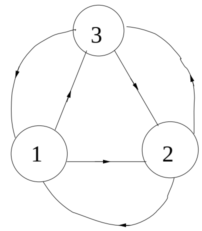
Chapter 13. Counting

Solution 13.1: A transition matrix for 3-disk pinball. *a) As the disk is convex, the transition to itself is forbidden. Therefore, the Markov diagram is*



,

with the corresponding transition matrix

$$\mathbb{T} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Note that $\mathbb{T}^2 = \mathbb{T} + 2$. *Suppose that* $\mathbb{T}^n = a_n \mathbb{T} + b_n$, *then*

$$\mathbb{T}^{n+1} = a_n \mathbb{T}^2 + b_n \mathbb{T} = (a_n + b_n)\mathbb{T} + 2a_n .$$

So $a_{n+1} = a_n + b_n\,,\ b_{n+1} = 2a_n$ *with* $a_1 = 1\,,\ b_1 = 0$.

b) From a) we have $a_{n+1} = a_n + 2a_{n-1}$. *Suppose that* $a_n \propto \lambda^n$. *Then* $\lambda^2 = \lambda + 2$. *Solving this equation and using the initial condition for* $n = 1$, *we obtain the general formula*

$$\begin{aligned} a_n &= \frac{1}{3}(2^n - (-1)^n)\,, \\ b_n &= \frac{2}{3}(2^{n-1} + (-1)^n)\,. \end{aligned}$$

c) $\mathbb{T}$ *has eigenvalue* 2 *and* -1 *(degeneracy* 2*). So the topological entropy is* $\ln 2$, *the same as in the case of the binary symbolic dynamics.* *(Yueheng Lan)*

Solution 13.2: Sum of A_{ij} is like a trace. *Suppose that* $A\phi_k = \lambda_k \phi_k$, *where* $\lambda_k\,, \phi_k$ *are eigenvalues and eigenvectors, respectively. Expressing the vector* $v = (1, 1, \cdots, 1)^t$ *in terms of the eigenvectors* ϕ_k, *i.e.,* $v = \Sigma_k d_k \phi_k$, *we have*

$$\begin{aligned} \Gamma_n &= \Sigma_{ij}[A^n]_{ij} = v^t A^n v \;=\; \Sigma_k v^t A^n d_k \phi_k \;=\; \Sigma_k d_k \lambda_k^n (v^t \phi_k) \\ &= \Sigma_k c_k \lambda_k^n\,, \end{aligned}$$

where $c_k = (v^t \phi_k) d_k$ *are constants.*

a) As $\mathrm{tr}\, A^n = \Sigma_k \lambda_k^n$, *it is easy to see that both* $\mathrm{tr}\, A^n$ *and* Γ_n *are dominated by the largest eigenvalue* λ_0. *That is*

$$\frac{\ln |\mathrm{tr}\, A^n|}{\ln |\Gamma_n|} = \frac{n \ln |\lambda_0| + \ln |\Sigma_k (\frac{\lambda_k}{\lambda_0})^n|}{n \ln |\lambda_0| + \ln |\Sigma_k d_k (\frac{\lambda_k}{\lambda_0})^n|} \to 1 \;\; \textit{as } n \to \infty\,.$$

b) The nonleading eigenvalues do not need to be distinct, as the ratio in a) is controlled by the largest eigenvalues only.

(Yueheng Lan)

Solution 13.4: Transition matrix and cycle counting. *a) According to the definition of $\mathbb{T}_{ij}$, the transition matrix is*

$$\mathbb{T} = \begin{pmatrix} a & c \\ b & 0 \end{pmatrix} .$$

b) All walks of length three $0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010$ *(four symbols!) with weights* $aaa, aac, acb, cba, cbc, baa, bac, bcb$ *. Let's calculate* $\mathbb{T}^3$*,*

$$\mathbb{T}^3 = \begin{pmatrix} a^3 + 2abc & a^2c + bc^2 \\ a^2b + b^2c & abc \end{pmatrix} .$$

