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QUANTUM CHROMODYNAMICS ON THE MASS SHELL

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I prove that in covariant gauges quark and gluon mass-shell renormalization constants obey Ward identities analogous to those of QED: $Z_1 = Z_2$ and $Z_4 = Z_3$. This implies that the mixing of ultraviolet and infrared singularities is not a peculiarity of dimensional regularization but a consequence of the Ward identities.

1. Introduction

The increasing phenomenological success of quark ideas raises hopes that the colour [1] Yang-Mills theories [2] might provide a theoretical underpinning for quark models. Theorists' strong aesthetic prejudice in favour of gauge theories is here complemented by two experimental facts: the fall-off of Yang-Mills effective coupling with energy [3] fits the observed deep inelastic scattering, and the ill-understood low energy behaviour might be the theory's principal virtue if it leads to confinement of colour particles [1].

The goal of the present and subsequent papers [4] is to explore this latter feature of non-Abelian gauge theories by studying quantum chromodynamics (QCD), a theory of equal-mass ($m \neq 0$) quarks of n colours interacting through N strictly massless gluons, with couplings related by a simple Lie algebra. QCD will be defined through its Feynman diagram expansion. Even though this conservative approach might miss much of the rich structure of Yang-Mills theories [5], it is not known whether it falls short of accounting for the observed quark phenomenology, and it is still important that the information contained in the QCD perturbation series be thoroughly examined.

Low energy behaviour of QCD involves two distinct unsolved infrared (IR) problems: *twinkling* * and *infrared slavery* [1]. Twinkling is a technical problem of properly defining the transition rates in the presence of coloured massless particles. (Its QED analogue was first solved by Block and Nordsieck [7]). Masslessness of the gluon precludes separation of a bare quark from its soft gluon cloud (the quark colour "twinkles"), and the transition rates must be expressed in terms of the *overall*

* The term "twinkling" was introduced by Cornwall and Tiktopoulos [6] in a slightly more restricted sense than the one used here.

colours of external particles taken together with their soft gluon clouds, and integrated over finite energy resolutions. A low energy gluon can be called soft only if its colour cannot be used to determine what particle it came from – this condition is satisfied automatically by the Lie algebra closed by coupling strengths.

QED suggests two methods for solving the twinkling problem. In the first approach [8,9] one should explicitly factorize all virtual gluon IR singularities in S-matrix elements, as well as the real soft gluon bremsstrahlung singularities in transition rates, and demonstrate the cancellation between the two. As a bonus one obtains the leading radiative corrections for small finite energy resolutions. Due to the negligible recoil of an electron in a soft photon emission, soft photons are uncorrelated and all QED IR singularities are described by exponentiation of one-loop IR singularities. In QCD all lowest non-trivial order calculations [10–13] show that IR factorizations and cancellations still go through, but with one essential difference – recoil of a soft gluon due to the emission of further soft gluons cannot ne neglected, and *soft gluons do not act independently* [9,14,15]. So far this difficulty had frustrated formulation of an all-order IR factorization procedure – it is very unlikely that one-loop IR integral will describe all IR singularities of QCD amplitudes. If it furthermore turns out that QCD is a confining theory, the IR factorization approach might be unnecessarily detailed in the sense that the leading radiative corrections so obtained will have no physical significance.

For these reasons the second method of proving IR finiteness of QED [16,17] might be more cogent to QCD. Here one forgoes explicit factorizations of virtual and real photon (gluon) IR singularities by looking directly at the transition rates and showing that if all indistinguishable processes are included, the (unrenormalized) transition rates are IR convergent (except for some exceptional values of external momenta). In other words, if a transition rate is finite in the lowest non-trivial order, the entire (unrenormalized) Feynman diagram series will not introduce any overall IR divergences. By avoiding explicit disentangling of IR singularities, Kinoshita's [16] approach also avoids complications due to the non-commutatively of QCD couplings, and in the one all-order example worked out so far [4,12] works the same way for QDE and QCD. Even though a general proof is lacking at the present time, I shall assume that QCD twinkling is a solved problem, and that the *unrenormalized QCD transition rates are infrared finite.*

Renormalization brings us to the second QCD infrared problem: does the behaviour of effective coupling and/or quark mass at low anergies give rise to infrared slavery (confinement)? While for QED the low energy coupling goes to a finite limit $\alpha = 137^{-1}$, the QCD coupling appears to be blowing up. It will be my aim to give this statement a precise meaning by exhibiting the coupling IR singularities as a formal perturbation series, in the hope that this will ultimately lead to a form of soft quark-quark potential useful for bound state calculations. The idea is to study QCD in three steps *.

* A very similar programme has been independently proposed by Sugamoto [13].

(i) Define the bare QCD in the usual way, as a Feynman diagram expansion in g_0. 2- and 3-external-leg diagrams evaluated at renormalization momenta yield renormalization constants. I shall prove that on the mass-shell renormalization constants satisfy Ward identities of QED type ($Z_1 = Z_2$, etc.), and show that these are respected only if the regularization permits mixing and mutual cancellations of ultraviolet (UV) and IR singularities.

(ii) As an *intermediate* step renormalize QCD on the mass-shell [13,18]; now QCD is a UV-finite formal power series in the mass-shell coupling g. Invariant regularization makes this a well-defined step whose aim is to uncover the low energy behaviour of the effective coupling. In both QED and QCD mass-shell renormalization induces IR singularities, but while in QED these cancel by Ward identity, in QCD they do not. The UV-IR connection established in (i) will now show that this IR singularity is controlled by the renormalization group for the pure Yang-Mills field. The formal expansion in g is reminiscent of the bare coupling expansion in QED; there both e_0 and the coefficients in the perturbation expansion are UV divergent, and the precise form of UV singularities of the coefficients yields the renormalization group [19] for the effective couplings at finite energies. In the mass-shell QCD both g and the coefficients in the perturbation expansion are IR divergent, and the precise form of the IR singularities of the coefficients controls the way in which the effective coupling g_{eff} diverges at low energies.

(iii) Finally, QCD should be rewritten in terms of some effective coupling characteristic of quark bound states, with the low energy controlled by (ii). I have no progress to report on this crucial phase of QCD confinement theory. In particular, while the steps (i) and (ii) clarify the role of the renormalization group for QCD low energy couplings, this information can be useful only if more is known about the behaviour of the renormalization group function $\beta(g)$ at large g.

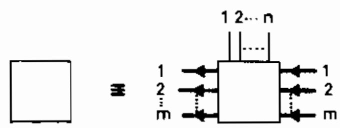
The present article, which is a detailed discussion of the results announced in the one-page preprint ref. [20], covers the step (i) of the above programme. In sect. 2 I review and extend 't Hooft's identities [21] from which in sect. 3 I derive the mass-shell identities. In this I use the original combinatoric approach of 't Hooft rather than functional formalism, because the diagrammatic notation is very convenient for perturbative calculations carried out in sect. 4. These are used in sect. 5 to verify the Ward identities and establish that the UV-IR singularity mixing is not a peculiarity of dimensional regularization, but a consequence of the Ward identities. In sect. 6 I discuss gauge dependence of the mass-shell renormalization constants. The appendix reviews the location of UV and IR divergences in the Schwinger and Feynman parametric representations, and gives the parametric space Feynman rules for QCD.

An application of the above results to QCD infrared problems has been outlined in ref. [12]. This will be discussed in detail in the subsequent article [4].

2. 't Hooft's identities

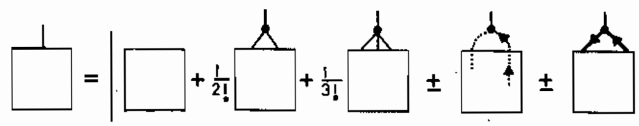
In this section I shall review the combinatorial derivation of 't Hooft's identities [21] and extend them to QCD Green functions with external gluons, quarks and ghosts. I shall follow the original approach of 't Hooft as developed by Lautrup [22].

The basic object in Lautrup's approach is the momentum space amputated Green function



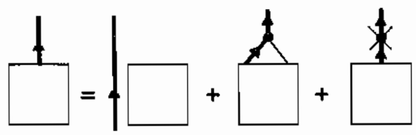
(2.1)

which is defined recursively by the Dyson-Schwinger equations

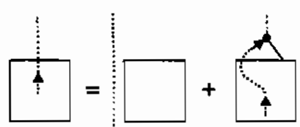


(2.2a)

(–sign for closed loops)



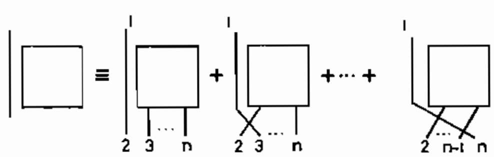
(2.2b)

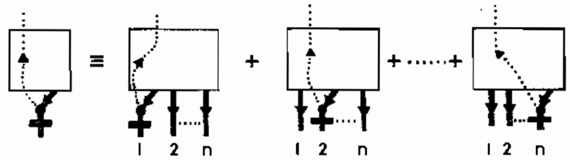


(2.2c)

and the requirement that all vacuum bubbles (disconnected diagrams with no external lines) vanish. The uninteresting external legs can be suppressed (as on the LHS of (2.1), or in (2.2)) and the box by itself can be thought of as Z, the generating functional for disconnected Green functions.

