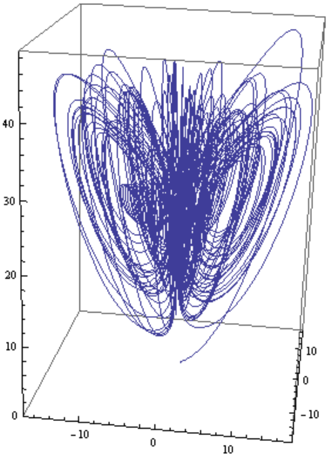
complex Lorenz equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma y_1 \\ -\sigma x_2 + \sigma y_2 \\ (\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - e y_2 \\ \rho_2 x_1 + (\rho_1 - z)x_2 + e y_1 - y_2 \\ -bz + x_1 y_1 + x_2 y_2 \end{bmatrix}$$

$\rho_1 = 28, \rho_2 = 0, b = 8/3, \sigma = 10, e = 1/10$

- A typical $\{x_1, x_2, z\}$ trajectory
- superimposed: a trajectory whose initial point is close to the relative equilibrium Q_1

attractor



example : dynamics confused

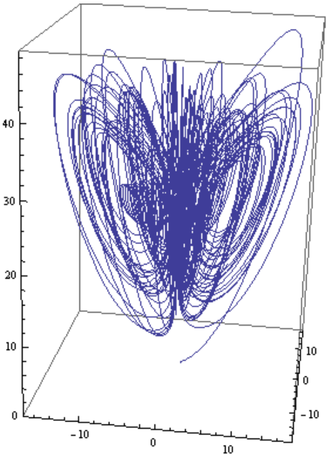
what to do?

it's a mess

the goal

reduce this messy strange attractor to something simple

attractor



example : dynamics symplified

what to do?

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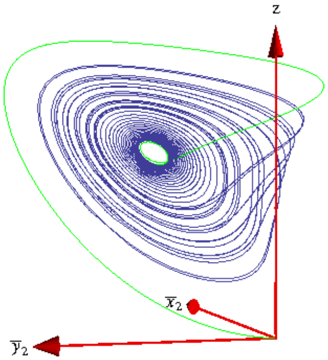
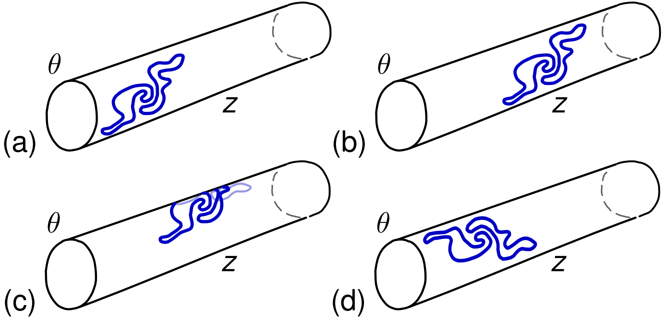
symmetry reduced state space



amazing!

$\mathrm{SO}(2)_z \times \mathrm{O}(2)_\theta$ relative periodic orbits of pipe flow



relative periodic orbit : recurs at time T_p, shifted by a streamwise translation, azimuthal rotation g_p

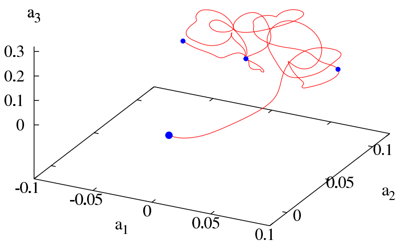
b) stream-wise recurrent

c) stream-wise, azimuthal recurrent

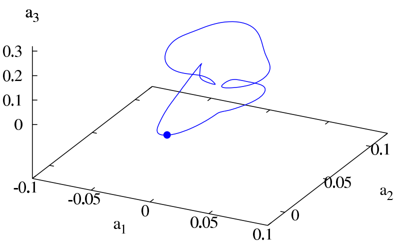
d) azimuthal flip recurrent

example : pipe flow relative periodic orbit

3 repeats, full space

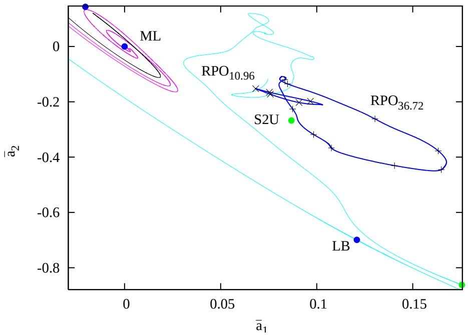


reduced space



triumph : all pipe flow solution in one happy family

example : relative periodic orbit in turbulent pipe flow



first relative periodic orbits embedded in turbulence for a pipe flow!

summary

symmetry reduction achieved!

