Chapter 4. Local stability

Solution 4.1: Trace-log of a matrix. *1) one method is to first check that this is true for any Hermitian matrix M. Then write an arbitrary complex matrix as sum $M = A + zB$, A, B Hermitian, Taylor expand in z and prove by analytic continuation that the identity applies to arbitrary M. (David Mermin)*

2) another method: evaluate $\frac{d}{dt}\det\left(e^{t\ln M}\right)$ by definition of derivative in terms of infinitesimals. [1] *(Kasper Juel Eriksen)*

3) check appendix M.1

4) This identity makes sense for a matrix $M \subset \mathbb{C}^{n\times n}$, if $|\prod_{i=1}^{n}\lambda_i| < \infty$ and $\{|\lambda_i| > 0, \forall i\}$, where $\{\lambda_i\}$ is a set of eigenvalues of M. Under these conditions there exist a nonsingular $O : M = ODO^{-1}$, $D = \mathrm{diag}[\{\lambda_i, i = 1, \ldots, n\}]$. If $f(M)$ is a matrix valued function defined in terms of power series then $f(M) = Of(D)O^{-1}$, and $f(D) = \mathrm{diag}[\{f(\lambda_i)\}]$. Using these properties and cyclic property of the trace we obtain

$$\exp(\mathrm{tr}\,(\ln M)) = \exp\left(\sum_i \ln\lambda_i\right) = \prod_i \lambda_i = \det(M)$$

5) Consider $M = \exp A$.

$$\det M = \det \lim_{n\to\infty}\left(\mathbf{1} + \frac{1}{n}A\right)^n = \lim_{n\to\infty}(\mathbf{1} + \frac{1}{n}\mathrm{tr}\,A + \ldots)^n = \exp(\mathrm{tr}\,(\ln M))$$

Solution 4.2: Stability, diagonal case. *The relation (4.17) can be verified by noting that the defining product (4.13) can be rewritten as*

$$\begin{aligned} e^{t\mathbf{A}} &= \left(\mathbf{U}\mathbf{U}^{-1} + \frac{t\mathbf{U}\mathbf{A}_D\mathbf{U}^{-1}}{m}\right)\left(\mathbf{U}\mathbf{U}^{-1} + \frac{t\mathbf{U}\mathbf{A}_D\mathbf{U}^{-1}}{m}\right)\cdots \\ &= \mathbf{U}\left(I + \frac{t\mathbf{A}_D}{m}\right)\mathbf{U}^{-1}\mathbf{U}\left(I + \frac{t\mathbf{A}_D}{m}\right)\mathbf{U}^{-1}\cdots = \mathbf{U}e^{t\mathbf{A}_D}\mathbf{U}^{-1}. \end{aligned} \tag{S.8}$$

Solution 4.3: State space volume contraction in Rössler flow. *Even if it were worth your while, the contraction rate cannot be linked to a computable fractal dimension. The relation goes through expanding eigenvalues, sect. 5.4. As the contraction is of order of 10^{-15}, there is no numerical algorithm that would give you any fractal dimension other than $D_H = 1$ for this attractor.*

Solution 4.4: Topology of the Rössler flow.

1. *The characteristic determinant of the stability matrix that yields the equilibrium point stability (4.28) yields*

$$\begin{vmatrix} -\lambda & -1 & -1 \\ 1 & a-\lambda & 0 \\ z^{\pm} & 0 & x^{\pm} - c - \lambda \end{vmatrix} = 0$$

[1] Predrag: to be fleshed out

$$\lambda^3 + \lambda^2(-a - x^\pm + c) + \lambda(a(x^\pm - c) + 1 + x^\pm/a) + c - 2x^\pm = 0\,.$$

Equation (4.47) follows after noting that $x^\pm - c = c(p^\pm - 1) = -cp^\mp$ *and* $2x^\pm - c = c(2p^\pm - 1) = \pm c\sqrt{D}$, *see (2.9).*

2. *Approximate solutions of (4.47) are obtained by expanding* $p^\pm$ *and* $\sqrt{D}$ *and substituting into this equation. Namely,*

$$\begin{array}{ll} \sqrt{D} & = 1 - 2\epsilon^2 - 2\epsilon^4 - 4\epsilon^6 - \ldots \\ p^- & = \epsilon^2 + \epsilon^4 + 2\epsilon^6 + \ldots \\ p^+ & = 1 - \epsilon^2 - \epsilon^4 - 2\epsilon^6 + \ldots \end{array}$$

In case of the equilibrium "$-$", close to the origin expansion of (4.47) results in

$$(\lambda^2 + 1)(\lambda + c) = -\epsilon\lambda(1 - c^2 - c\lambda) + \epsilon^2 c(\lambda^2 + 2) + o(\epsilon^2)$$

The term on the left-hand side suggests the expansion for eigenvalues as

$$\lambda_1 = -c + \epsilon a_1 + \ldots\,, \quad \lambda_2 + i\theta_2 = \epsilon b_1 + i + \ldots \quad .$$

after some algebra one finds the first order correction coefficients $a_1 = c/(c^2+1)$ *and* $b_1 = (c^3 + i)/(2(c^2 + 1))$. *Numerical values are* $\lambda_1 \approx -5.694$, $\lambda_2 + i\theta_2 \approx 0.0970 + i1.0005$.

In case of p^+, *the leading order term in (4.47) is* $1/\epsilon$. *Set* $x = \lambda/\epsilon$, *then expansion of (4.47) results in*

$$x = c - \epsilon x - \epsilon^2(2c - x) - \epsilon^3(x^3 - cx^2) - \epsilon^4(2c - x(1 + c^2) + cx^2) + o(\epsilon^4)$$

Solve for real eigenvalue first. Set $x = c + \epsilon a_1 + \epsilon^2 a_2 + \epsilon^3 a_3 + \epsilon^4 a_4 + \ldots$. *The subtle point here is that leading order correction term of the real eigenvalue is* ϵa_1, *but to determine leading order of the real part of complex eigenvalue, one needs all terms* a_1 *through* a_4.

Collecting powers of ϵ *results in*

$$\begin{array}{lll} \epsilon: & a_1 + c = 0 & a_1 = \; -c \\ \epsilon^2: & c + a_1 + a_2 = 0 & a_2 = \; 0 \\ \epsilon^3: & a_1 - a_2 - a_3 = 0 & a_3 = \; -c \\ \epsilon^4: & c + c^2 a_1 - a_2 + a_3 + a_4 = 0 & a_4 = \; c^3\,. \end{array}$$

hence

$$\mu_1 = \epsilon x = a - a^2/c + o(\epsilon^3) \approx 0.192982\,.$$

To calculate the complex eigenvalue, one can make use of identities $\det A = \prod \lambda = 2x^+ - c$, *and* $\operatorname{tr} A = \sum \lambda = a + x^+ - c$. *Namely,*

$$\lambda_2 = \frac{1}{2}\left(a - cp^- - \lambda_1\right) = -\frac{a^5}{2c^2} + o(\epsilon^5) \approx -0.49 \times 10^{-6}\,,$$

$$\theta_2 = \sqrt{\frac{2x^+ - c}{\lambda_1} - \lambda_2^2} = \sqrt{\frac{a+c}{a}}\left(1 + o(\epsilon)\right) \approx 5.431\,.$$

(Rytis Paškauskas)