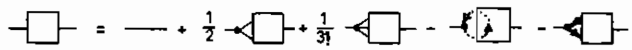
The Dyson-Schwinger equations have a simple intuitive interpretation: (2.2a) says that a gluon can either go through without interacting, or end in a 3-gluon, 4-gluon, ghost or quark vertex. The factors $1/2!$, $1/3!$ assure a correct combinatoric weight for each Feynman diagram. Green functions are symmetric in external gluon lines and antisymmetric in all in- or out-quark lines. The following two examples illustrate how the compact notation relates to the full Green function notation



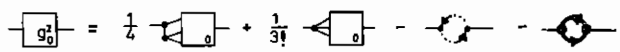


(2.3)

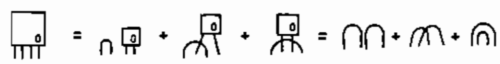
The box is really just the sum of all Feynman diagrams with a given number of external legs. Let me illustrate how they are generated by (2.2) with a simple example, the gluon self-energy at one-loop level. (2.2a) gives



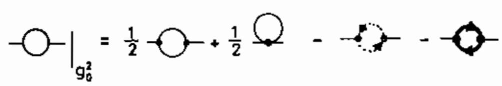
Using (2.2) again and keeping only g_0^2 terms



where the subscript 0 on 4-gluon Green functions means that they are of order zero in the coupling constant, i.e. completely disconnected. From (2.2a)



Substituting this above we obtain the g_0^2 order gluon propagator



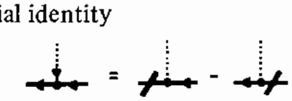
with all correct signs and combinatoric factors.

In this manner the iteration of (2.2) generates all Feynman diagrams: colours of internal lines are summed over and in loops the loop momenta are integrated with a factor

$$\int \frac{\mathrm{d}^{4-\epsilon} l}{(2\pi)^{4-\epsilon}}$$

for each loop. This, together with vertex and propagator factors defined in figs. 1 and 2 yields all contributing Feynman integrals (I follow the conventions of Bjorken and Drell [23]). I shall discuss how the dimensional regularization takes care of both UV and IR singularities in sect. 5; for the purposes of the present and the next section I merely assume that there exists a regularization which allows shifts of integration momenta.

The QED combinatorial proof of the Ward-Takahashi identities follows from a trivial identity



(2.4)

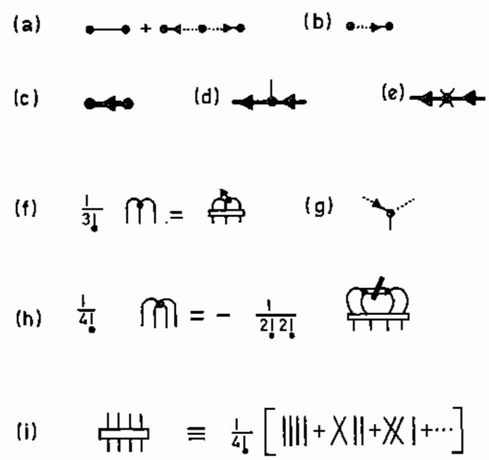
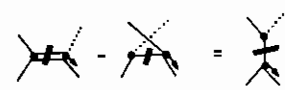


Fig. 1. Feynman rules for QCD. Propagators: (a) gluon in covariant gauges, (b) ghost, (c) quark. Vertices: (d) quark-gluon, (e) quark mass counterterm, (d) 3-gluon, (g) ghost-gluon, (h) 4-gluon. (i) Symmetrization symbol – combinatorial factor insures idempotency, i.e. that two consecutive symmetrizations are the same as one. For the notation see figs. 2 and 3.

i.e., $i\not k = i(\not k + \not p - m) - i(\not p - m)$. In QCD this identity applies as well (Lautrup's slashed line notation now keeps track of colour indices), but further identities for the insertion of k^μ into 3- and 4-gluon vertices are needed. As such vertices are symmetric in all gluons, it is convenient to introduce a symmetrization symbol [24,25], and write 3- and 4-vertices in a compact diagrammatic form as in fig. 1. To extend (2.4) to QCD one also needs the Lie algebra relations between quark-gluon and gluon-gluon couplings $(T_i)^a_b$ and $-iC_{ijk}$. The diagrammatic methods for this have been developed in ref. [24] (see also (A.9)). In the notation of fig. 2, which combines momentum and colour factors, some examples of such relations are

 (2.5a)

 (2.5b)

 (2.5c)

As a slashed line means that the momentum-space propagator is absent, these are purely group-theoretic relations. Another convenient diagrammatic relation is a statement of momentum conservation ($k_1^\mu + k_2^\mu = -k_3^\mu$)

 (2.6)

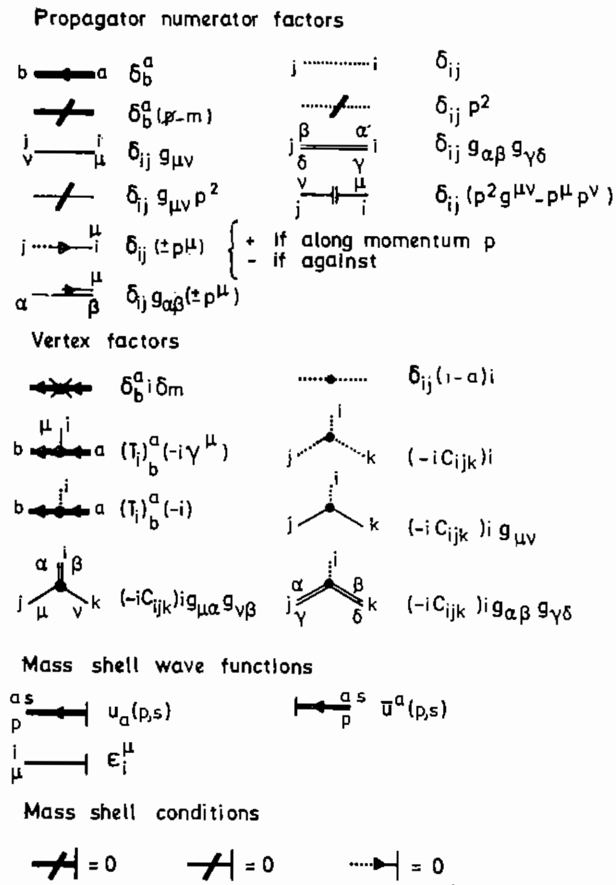


Fig. 2. Combined diagrammatic notation for the group-theoretic and momentum space factors. For each numerator factor sandwiched between two dots, an additional $i\,(\not p - m)^{-1}$ factor for a quark, and $-ip^{-2}$ factor for a gluon or associated auxillary particles. In the Feynman-parametric space, rule 5 of the appendix, replace p^{μ} numerator factor for internal line i by D_i^{μ}.

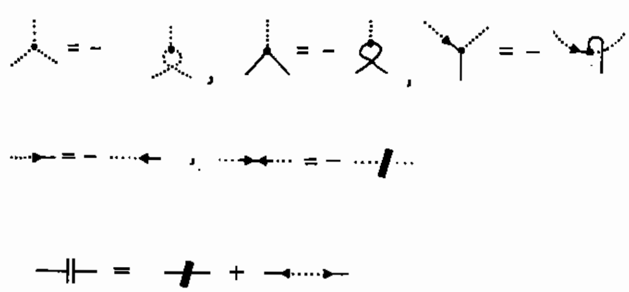
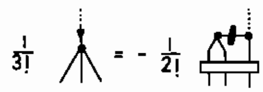


Fig. 3. Sign conventions, some relations. Gluon colours are read anticlockwise around the vertex C_{ijk}, and care must be exercised that interchanges of legs are compensated by minus signs in diagrammatic equations.