- families of solutions are mapped to a single solution
  - relative equilibria become equilibria
  - relative periodic orbits become periodic orbits

conclusion

- symmetry reduction by method of slices:
  efficient, allows exploration of high-dimensional flows
  hitherto unthinkable

to be done

- construct Poincaré sections
- use the information quantitatively (periodic orbit theory)

take-home message

if you have a symmetry

use it!

without symmetry reduction, no understanding of fluid flows, nonlinear field theories possible

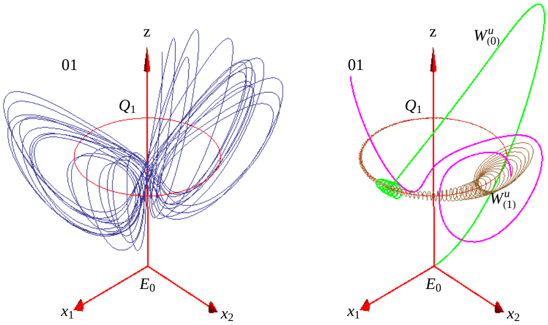
amazing theory! amazing numerics! hope...

"Ask your doctor if taking a pill to solve all your problems is right for you."

Das Gebot

what I teach you now you must do

continuous symmetry induces drifts



- generic chaotic trajectory (blue)
- E_0 equilibrium
- E_0 unstable manifold - a cone of such (green)
- Q_1 relative equilibrium (red)
- Q_1 unstable manifold, one for each point on Q_1 (brown)
- relative periodic orbit $\overline{01}$ (purple)

example : $\mathrm{SO}(2)$ invariance

complex Lorenz equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma y_1 \\ -\sigma x_2 + \sigma y_2 \\ (\rho_1 - z) x_1 - \rho_2 x_2 - y_1 - e y_2 \\ \rho_2 x_1 + (\rho_1 - z) x_2 + e y_1 - y_2 \\ -bz + x_1 y_1 + x_2 y_2 \end{bmatrix}$$

invariant under a $\mathrm{SO}(2)$ rotation by finite angle ϕ:

$$g(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

example : $\mathrm{SO}(2)$ invariance of complex Lorenz equations

complex Lorenz equations equations are invariant under $\mathrm{SO}(2)$ rotation by finite angle ϕ:

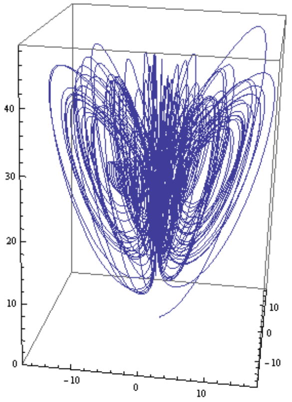
$$g(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{SO}(2)$ has one generator of infinitesimal rotations

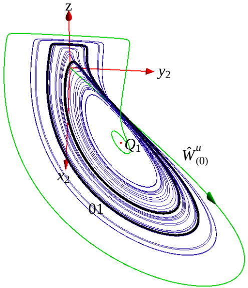
$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

die Lösung : complex Lorenz flow reduced

full state space



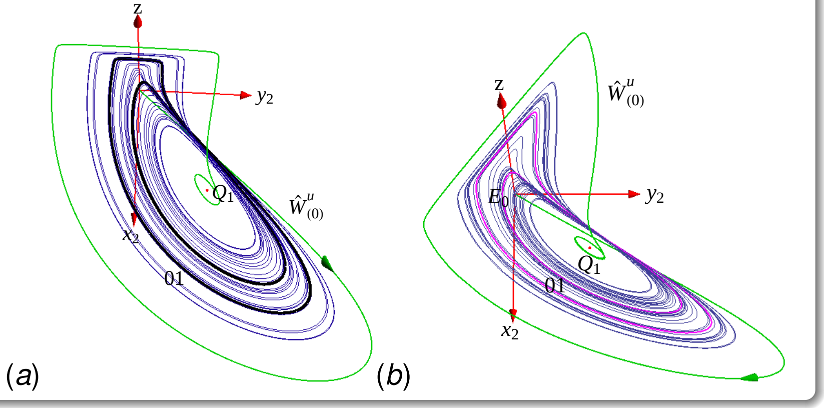
reduced state space



ergodic trajectory was a mess, now the topology is reveled
relative periodic orbit $\overline{01}$ now a periodic orbit

