Chapter 2. Go with the flow

evolution!group
group!evolution
ordinary differential equations!almost
differential equations!ordinary!almost
gradient!system
dynamical!system!gradient

Solution 2.1: Trajectories do not intersect. *Suppose that two trajectories C_x and C_y intersect at some point z. We claim that any points $\tilde{x}$ on C_x is also a point on C_y and vice versa. We only need to prove the first part of the statement. According to the definition of C_x, there exist $t_x\,, t_y\,, t_1 \in \mathbb{R}$ such that $f^{t_x}(x) = z\,, f^{t_y}(y) = z\,, f^{t_1}(x) = \tilde{x}$. It is easy to check that $f^{t_y - t_x + t_1}(y) = \tilde{x}$. So, $\tilde{x} \in C_y$. Therefore, if two trajectories intersect, then they are the same trajectory.*

(Yueheng Lan)

Solution 2.2: Evolution as a group. *Let's check the basic defining properties of a group. The members of the set are $f^t\,, t \in \mathbb{R}$ and the "product law" is given by '$\circ$'.*

- *As $f^{t+s} = f^t \circ f^s$, the set is closed,* i.e., *the product of any two members generates another member of the set.*
- *It is associative, as $(f^t \circ f^s) \circ f^r = f^{t+s+r} = f^t \circ (f^s \circ f^r)$.*
- *$I = f^0$ is the identity, as $f^t \circ f^0 = f^t$.*
- *f^{-t} is the inverse of f^t, as $f^{-t} \circ f^t = I$.*

So, $\{f^t, \circ\}_{t\in\mathbb{R}}$ forms a group. As $f^t \circ f^s = f^{t+s} = f^s \circ f^t$, it is a commutative (Abelian) group.

Any Abelian group can replace the continuous time. For example, $\mathbb{R}$ can be replaced by $\mathbb{Z}_6$. To mess things up try a non-commutative group.

(Yueheng Lan)

Solution 2.3: Almost ODE's. *What is an ODE on $\mathbb{R}$? An ODE is an equality which reveals explicitly the relation between function $x(t)$ and its time derivatives $\dot{x}, \ddot{x}, \cdots$,* i.e., *$F(t, x, \dot{x}, \ddot{x}, \cdots) = 0$ for some given function F. Let's check the equations given in the exercise.*
(a) $\dot{x} = \exp(\dot{x})$ is an ODE.
(b) $\dot{x} = x(x(t))$ is not an ODE, as $x(x(t))$ is not a known function acting on $x(t)$.
(c) $\dot{x} = x(t+1)$ is not an ODE, as $x(t+1)$ is not a value at current time. Actually, it is a difference-differential equation.

(Yueheng Lan)

Solution 2.4: All equilibrium points are fixed points. *Given a vector field $v(x)$, the state space dynamics is defined by*

$$\frac{d}{dt}x(t) = v(x(t))\,. \tag{S.3}$$

An equilibrium point a of v is defined by $v(a) = 0$, so $x(t) = a$ is a constant solution of (S.3). For the flow f^t defined by (S.3), this solution satisfies $f^t(a) = a\,, t \in \mathbb{R}\,.$ So, it is a fixed point of the dynamics f^t.

(Yueheng Lan)

Solution 2.5: Gradient systems.

1. *The directional derivative*

$$\frac{d}{dn}\phi = n \cdot \nabla\phi$$

produces the increasing rate along the unit vector n. So, along the gradient direction $\nabla\phi/|\nabla\phi|$, ϕ has the largest increasing rate. The velocity of the particle has the opposite direction to the gradient, so ϕ deceases most rapidly in the velocity direction.

2. *An extremum a of ϕ satisfies $\nabla\phi(a) = 0$. According to exercise 2.4, a is a fixed point of the flow.*

3. *Two arguments lead to the same conclusion here.*

First, near an equilibrium point, the equation is always linearizable. For gradient system, after orthogonal transformation it is even possible to write the linearized equation in diagonal form so that we need only to consider one eigendirection. The corresponding scalar equation is $\dot{x} = \lambda x$. Notice that we moved the origin to the equilibrium point. The solution of this equation is $x(t) = x(0)\exp(\lambda t)$, for $\lambda \neq 0$. if $x(0) \neq 0$, it will take infinite amount of time (positive or negative) for $x(t) \to 0$. For $\lambda = 0$, the approach to zero is even slower as then only higher orders of x take effect.

The second argument seems easier. We know that the solution curve through an equilibrium point is the point itself. According to exercise 2.1, no other solution curve will intersect it, which means that if not starting from the equilibrium point itself, other point can never reach it.

4. *On a periodic orbit, the velocity is bounded away from zero. So ϕ is always decreasing on a periodic orbit, but in view of the periodicity, we know that this can not happen (at each point, there is only one value of ϕ.). So, there is no periodic orbit in a gradient system.*

(Yueheng Lan)

Solution 2.8: Rössler system. *You will probably want the matlab function* `ode45` *to do this. There are several others which perform better in different situations (for example* `ode23` *for stiff ODEs), but* `ode45` *seems to be the best for general use.*

To use `ode45` *you must create a function, say '*`rossler`*', which will take in a time and a vector of* `[x,y,z]` *and return* `[xdot, ydot, zdot]`*. Then the command would be something like*

```
ode45([tmin, tmax], [x0 y0 z0], @rossler)
```

(Jonathan Halcrow)

Solution 2.9: Equilibria of the Rössler system.

1. *Solve $\dot{x} = \dot{y} = \dot{z} = 0$, to get $x = az$, $y = -z$ and $x^2 - cx + ab = 0$. There are two solutions of a quadratic equation, hence there are two equilibrium points:*

$$x^{\pm} = az^{\pm} = -ay^{\pm} = (c \pm \sqrt{c^2 - 4ab})/2 \,. \qquad \text{(S.4)}$$

2. *That above expressions are exact. However, it pays to think of $\epsilon = a/c$ as a small parameter in the problem. By subsitution from (2.19),*

$$x^{\pm} = cp^{\pm}, \quad y^{\pm} = -p^{\pm}/\epsilon, \quad z^{\pm} = p^{\pm}/\epsilon. \qquad \text{(S.5)}$$

Expanding $\sqrt{D}$ in ϵ yields $p^- = \epsilon^2 + o(\epsilon^3)$, and $p^+ = 1 - \epsilon^2 + o(\epsilon^3)$. Hence *helium!collinear*

$$
\begin{array}{ll}
x^- = a^2/c + o(\epsilon^3), & x^+ = c - a^2/c + o(\epsilon^3), \\
y^- = -a/c + o(\epsilon^2), & z^+ = c/a + a/c + o(\epsilon^2), \\
z^- = a/c + o(\epsilon^2), & z^+ = c/a - a/c + o(\epsilon^2).
\end{array}
\tag{S.6}
$$

For $a = b = 0.2$, $c = 5.7$ in (2.15), $\epsilon \approx 0.035$, so

$$
\begin{array}{l}
(x^-, y^-, z^-) = (\,0.0070,\, -0.0351,\, 0.0351\,)\,, \\
(x^+, y^+, z^+) = (\,5.6929,\, -28.464,\, 28.464\,)\,.
\end{array}
\tag{S.7}
$$

(Rytis Paškauskas)

Solution 2.11: Classical collinear helium dynamics. *An example of a solution are A. Prügel-Bennett's programs, available at* `ChaosBook.org/extras`.