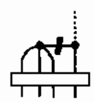
't Hooft's identities for bare 3- and 4-gluon vertices now follow

(2.7a)

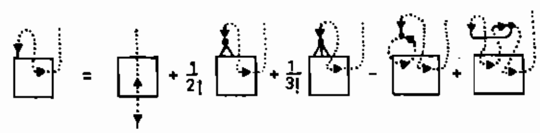


(2.7b)

as well as a further identity

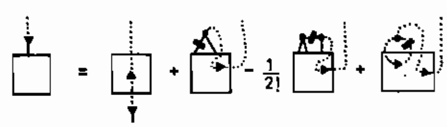
 $= 0$ (2.8)

Now apply the pure Yang-Mills Dyson-Schwinger equation ((2.2a) without quarks) to a particular Green function



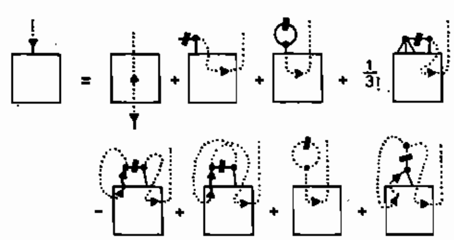
(2.9)

where all suppressed external legs are gluons (no other ghost lines than the one explicitly drawn). A ghost vertex can lie either on a ghost loop or on the incoming ghost line, hence the last two terms in (2.9). They add up by the momentum conservation (2.6). On the second and third terms we apply the identities (2.7), and the $k^\mu k^\nu$ term from (2.7a) cancels most of the left-hand side of (2.9) by the Dyson-Schwinger equation (2.2c). (That was the reason why I started with the Green function of (2.9).) This all leads to



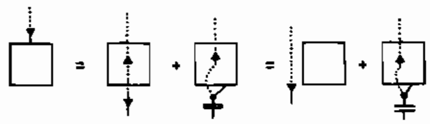
(2.10)

Now use again the Schwinger-Dyson equation (2.2a) for the second term, and (2.2c) for the last term



(2.11)

The third and seventh term vanish by the antisymmetry of C_{ijk}, the fourth by (2.8), and the fifth, sixth and eight cancel by the Jacobi relation (2.5a), yielding *'t Hooft's identity* [21] for the pure Yang-Mills theory

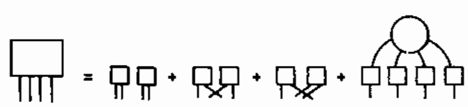


(2.12)

This is valid in *all covariant gauges* because the transverse projection appearing in (2.7a) assured that the $k^\mu k^\nu$ part of the gluon propagator does not affect any of the cancellations that went into the proof.

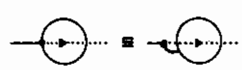
(2.12) is easily extended in two ways. First, the full QCD (2.2) together with identities (2.4) and (2.5c) incorporates the quarks. Second, if there are external ghost lines going through the box, each application of the Dyson-Schwinger equations (steps (2.9) and (2.11)) yields an extra term corresponding to the possibility of attaching a gluon to those lines. The resulting extended 't Hooft's identity for QCD Green functions with external gluons, quarks and ghosts is given in fig. 4. This is the main result of this section. Its meaning is that no longitudinal gluon can couple to mass-shell quarks and gluons [21].

For renormalization purposes it is necessary to introduce also amputated connected Green functions (I shall denote them by circles) which are obtained from "boxes" by dropping all disconnected diagrams and all self-energy corrections to external lines. For example



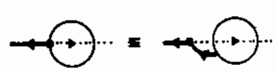
(2.13)

To rewrite 't Hooft's identities in terms of these it is convenient to introduce the notation



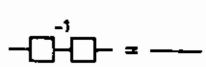
(2.14a)

(note that this fixes a sign convention)

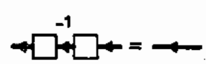


(2.14b)

and the inverse self-energies defined by



(2.15a)



(2.15b)

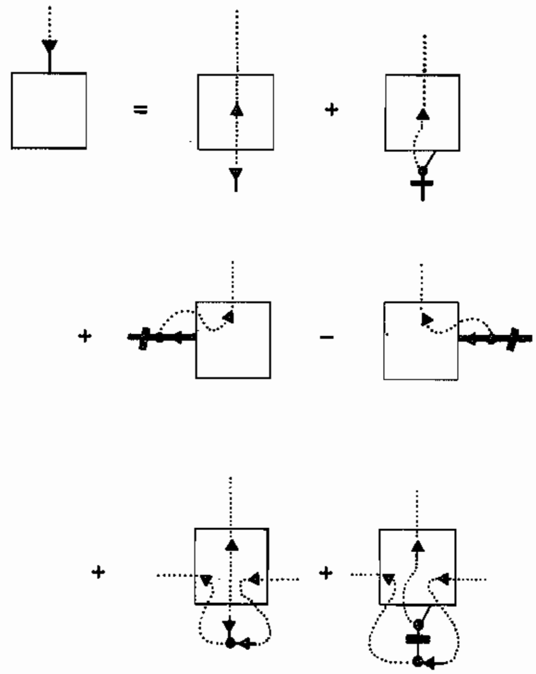
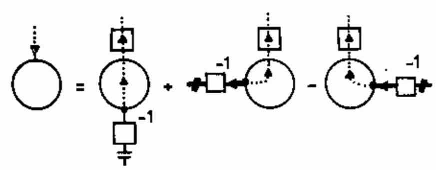


Fig. 4. 't Hooft's identity for QCD Green functions with external quarks, gluons and ghosts.

In terms of connected amputated Green functions (with external quarks and gluons, but no ghosts) 't Hooft's identity is given by



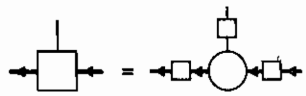
(2.16)

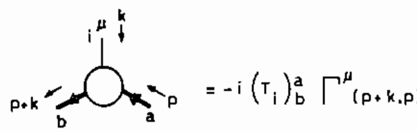
(This suppressed external leg notation is related to the full notation as in (2.3).) Further expansion in terms of one-particle-irreducible Green functions leads to the Lee-Kluberg-Stern-Zuber [26] identities. (In the derivation of (2.16) I have used the transversality of the gluon propagator – this will be justified by (3.5).)

3. Ward identities

In this paper the term "Ward identities" will be reserved for the relations between renormalization constants evaluated on the mass-shell.

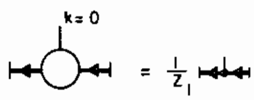
Consider $\partial(k \cdot G)/\partial k_\nu$, where G_μ is a quark-quark-gluon Green function (colour and spinor indices suppressed). G_μ is related to the proper vertex Γ_μ by (2.13)


(3.1)

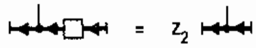


(3.2)

The quark mass-shell renormalization constants are defined by



(3.3a)

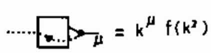


(3.3b)

where the quark mass counter-term had been used in (3.3b), so the extra propagator picks out the first derivative of the self-energy, $Z_2 = (p_\mu/m)\, \partial S/\partial p^\mu$, evaluated on the mass-shell. (In sect. 4 such renormalization constants are computed explicitly to the one-loop level.) 't Hooft's identity fig. 4 for the 2-gluon Green function is

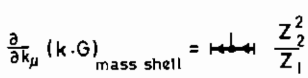

(3.4)

In *covariant gauges* Lorentz invariance requires



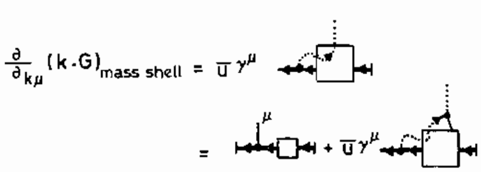
(3.5)

so the second term in (3.4) vanishes. Hence $k \cdot G$ has no contribution from the external gluon self-energy, and



(3.6)

Now re-evaluate (3.6) using 't Hooft's identity:



(3.7)

I have used k momentum routing of (3.2), noting that after differentiating fig. 4,

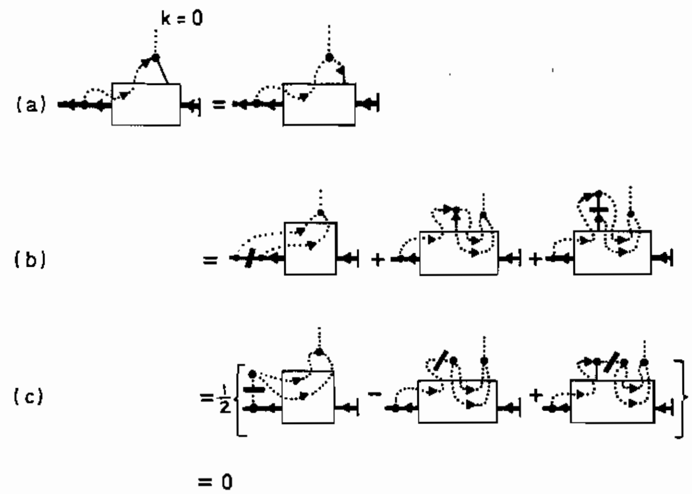
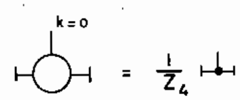


Fig. 5. Proof of vanishing of the second term in (3.7). (a) by (2.6) and $k^\mu = 0$. (b) by 't Hooft's identity fig. 4. (c) by antisymmetry of the top C_{ijk} vertex, Lie algebra (2.5) and momentum conservation (2.6). The second term cancels the remaining two by (2.2c).