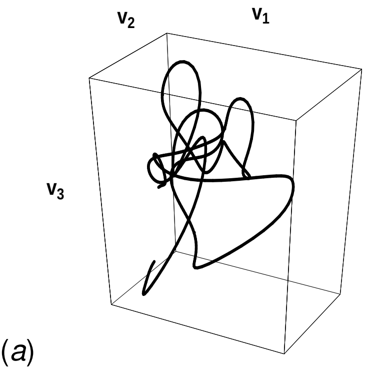
slice charts are local

portrait of complex Lorenz flow in a single slice hyperplane



any choice of the slice $\hat{x}'$ exhibit flow discontinuities

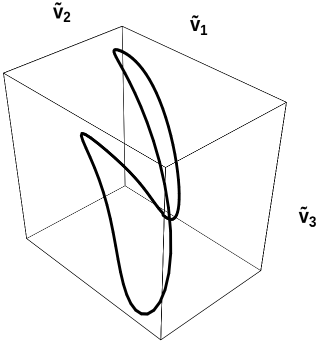
in full state space



a relative periodic orbit of the Kuramoto-Sivashinsky flow, 128d state space traced for four periods T_p, projected on

full state space coordinate frame $\{v_1, v_2, v_3\}$; a mess

in slice



(*b*)

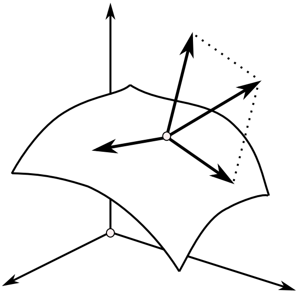
a relative periodic orbit of the Kuramoto-Sivashinsky flow projected on

a slice $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ frame

how relativists do it : connections or ‘gauge fixing’

2-continuous parameter symmetry :
each state space point x owns 3 tangent vectors

local tangent space



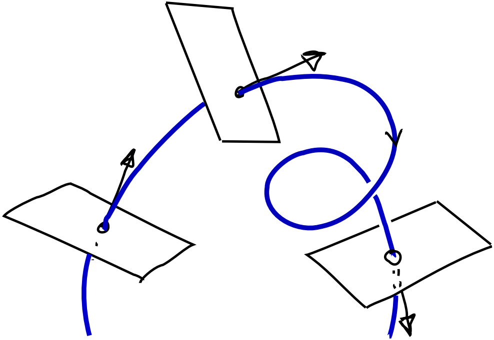
$v(x)$ along the time flow

$t^{(1)}(x)$, $t^{(2)}(x)$ along infinitesimal symmetry shifts

Kim Jong Il gauge

follow flow $\hat{v}(x)$ normal to group tangent directions

method of “connections”



never stray along the group directions, always move orthogonally to the group orbit

North Korean gauge :
slacking along non-shape-changing directions is forbidden

sophisticates do it : Faddeev-Popov gauge fixing

the equivalence principle

integrate over classes of gauge equivalent fields
instead of all fields A_μ^a

the representative in the class of equivalent fields is fixed by a gauge condition,

$$\partial_\mu A_\mu^a = 0\,,$$

a plane intersected by the gauge orbits

$$A_\mu = A_\mu^a t_a \to A_\mu^\Omega = \Omega A_\mu \Omega^{-1} + \partial_\mu \Omega \Omega^{-1}$$

- abelian orbits intersect the plane at the same angle
- non-abelian intersection angle depends on the field

Zutiefst Nutzlos

elegant, deep and useless : no symmetry reduction

Die Faulheit

drifting is energetically cheap

flows are lazy, rather than doing work, solutions drift along non-shape-changing symmetry directions

make Phil Morrison happy

call this

Cartan derivative

$$g^{-1}\dot{g}\,x = e^{-\phi\cdot\mathbf{T}}\,\frac{d}{d\,\tau}e^{\phi\cdot\mathbf{T}}x = \dot{\phi}\cdot t(x)$$