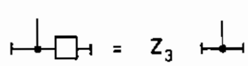
everything vanishes on the mass-shell, except the $(\partial/\partial k^\mu)(\not p + \not k - m) = \gamma^\mu$ term. The second line was obtained by expanding the ghost by (2.2c). From (3.6) and (3.7) it follows that $1 - Z_2/Z_1$ is proportional to the second term of (3.7), but it vanishes as proven in fig. 5. Hence the mass-shell quark renormalizations obey the same Ward identity in QED and QCD

$$Z_1 = Z_2 \,. \tag{3.8}$$

The Ward identity for gluon renormalizations is derived in the same way. The gluon mass-shell renormalization constants are defined by

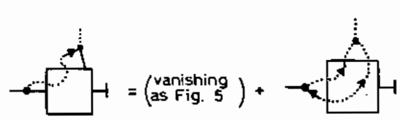


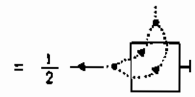
(3.9a)



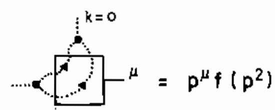
(3.9b)

To avoid ambiguities due to the vanishing of the bare vertex as all momenta vanish, I compute them at $k = 0, p \neq 0$ (the other possible choice consistent with $p_i^2 = 0$ would be $p_1 = p_2 = -\frac{1}{2} p_3 \neq 0$). By the same argument as for the quarks, $1 - Z_3/Z_4$ is proportional to




(3.10)

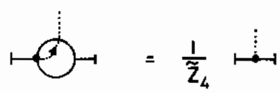
by the antisymmetry of the top C_{ijk} vertex and (2.6). In the covariant gauges by Loretnz invariance

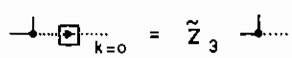

(3.11)

vanishes on the mass shell. Hence the gluon renormalizations also obey a simple Ward identity

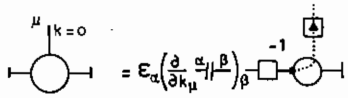
$$Z_4 = Z_3 \,. \tag{3.12}$$

The above argument cannot be repeated for the ghost renormalizations, as 't Hooft's identity fig. 4 for the ghost-ghost-gluon Green function yields no mass-shell vanishing factors on the external legs. Fortunately, the ghost renormalizations are already related by the Slavnov-Taylor [27] identities. The ghost mass-shell renormalization constants are defined by


(3.13a)


(3.13b)

The Slavnov-Taylor identities follow from (2.16). Consider $\partial(k\cdot\Gamma)/\partial k^\mu$ on the mass-shell, where $\Gamma_{\alpha\beta\gamma}$ is the 3-gluon proper vertex. Again only the derivative of $g^{\mu\nu}(p+k)^2 - (p+k)^\mu(p+k)^\nu$ survives, leading to the Slavnov-Taylor identity:


(3.14a)

$$\frac{1}{Z_4} = \frac{1}{Z_3}\frac{\tilde{Z}_3}{\tilde{Z}_4} \,. \tag{3.14b}$$

This together with (3.12) yields the Ward identity for ghost renormalizations

$$\tilde{Z}_4 = \tilde{Z}_3 \,. \tag{3.15}$$

Two remarks are in order. First, in deriving the above Ward identities I have used, beyond the assumptions necessary for 't Hooft's identity (that there exists a regularization procedure which allows shifts of integration momenta), the assumption of Lorentz invariance in (3.5) and (3.11), so it is possible that general non-covariant gauges obey more complicated Ward identities. Second, the derivation did not rely on any assumptions beyond those used for the original Slavnov-Taylor identities (those were also derived on the mass-shell). That the renormalization constants also obey the stronger Ward identities (3.8) and (3.12) was first noted only in explicit calculations [18,20].

4. One-loop contributions to the renormalization constants and magnetic moment

This section is rather technical, and a reader interested primarily in conceptual consequences of QCD Ward identities should proceed straight to the next section. Here I shall write down a series of one-loop integrals which will be used in sect. 5 for verification of the Ward identities. I calculate in covariant gauges and $4-\epsilon$ dimensions (ϵ not assumed small), but the *dimensional regularization [28–30] is not essential* to my calculations. I shall write the integrals directly in the Feynman (or Schwinger) parametric form using the rules summarized in the appendix. The transition from momentum space to the parametric representation assumes only existence of a regularization scheme which allows shifts of integration momenta, and the *integrand* of a parametric integral is unambigously defined in any dimension, including four. This enables us [31] to split Feynman parametric integrals into their UV, IR and finite parts. The non-UV parts so defined are simply related to the renormalized amplitudes (renormalized on- or off-mass-shell). The divergent parts can be manipulated as unevaluated integrals: UV parts absorbed into renormalizations, and IR parts kept around until they mutually cancel or combine into something simpler. I mention this only as one possible alternative to the dimensional regularization: for the integrals at hand the dimensional regularization is more elegant, and I shall not use the finite part prescription of ref. [31] here *. The analytic continuations needed to evaluate dimensionally regularized singularities ** will be discussed

* Two remarks about the method developed in ref. [31]. First, the UV, IR, finite part decomposition doesn't respect the Ward identities, in the sense that even though $L + B = 0$, for the finite parts $\Delta L + \Delta B \neq 0$. Second, in all QED integrals the separation of UV and IR is automatic, essentially due to the non-zero electron mass. For QCD the method of ref. [31] has to be modified for purely gluonic diagrams, as there is no automatic splitting of integrals into sums of UV and IR singular terms.

** The often repeated statement that in the dimensional regularization UV divergences appear in $\Gamma(\frac{1}{2}\epsilon)$ factors, and IR divergencies arise from parameter integrations is true only for one-loop Feynman parametric integrals. In general all singularities arise from parametric space, UV divergences from vanishing of the parametric function U, and IR divergences from vanishing of $V(z_i, p_i, m_i)$. (See appendix and ref. [31] for details).

in the next section: they do not affect the form of the *integrands* computed below.

On the mass-shell, the quark vertex renormalization L, the colour magnetic moment a, the mass counter-term δm, the wave-function renormalization B as well as the gluon wave-function renormalization C are defined by (I follow the notational conventions of Bjorken and Drell [23])

$$
\begin{aligned}
&1 + L = \frac{1}{Z_1} = F_1(0) \,, \\
&a = \frac{M}{1+L} \,, \\
&\delta m = \Sigma \,, \\
&B = \frac{1}{1 - Z_2} = \frac{p^\mu}{m} \frac{\partial \Sigma}{\partial p^\mu} \,, \\
&C = -g_0^2 \pi(0) \,,
\end{aligned}
\tag{4.1}
$$

where the unrenormalized quark from factors are computed from (3.2)

$$\bar{u}\Gamma^\mu(p + \tfrac{1}{2}q, p - \tfrac{1}{2}q)\, u = F_1(q^2)\, \bar{u}\gamma^\mu u + F_2(q^2)\, \bar{u}\, \frac{i\sigma^{\mu\nu}q_\nu}{2m} u$$

(from now on, $\bar{u}$ and u will be usually suppressed, and $m = 1$), and the (unrenormalized) magnetic moment $M = F_2(0)$ is computed by dropping from Γ^μ the numerator terms proportional to q^μ, $q^\mu \not{p}$, γ^μ, $p^\mu \not{q}$ or quadratic and higher in q

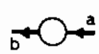
$$\Gamma^\mu \to Ap^\mu + Bp^\mu \not{p} + C\gamma^\mu \not{p} + \tfrac{1}{2} D\gamma^\mu \not{q} + \tfrac{1}{2} E\gamma^\mu \not{q}$$

and replacing this remainder by its magnetic moment projection

$$M = -(A + B + C + D + E) \,.$$

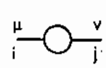
$\Sigma(p)$ is the (one-particle-irreducible) quark self-energy

$$-i\delta_b^a \Sigma(p) = \tag{4.2}$$



and $\Pi(q^2)$ the (one-particle-irreducible) gluon self-energy

$$= \tag{4.3}$$



$$\Pi^{\mu\nu}(q) = (q^\mu q^\nu - g^{\mu\nu}q^2)\, \Pi(q^2)$$

As in QED, everything is expanded in a formal power series *

$$M = \sum_{n=1}^{\infty} M^{(2n)}\left(\frac{\alpha_0}{\pi}\right)^n , \qquad \text{etc.,} \tag{4.4a}$$

where

$$\alpha_0 = \frac{g_0^2}{(4\pi)^{1-\epsilon/2}}\, m^{-\epsilon} .$$

g_0 was introduced by rescaling all couplings $T_i \to g_0 T_i$, $C_{ijk} \to g_0 C_{ijk}$. This is merely a device for counting vertices. g_0 can be thought of as a coupling constant only if the underlying group is simple, and the T_i's have a fixed normalization (see ref. [24]). The $(2n)$th order contribution is computed from all n-loop one-particle-irreducible graphs G

$$M^{(2n)} = \sum_G W_G M_G , \tag{4.4b}$$

where the group theoretic weight W_G contains all colour summations over T_i and $-iC_{ijk}$, and M_G is the momentum-space Feynman integral for the graph G.

Group theoretic weights are trivially dealt with. The contributing 1-loop graphs are drawn in fig. 6. For quark graphs the theoretic weights are

(a) $(T_i)_b^a W_a = (T_j)_b^c (T_i)_c^d (T_j)_d^a$,

(b) $(T_i)_b^a W_b = (-iC_{ikj})(T_k)_b^d (T_j)_d^a$,

(c) $(T_i)_b^a W_c = (T_i)_b^c (T_j)_c^d (T_j)_d^a$,

or, in the diagrammatic notation of ref. [24], simply the diagrams fig. 6a, b, and c by themselves. Multiplying the Lie algebra commutator (A.9a)

$$(T_i)_b^c (T_j)_c^d - (T_j)_b^c (T_i)_c^d = (-iC_{ijk})(T_k)_b^d \tag{4.5}$$

by $(T_j)_d^a$ we obtain a relation between weights

$$W_c - W_a = W_b \tag{4.6a}$$

and similarly

$$W_d = W_e = W_f = \tfrac{1}{2} W_b . \tag{4.6b}$$

* The only deviation from the Bjorken and Drell notation is

$$g_0^2 \pi \equiv \sum_{n=1}^{\infty} \pi^{(2n)} (\alpha_0/\pi)^n .$$

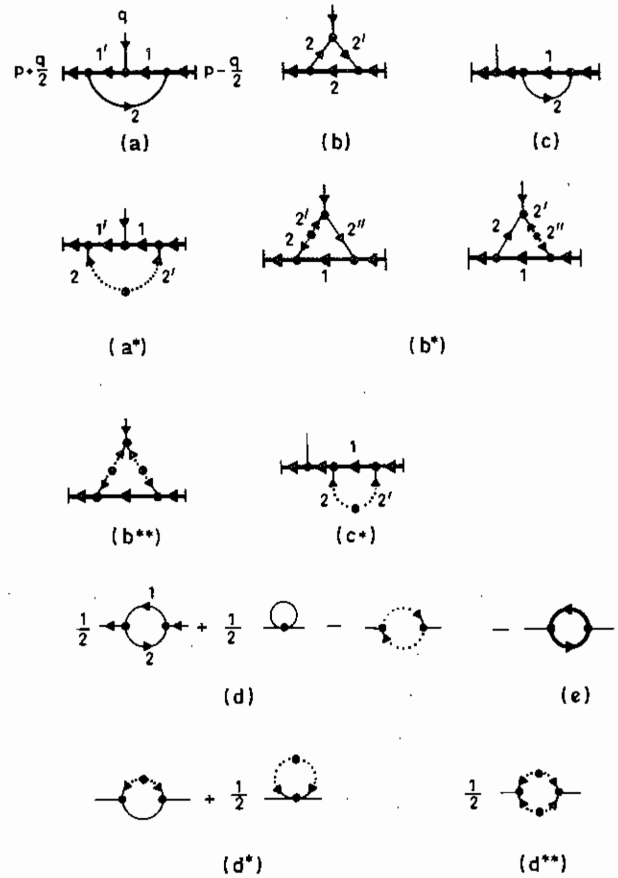


Fig. 6. Diagrams contributing to quark renormalizations and magnetic moment. (a)–(d) Feynman gauge contributions. (a*)–(d**) covariant gauges. Arrows on the lines indicate momenta directions, and numbers label the corresponding Feynman parameters. z_1 and $z_{1'}$ appear only as a sum $z_1 + z_{1'}$; in the parametric integral we set $z_1 + z_{1'} \to z_1$. (b**) vanishes for both L and M by 't Hooft's identity (2.7a).

This is all the group theory needed for the verification of the Ward identity (3.8). (The above example shows how much quicker it is to compute weights diagrammatically than algebraically: the diagrammatic notation of ref. [24] keeps automatically track of all dummy indices and signs.)

Next I write down the Feynman parametric integrals for all graphs of fig. 6, using the rules of the appendix. In covariant gauges the integrals for the diagrams a and b are of form

$$\Gamma^\mu = -\tfrac{1}{4}\,\Gamma(\tfrac{1}{2}\,\epsilon)\int \mathrm{d}z_G\,\delta(1 - z_1 - z_2)\Big[(1 + \tfrac{1}{2}\,\epsilon)\,\frac{\epsilon}{2}\;\frac{(a-1)\,F_0^*}{V^{2+\epsilon/2}}$$

$$+\frac{\epsilon}{2}\frac{F_0+(a-1)F_1^*}{V^{1+\epsilon/2}}+\frac{F_1+(a-1)F_2^*}{V^{\epsilon/2}}\Bigg] . \tag{4.7}$$

For the quark graphs a, b and c on the mass-shell $V = m^2 z_1 - p^2 z_1 z_2 = z_1^2$, and scalar currents have been substituted by $A_1 = 1 - z$, $A_2 = -z$, $(z = z_1)$. The starred numerators F_i^* arise from $k^\mu k^\nu/k^2$ gauge terms – note that they have different phase-space factors $\mathrm{d}z_G$.

Diagram a

fig. 6a: $\mathrm{d}z_G = \mathrm{d}z_1 \mathrm{d}z_2 z_1$,

$$\text{for } L_a, \quad F_0 = -2(-2+2z+z^2)+\epsilon z^2 , \quad F_1 = -2(1-\tfrac{1}{2}\epsilon)^2 ;$$

$$\text{for } M_a, \quad F_0 = -4z(1-z)+2\epsilon z(2-z) , \quad F_1 = 0 . \tag{4.8}$$

fig. 6a*: $\mathrm{d}z_G = \mathrm{d}z_1 \mathrm{d}z_2 z_1 z_2$,

$$\text{for } L_a, \quad F_0^* = -z^2(2-z)^2 , \quad F_1^* = 2-(6-\epsilon)(2-z)z ,$$

$$F_2^* = -\tfrac{1}{4}(4-\epsilon)(6-\epsilon) ;$$

$$\text{for } M_a, \quad F_0^* = F_1^* = F_2^* = 0 . \tag{4.9}$$

(Some details of the calculation from fig. 6a are given in ref. [32], sects. 3 and 5.)

Diagram b

fig. 6b: $\mathrm{d}z_G = \mathrm{d}z_1 \mathrm{d}z_2 z_2$,

$$\text{for } L_b, \quad F_0 = (6-2\epsilon)z^2 , \quad F_1 = -6+2\epsilon ;$$

$$\text{for } M_b, \quad F_0 = 4z(1-z)+2\epsilon z^2 , \quad F_1 = 0 . \tag{4.10}$$

fig. 6b*: $\mathrm{d}z_G = \frac{1}{2}\,\mathrm{d}z_1 \mathrm{d}z_2 z_2^2$,

$$\text{for } L_b, \quad F_0^* = 0 , \quad F_1^* = -(3-\epsilon)(2-z)z ,$$

$$F_2^* = -\tfrac{1}{2}(3-\epsilon)(6-\epsilon) ;$$

$$\text{for } M_b, \quad F_0^* = 2z^3(2-z) , \quad F_1^* = -4z+z^2(8-\epsilon) , \quad F_2^* = 0 . \tag{4.11}$$

The quark mass counter-term and the quark wave-function renormalization are computed from fig. 6c as in ref. [32], sect. 3.

Diagram c

$$\delta m_c = \tfrac{1}{4}\,\Gamma(\tfrac{1}{2}\epsilon)\int \mathrm{d}z_1 \mathrm{d}z_2 \delta(1-z_1-z_2)\left[\frac{\epsilon}{2}\,\frac{(a-1)\,z_2 F_0^*}{V^{1+\epsilon/2}}+\frac{F_0+(a-1)\,z_2 F_1^*}{V^{\epsilon/2}}\right],$$

$$B_c = \tfrac{1}{4}\Gamma(\tfrac{1}{2}\epsilon)\int \mathrm{d}z_1 \mathrm{d}z_2 \delta(1-z_1-z_2)\left[\frac{\epsilon}{2}\,\frac{(a-1)\,z_2 E_0^*}{V^{1+\epsilon/2}} + \frac{E_0 + (a-1)\,z_2 E_1^*}{V^{\epsilon/2}}\right.$$

$$\left. + (1+\tfrac{1}{2}\epsilon)\,\epsilon G\,\frac{(a-1)\,z_2 F_0^*}{V^{2+\epsilon/2}} + \epsilon G\,\frac{F_0 + (a-1)\,z_2 F_1^*}{V^{1+\epsilon/2}}\right] \qquad (4.12)$$

with $G = z_1 z_2$, and

fig. 6c: $F_0 = 2 + (2-\epsilon)\,z$, $E_0 = -(2-\epsilon)(1-z)$;

fig. 6c*: $F_0^* = z^2(2-z)$, $F_1^* = -1 + (3-\frac{1}{2}\epsilon)\,z$,

$E_0^* = (5-3z)\,z$, $E_1^* = 1 - \frac{1}{2}\epsilon + (3-\frac{1}{2}\epsilon)\,z$.

Diagrams d

$$\Pi_d(q^2) = -\tfrac{1}{4}\Gamma(\tfrac{1}{2}\epsilon)\int \mathrm{d}z_1 \mathrm{d}z_2 \delta(1-z_1-z_2)\,\frac{I}{V(q^2)^{\epsilon/2}}\,, \qquad (4.16)$$

$$V(q^2) = -q^2 z_1 z_2\,,$$

figs. 6d, d*, d**: $I = 4(1-\frac{1}{2}\epsilon)\,z_1 z_2 + 1 + \frac{1}{2}\epsilon + (1-a)(\frac{1}{2}-\epsilon) + \frac{1}{8}(1-a)^2\epsilon$.

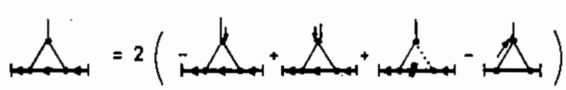
Diagram e

$$\Pi_e(q^2) = \tfrac{1}{4}\,\Gamma(\tfrac{1}{2}\epsilon)\int \mathrm{d}z_1 \mathrm{d}z_2 \delta(1-z_1-z_2)\,\frac{8 z_1 z_2}{V^{\epsilon/2}}\,, \qquad (4.17)$$

$$V = 1 - q^2 z_1 z_2\,.$$

From the computation point of view it is interesting to see how the diagrammatic notation of figs. 1–3, 't Hooft's identities (2.4), (2.7) and momentum conservation (2.6) simplify the above calculations. Two examples:

By (2.4), (2.7) and (2.6) diagram b can be reduced to

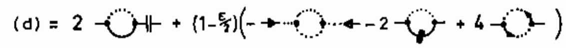

(4.18)

The numerator operator $\mathbb{F}^{\mu}$ (defined in the appendix) can now be read off directly from the above diagrammatic expression. The contribution to L_b from $p \cdot \Gamma(p, p)$ is

$$\mathbb{F} = 2[-0 + 0 + (\not{D}_1^2 - 1) - (p \cdot D_2)\,\gamma^{\mu}(\not{D}_1 + 1)\,\gamma^{\mu}]\,,$$

which then, by the rules of the appendix, yields F_0 and F_1 listed in (4.10).

Similarly, fig. 6d can be diagrammatically reduced to

 (4.19)

At this stage transversality is not manifest, except for the first term. The corresponding Feynman-parametric integral is

$$\pi_d^{\mu\nu}(q^2) = \tfrac{1}{4}\Gamma(\tfrac{1}{2}\epsilon) \int \mathrm{d}z_1 \mathrm{d}z_2 \delta(1 - z_1 - z_2) \frac{1}{V^{\epsilon/2}} \left\{ F_0^{\mu\nu} + \frac{V}{\frac{1}{2}\epsilon - 1} F_1^{\mu\nu} \right\} \tag{4.20}$$

(group theory already factored out by (4.4)), where F_i can again be simply read off (4.19) from the rules of the appendix

$$F_0^{\mu\nu} = (g^{\mu\nu}q^2 - q^\mu q^\nu)2 + (1 - \tfrac{1}{2}\epsilon)[q^\mu q^\nu - (A_1^2 + A_2^2)\, q^2 g^{\mu\nu} + 4A_1 A_2 q^\mu q^\nu] \,,$$

$$F_1^{\mu\nu} \cdot \qquad -\tfrac{1}{2}(1 - \tfrac{1}{2}\epsilon)[\quad -2(4-\epsilon) \quad g^{\mu\nu} + \quad 4g^{\mu\nu} \quad] \,,$$

and the *integrand* becomes explicitly transverse

$$F_0^{\mu\nu} + \frac{z_1 z_2 q^2}{1 - \frac{1}{2}\epsilon} F_1^{\mu\nu} = (q^\mu q^\nu - g^{\mu\nu}q^2)[-1 - \tfrac{1}{2}\epsilon - 4z_1 z_2(1 - \tfrac{1}{2}\epsilon)] \,.$$

This miracle has occurred prior to Feynman parameter integrations because Feynman parametrization already exploits the freedom of momentum integration shifts. If the seagull (diagram with a 4-gluon vertex in fig. 6d) were omitted, (4.20) would become transverse only *after* parametrix space integrations.

5. Regularization of infrared and ultraviolet divergences

In the derivation of QCD Ward identities I have assumed that there exists a regularization which

a) allows shifting of integration momenta;

b) preserves Lorentz invariance.

In this section I shall establish that any regularization which respects QCD Ward identities also

c) mixes infrared and ultraviolet singularities;

d) sets equal to zero any Feynman integral free of an intrinsic scale.

(An integral has no intrinsic scale if all internal propagators are massless, and for all combinations of external momenta flowing through the diagram, $p_i^2 = p_i \cdot p_j = 0$.)

This will follow from the verification of the Ward identities by the one-loop integrals of the preceding section. I shall discuss them at some length, but the essential observation is this: we know from QED that the integrals *a* and *c* of fig. 6 are IR divergent, and that apart from the group theoretic weights, their IR divergences are the same and have the same gauge dependence. However, the integral b (fig. 6) has a different IR divergence, e.g., in *Feynman gauge b is IR finite, but a and c are IR divergent.* We also know from explicit verifications of off-mass-shell Slavnov-Taylor identities that their *UV parts differ.* Hence for $Z_1 = Z_2$ to be true, UV and IR singularities must add up. We already knew that the dimensional regularization does precisely that [29,33–35], but now we learn that this UV-IR mixing is a general consequence of the gauge nature of the theory. Pursuing the same line of thought we conclude that $Z_4 = Z_3$ also implies UV-IR cancellations, because we know from the Slavnov-Taylor identities that the UV singularities alone do not satisfy $Z_4 = Z_3$. Explicit evaluation shows that this requires the gluonic contributions to the mass-shell Z_3 to vanish. On the surface this UV-IR cancellation looks quite different from those that took place in Z_1 and Z_2, but a closer look reveals that it occurs by the same mechanism as the UV-IR cancellations in the Z_1 (or Z_2) gauge terms; furthermore, it is the same mechanism that leads to the vanishing of the tadpole diagram in fig. 6d, on mass-shell or off mass-shell. Therefore UV-IR cancellations occur the same way for QCD quark integrals and purely gluonic integrals, on or off the mass-shell.

To demonstrate the above observations in more detail, I proceed to evaluate the integrals of the preceding section. Contrasted with the starting expressions, the results are strikingly simple:

$$
\begin{aligned}
&L_a = L_b = -B_c = \delta m_c = \tfrac{1}{4}\Gamma(\tfrac{1}{2}\epsilon)\frac{3-\epsilon}{1-\epsilon}, \\
&M_a = \tfrac{1}{2}\Gamma(1+\tfrac{1}{2}\epsilon), \\
&M_b = -\tfrac{1}{4}\Gamma(\tfrac{1}{2}\epsilon)\frac{-2-\epsilon}{1-\epsilon} \qquad \text{(IR singular only)}, \\
&C_d = 0, \\
&C_e = -\Gamma(\tfrac{1}{2}\epsilon)\,\tfrac{1}{3}.
\end{aligned}
\tag{5.1}
$$

The first surprise is that they are all gauge invariant. It is also amusing to note the simplicity of the Schwinger correction M_a. This is IR finite because the quark mass sets the scale for the energy of the virtual gluon; for M_b such mass scale is lacking, and arbitrarily soft virtual gluons contribute.

The above dimensional evaluation was straightforward, but a closer look reveals that most of the integrals have to be defined by a two-fold analytic continuation, both from above and below four dimensions [29,30,34] and this might cause a certain amount of worry – are integrals defined by such continuations really unique? Furthermore, IR singularities in quark renormalization constants look different from the $q^2 \to 0$ singularity in gluon renormalization constants – can they all be called IR singularities and regularized in the same way? To answer these questions, I shall re-evaluate (5.1) keeping this time track of IR singular terms, and leaving open the option of regularizing them by other methods.

In quark renormalization constants the IR singularities arise from the $z_1 \to 0$ end of integration. I label them by defining

$$-\tfrac{1}{4}\,\Gamma(1+\tfrac{1}{2}\,\epsilon)\int_0^1 \frac{\mathrm{d}z}{z^{1+\epsilon}} = \tfrac{1}{8}\,\Gamma(\tfrac{1}{2}\,\epsilon) + I\,. \tag{5.2}$$

In the dimensional regularization (5.2) is defined by analytic continuation from $\epsilon < 0$, and $I = 0$. In other regularizations $I \neq 0$; for example, if the gluon were massive ($V = \lambda^2 z_2 + z_1^2$ on the mass shell, $\lambda^2 \to 0$), in the $\epsilon \to 0$ limit the left-hand side of (5.2) would be replaced by

$$-\tfrac{1}{4}\int_0^1 \mathrm{d}z\,\frac{z}{z^2+\lambda^2} = \tfrac{1}{4}\ln\lambda\,. \tag{5.3}$$

Re-evaluation of (5.1) now separates our IR singular terms as factors of I with coefficients

$$\begin{aligned}
&L_a \to 4 + 2(1+\epsilon)(1-a)\,,\\
&L_b \to \quad (-3+\epsilon)(1-a)\,,\\
&B_c \to -4 - 2(1+\epsilon)(1-a)\,,\\
&\delta m_c \to 0\,,\\
&M_a \to 0\,,\\
&M_b \to 4 + \tfrac{1}{2}\,\epsilon(1-a)\,.
\end{aligned} \tag{5.4}$$

(Note that IR and UV singularities are separately gauge *dependent.*)

The Ward identity (3.8), rewritten in terms of the one-particle-irreducible renormalization constants, states

$$L + B = 0\,. \tag{5.5}$$

Using (4.6a), (5.1) and (5.4), explicit one-loop calculation yields

$$W_a L_a + W_b L_b + W_c B_c = W_b I[-4 - (5+\epsilon)(1-a)]\,. \tag{5.6}$$

Hence, the Ward identity is violated unless

(a) $W_b = 0$; QED allows arbitrary IR regularization.

(b) $I = 0$; dimensional regularization is a consistent regularization of QCD IR.

As argued above, the crucial feature is the mutual cancellation of UV and IR singularities.

In the gluon renormalization constants the IR singularities arise from $q^2 \to 0$ limit. The reason why this looks so different from the $z_1 \to 0$ singularity of the quark renormalization constants is that in the Feynman-parametric representation the quark mass provides a natural way of decomposing the integral into a sum of an UV part (F_1 term) and an IR part (F_0 term). For the purely gluonic integral (4.16) there is no such intrinsic UV-IR separation. To see that the $q^2 \to 0$ limit reallt gives rise to $1/\epsilon$ IR singularity it is instructive [18] to rewrite (4.16) in exponential form (this is related to the Schwinger-parametric form by a rescaling of z_i – see the appendix and (A.4))

$$\pi_d(q^2) = -\tfrac{1}{4}\, i^{\epsilon/2} \int_0^1 \mathrm{d}z_1 \mathrm{d}z_2 \delta(1 - z_1 - z_2)\, I \int_0^\infty \frac{\mathrm{d}z}{z^{1-\epsilon/2}} \exp[-iz(-q^2 z_1 z_2)] \ . \tag{5.7}$$

If $q^2 \neq 0$, this has an UV singularity from the $z \to 0$ region, but the potential IR singularities from the $z \to \infty$ end are damped by the exponential. The integral can be defined by the analytic continuation from $\epsilon > 0$. However, if $q^2 = 0$, an IR singularity appears from $z \to \infty$ end, and the integral is uniquely defined [36] by

$$\int_0^\infty \frac{\mathrm{d}z}{z^{1-\epsilon}} \equiv \int_0^1 \frac{\mathrm{d}z}{z^{1-\epsilon}} + \int_1^\infty \frac{\mathrm{d}z}{z^{1-\epsilon}} = \frac{1}{\epsilon} - \frac{1}{\epsilon} = 0 \ , \tag{5.8}$$

where the first (UV singular) piece is continued from $\epsilon > 0$ and the second (IR singular) is continued from $\epsilon < 0$. IR singularity is again a simple pole of $1/\epsilon$ type, and while separately UV and IR poles have complicated gauge dependence, their mutual cancellation is total. The cancellation occurred because with $V \equiv 0$ (all $m_i^2 = 0$, all $q_i \cdot q_j = 0$) any Feynman integral, no matter how complicated, vanishes due to the factorization of the overall scale integral (5.8). Hence it is not necessary to compute gluonic contributions to the 3-vertex renormalization Z_4 – we already know that those will vanish as well. To see whether the regularization (5.8) is dictated by the Ward identity $Z_4 = Z_3$, we can try other IR regularizations (such as cutting off the upper end of the integral (5.8)) by defining

$$-\tfrac{1}{4} \int_1^\infty \frac{\mathrm{d}z}{z^{1-\epsilon}} = \frac{1}{4\epsilon} + I' \ . \tag{5.9}$$

Again it is not necessary to compute Z_4 explicitly: we know from the Slavnov-Taylor identity for UV poles that $[Z_1/Z_2]_{\mathrm{UV}} = [Z_4/Z_3]_{\mathrm{UV}}$, hence the IR singularities have a different gauge dependence, just like in (5.4), and again $I' = 0$ by the Ward identities.

One might still be sceptical about the claim that IR singularities of (5.2) and (5.9) should be treated on equal footing; even if they are defined as singularities of

Schwinger-parametric integrals for $z_i \to \infty$, in (5.2) they come from the region $z_1 << z_2$, $z_2 \to \infty$ and in (5.9) from $z_1 + z_2 \to \infty$. Therefore it is instructive to note that a portion of quark IR singularities can be defined either by (5.2) or by (5.9). In (4.7) to (4.15) I have evaluated gauge terms F_i^* by parametric rules, and continued IR singularities by (5.2). Explicit evaluation shows that all gauge-dependent terms fully vanish by UV-IR cancellation. That is easy to understand by applying the Feynman and 't Hooft identities (2.4) and (2.7) prior to Feynman parametrization. For example, for diagram fig. 6a*

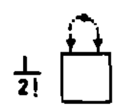
$(a^*) = -$  $= \text{const.} \int \frac{dk^{4-\varepsilon}}{(k^2)^2}$ (5.10)

This integral lacks internal scale and is regularized to zero by (5.8). Hence (5.2) and (5.9) are equivalent regularizations of the IR *singular* parts (the definitions of associated finite parts are inequivalent; for (5.2) the finite part arises from the decomposition of the parametric integral in $F_0 : F_1$, ... terms and for (5.9) from the *ad hoc* lower limit of integration).

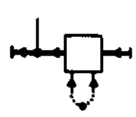
UV and IR singularities of diagrams of arbitrarily high order will be treated in the subsequent paper [4] along the lines of ref. [31].

6. Gauge dependence of renormalization constants

In the notation of sect. 2, the term linear in $(1 - a)$ for an arbitrary Green function is of the form (external legs suppressed) [22]

$\frac{1}{2!}$  (6.1)

In QED the linear term yields the full gauge dependenee of the renormalization constants. Consider the gauge contribution to Z_2 of (3.3b) and apply the Ward-Takahashi identities (QED version of fig. 4)

 $=$  $= -ie_0^2 \lambda(0) Z_2^0$  (6.2)

$$\lambda(0) = \frac{1-a}{(2\pi)^{4-\epsilon}} \int \frac{d^{4-\epsilon}k}{(k^2)^2} \,, \tag{6.3}$$

the gauge term has factored out explicitly. Carried out to all orders, this leads to the Johnson-Zumino [38] result

$$Z_2^{1-a} = \exp[-ie_0^2\lambda(0)]\, Z_2^0 \,. \tag{6.4}$$

As before, $\lambda(0) = 0$ in dimensional regularization, and in covariant gauges the QED renormalization constants are gauge independent. The above exponentiation is of some relevance to the study of IR behaviour, because if the IR part of $\lambda(0)$ is separated out by (5.9), (6.4) tells us that the gauge dependence of the IR singularities of QED renormalization constants is given by the exponentiation of a *one-loop* IR singularity. In a similar way an understanding of gauge dependence of QCD renormalization constants sheds light on the form of QCD IR singularities.

For pure Yang-Mills theory gauge independence of all mass-shell renormalization constants is trivial – by the results of the preceding section, all $Z_i = 1$. Parenthetically, this *does not* mean that mass-shell QCD is not renormalized; rather it means that mass-shell renormalization has transmuted all UV singularities into IR singularities of the precisely same form. I shall discuss the renormalization in the subsequent paper [4].

For QCD the situation is not so trivial, due to the presence of massive quark loops. For QCD I have verified the gauge independence of renormalization constants only at one- and two-loop levels. The gauge terms vanish by the general properties of QCD regularizations listed at the beginning of sect. 5.

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Appendix

Feynman-parametric integrals

Here I shall sketch the Feynman-parametric methods for isolating UV and IR divergences (discussed in detail in ref. [31]) and state the rules for constructing QCD momentum integrals directly in Feynman parametric form. In the context of the present article this is merely a convenience which streamlines the one-loop calculations of sect. 4, each of which could be easily done by the usual two-step procedure of first writing a dimensionally regularized momentum integral, and then introducing Feynman parameters. However, parametric methods will be essential for the analysis of arbitrary order Feynman integrals undertaken in the subsequent paper.

Schwinger parametrization. Schwinger parametrization starts with a replacement of the denominator of a Feynman propagator by a parametric integral

$$\frac{-i}{p^2 - m^2} = \int_0^\infty \mathrm{d}z \; \mathrm{e}^{-iz(p^2 - m^2)} \,. \tag{A.1}$$

If $p^2 \to \infty$, the dominant contribution is from $z \to 0$. From now on $z \to 0$ *will be referred to as the ultraviolet limit* in the parameter space. If $p^2 \to m^2$, the integral is not damped for $z \to \infty$: *this is the infrared limit* in the parameter space. Parametrizing in this way all propagators, shifting, diagonalizing and integrating over all internal momenta one arrives at the Schwinger-parametric representation of the Feynman integral associated with diagram G

$$\int \frac{\mathrm{d}z_\mathrm{G}}{U^{2-\epsilon/2}} \left(F_0 + \frac{F_1}{iU} + \frac{F_2}{(iU)^2} + \ldots \right) \mathrm{e}^{-iV} \,. \tag{A.2}$$

U is the Jacobian of the transformation from the momentum integral to the parametric integral. It is a topological function of the loop structure of G, and can be read off diagram G by elegant group-theoretic formulas. (For one loop $U = z_\mathrm{G}$, for two loops $U = z_\alpha z_\beta + z_\alpha z_\gamma + z_\beta z_\gamma$, etc). It vanishes in the UV region of the parametric space, making the integrand of (A.2) singular. *Vanishing of the parametric function U gives rise to ultraviolet singularities.*

All dependence on external momenta and internal masses is contained in the functions F_i and V, which are themselves expressible in terms of pure parametric functions A_i and B_{ij}.

The scalar current A_i (a set of those for each independent external momentum p) is a fraction ($|A_i| \leqslant 1$) of the external momentum p flowing through the line i. The parameter z_i plays the role of "resistance" of the line i, in the sense that as $z_i \to \infty$ (faster than all other z_j), $A_i \to 0$ (no current flows through a line of infinite resistance). If i is a gluon (photon) line, $A_i \to 0$ *describes the infrared regime* (the gluon has vanishing momentum). This affects the function V, which, if it vanishes for some $z_i \to \infty$, fails to damp the high end of the integral (A.2). For example, for the mass-shell quark vertex with a single quark line going through and a cloud of gluons

$$V = \sum_{i=\text{quark}} z_i (m^2 - A_i p^2) \,, \qquad p^2 \doteq m^2 = 1 \tag{A.3}$$

and when all gluons $z_i \to \infty$, all the momentum flows through the quark line ($A_i \to 1$ for quarks) and (A.3) vanishes. *Vanishing of the parametric function V as some $z_i \to \infty$ gives rise to infrared singularities* (the possibility of V vanishing on some hypersurface will not concern us).

B_{ij} arises from integration over $r_i^\mu r_j^\nu$, where r_i is the sum of internal integration momenta flowing through line i. It describes the change in scalar current A_i through line i due to the change of resistance z_j of line j, and in ref. [39] it was shown that *B_{ij} can be used as the basic block for constructing all other parametric functions.*

F_0, F_1, F_2 ... arise from the numerator terms with the 0, 1, 2, ... factors of the form $r_i^\mu r_j^\nu$, and the corresponding terms in the integrand of (A.2) have successively more severe UV singularities. F_i by themselves are non-singular functions of B_{ij}, A_i, p and m_i, and can only moderate UV or IR singularities, not enhance them.

To summarize, in Schwinger parametrization UV singularities arise from *vanishing of U in the $z_i \to 0$ ultraviolet region,* and IR singularities *arise from vanishing of V in the $z_i \to \infty$ infrared region.* (Full analysis of overall and sub-divergences is carried out in ref. [31]).

Feynman parametrization. Evaluation of (A.2) starts by the observation that the parameters z_i have no intrinsic scale, so that an overall scale z

$$z_i \to z z_i\,, \qquad \sum z_i = 1$$

can be integrated out by reversing (A.1)

$$\int_0^\infty \mathrm{d}z\, z^{r-1} \mathrm{e}^{-izV} = \frac{(-i)^r \Gamma(r)}{V^r}\,. \tag{A.4}$$

The integral brought to this form is called *Feynman-parametric.* By analogy with (A.1) $1/V^r$ is a generalized propagator for the entire graph with effective mass $\Sigma z_i m_i^2$ and effective squared momentum

$$G = -\frac{1}{U} \sum z_i z_j B'_{ij}(q_i \cdot q_j)\,,$$

where $V = \Sigma z_i m_i^2 - G$. In Feynman-parametric space the integration region is compact and *infrared divergences arise* from integrand singularities ($1/V \to \infty$) *on the edges of the integration region* ($\Sigma z_i \to 1$ for some set of lines). Detailed analysis [31] shows that V will not vanish sufficiently rapidly unless the effective mass $\Sigma z_i m_i^2$ vanishes as well.

QCD Feynman rules

The rules for writing down a QCD-invariant amplitude are the same as those given for QED in sect. 2 of ref. [39], with rules 4 to 6 replaced by the following rules 4 to 7.

Rule 4. Construct the parametric integral

$$\left(\frac{i}{(4\pi)^{2-\epsilon/2}}\right)^n (-1)^N \Gamma(N - 2n + \tfrac{1}{2}\, n\epsilon)\, \frac{\int \mathrm{d}z_G \delta(1 - z_G)}{U^{2-\epsilon/2}\, V^{N-2n+n\epsilon/2}} \tag{A.5a}$$

(Feynman-Chisholm representation)

or

$$\left(\frac{i}{(4\pi i)^{2-\epsilon/2}}\right)^n (-i)^N \int \frac{\mathrm{d}z_G}{U^{2-\epsilon/2}}\, \mathrm{e}^{-iV} \tag{A.5b}$$

(Schwinger-Nambu representation).

Here N is the number of internal lines, n is the number of independent loops, $4-\epsilon$ is the number of space-time dimensions and the integration domain is given by

$$\mathrm{d}z_G = \prod_{j=1}^{N} \int_0^\infty \mathrm{d}z_j\,, \qquad z_G = \sum_{j=1}^{N} z_j\,. \tag{A.6}$$

Rule 5. Multiply (4.1) by factors associated with the remaining elements of the diagram G:

(a) for each mass-shell quark line a factor $\sqrt{Z_2}$;

(b) for each mass-shell gluon line a factor $\sqrt{Z_3}$

(omit factors $\sqrt{Z_2}$ and $\sqrt{Z_3}$ if constructing an unrenormalized amplitude);

(c) for each internal quark j line a factor $i\,(\not{D}_j + m)$ times the numerator factor of fig. 2;

(d) for each internal gluon or gluon auxillary line a factor $-i$ times the numerator factor of fig. 2;

(e) for each vertex a factor from fig. 1 or fig. 2;

(f) for each closed quark loop a factor -1;

(g) for each closed ghost loop a factor -1;

(h) the combinatoric factor associated with the diagram G. This arises from the iteration of (2.2). Alternatively, it can be computed by the rules of Diagrammar [40];

(i) renormalize charge by $g = (Z_2/Z_1)\sqrt{Z_2}\,g_0$ (skip this step if constructing an unrenormalized amplitude).

Rule 6. Let us denote by $I\!F$ the product of γ^μ, D_i^μ, $g^{\mu\nu}$ and $(\not{D}_j + m)$ from vertices and propagators, appropriate external quark spinor factors and gluon polarizations, and combinatorial and loop factors. Then the action of $I\!F$ on the integral (A.5) is defined by

$$I\!F \frac{\Gamma(m)}{V^m} = F_0 \frac{\Gamma(m)}{V^m} + F_1 \frac{\Gamma(m-1)}{V^{m-1}} + F_2 \frac{\Gamma(m-2)}{V^{m-2}} + \ldots \tag{A.7a}$$

(Feynman-Chisholm representation),

$$I\!F\, \mathrm{e}^{-iV} = \left[F_0 + \frac{1}{iU} F_1 + \left(\frac{1}{iU}\right)^2 F_2 + \ldots\right] \mathrm{e}^{-iV} \tag{A.7b}$$

(Schwinger-Nambu representation),

where the subscript k of F_k stands for the number of contractions. By contraction we mean picking out a pair of D_i^μ, D_j^ν from $I\!F$, replacing them by $g^{\mu\nu}$, putting a factor $-\frac{1}{2} B_{ij}$ in front and summing the result of this operation over all distinct pairs. Non-contracted D_i^μ are then replaced by $Q_i'^\mu$. Summations over Lorentz indices and spinor traces have to be evaluated by the rules of dimensional regularization [28,30] such as

$$g_\mu^\mu = 4 - \epsilon\,,$$

$$\gamma^\mu \gamma_\mu = 4 - \epsilon ,$$

$$\gamma^\mu \not a \gamma_\mu = (\epsilon - 2) \not a ,$$

$$\gamma^\mu \not a \not b \gamma_\mu = 4(a \cdot b) - \epsilon \not b \not a ,$$

$$\gamma^\mu \not a \not b \not c \gamma_\mu = -2 \not c \not b \not a - \epsilon \not a \not b \not c . \tag{A.8}$$

Rule 7. Let us denote by W_G the group-theoretical weight of the diagram G, i.e. the product of all $-iC_{ijk}$ and $(T_i)^a_b$ factors from vertices and δ_{ij} and δ^a_b factors from the internal lines. The only rules needed for general, symmetry-group-independent considerations are the Lie algebra relations between coupling strengths. In the diagrammatic notion of ref. [24] they are

(A.9a)

(A.9b)

A method for calculation of W_G for specific Lie algebras is given in ref. [24]. The basic idea is that a *full description* of QCD couplings algebra is given by enumeration of the types of couplings between quarks. If nothing beyond colour conservation is assumed, the algebra is SU(n). For each additional type of coupling (quark-antiquark, three quarks, etc.), an invariance relation reduces the number of independent gluon colours, restricting the algebra to some sub-algebra of SU(n). The weight W_G is a combinatoric number which counts the number of distinct colourings of the diagram with the allowed quark and gluon colours (each colouring has the same momentum space integral), and it can be computed from the invariance relations without recourse to any explicit matrix representations for C_{ijk} and $(T_i)^a_b$.

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