There are altogether 8 *terms, corresponding exactly to the terms in all the walks.*

c) Let's look at the following equality

$$\mathbb{T}^n_{ij} = \Sigma_{k_1,k_2,\cdots,k_{n-1}} \mathbb{T}_{ik_1} \mathbb{T}_{k_1k_2} \cdots \mathbb{T}_{k_{n-1}j} .$$

Every term in the sum is a possible path from i *to* j*, though the weight could be zero. The summation is over all possible intermediate points* ($n-1$ *of them). So,* $\mathbb{T}^n_{ij}$ *gives the total weight (probability or number) of all the walks from* i *to* j *in n steps.*

d) We take $a = b = c = 1$ *to just count the number of possible walks in* n *steps. This is the crudest description of the dynamics. Taking* a, b, c *as transition probabilities would give a more detailed description. The eigenvlues of* $\mathbb{T}$ *is* $(1 \pm \sqrt{5})/2$*, so we get* $N(n) \propto (\frac{1+\sqrt{5}}{2})^n$*.*

e) The topological entropy is then $\ln \frac{1+\sqrt{5}}{2}$*.* *(Yueheng Lan)*

Solution 13.6: "Golden mean" pruned map. *It is easy to write the transition matrix* $\mathbb{T}$

$$\mathbb{T} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} .$$

The eigenvalues are $(1 \pm \sqrt{5})/2$*. The number of periodic orbits of length* n *is the trace*

$$\mathbb{T}^n = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n} .$$

(Yueheng Lan)

Solution 13.5: 3-disk prime cycle counting. *The formula for arbitrary length cycles is derived in sect. 13.4.*

Solution 13.44: Alphabet {0,1}, prune $\_1000\_$, $\_00100\_$, $\_01100\_$.

step 1. *$\_1000\_$ prunes all cycles with a $\_000\_$ subsequence with the exception of the fixed point $\overline{0}$; hence we factor out $(1-t_0)$ explicitly, and prune $\_000\_$ from the rest. Physically this means that x_0 is an isolated fixed point - no cycle stays in its vicinity for more than 2 iterations. In the notation of exercise 13.18, the alphabet is {1, 2, 3; $\overline{0}$}, and the remaining pruning rules have to be rewritten in terms of symbols 2=10, 3=100:*

step 2. *alphabet {1, 2, 3; $\overline{0}$}, prune $\_33\_$, $\_213\_$,$\_313\_$. Physically, the 3-cycle $\overline{3}=\overline{100}$ is pruned and no long cycles stay close enough to it for a single $\_100\_$ repeat. As in exercise 13.7, prohibition of $\_33\_$ is implemented by dropping the symbol "3" and extending the alphabet by the allowed blocks 13, 23:*

step 3. *alphabet {1, 2, $\underline{13}$, $\underline{23}$; $\overline{0}$}, prune $\_2\underline{13}\_$, $\_\underline{23}\,\underline{13}\_$, $\_\underline{13}\,\underline{13}\_$, where $\underline{13}=13$, $\underline{23}=23$ are now used as single letters. Pruning of the repetitions $\_\underline{13}\,\underline{13}\_$ (the 4-cycle $\overline{13}=\overline{1100}$ is pruned) yields the*

Result: *alphabet {1, 2, $\underline{23}$, $\underline{113}$; $\overline{0}$}, unrestricted 4-ary dynamics. The other remaining possible blocks $\_2\underline{13}\_$, $\_\underline{2313}\_$ are forbidden by the rules of step 3. The topological zeta function is given by*

$$1/\zeta = (1-t_0)(1-t_1-t_2-t_{23}-t_{113}) \tag{S.44}$$

for unrestricted 4-letter alphabet {1, 2, $\underline{23}$, $\underline{113}$}.

Solution 13.8: Spectrum of the "golden mean" pruned map.

1. *The idea is that with the redefinition $2=10$, the alphabet {1,2} is unrestricted binary, and due to the piecewise linearity of the map, the stability weights factor in a way similar to (16.10).*
2. *As in (17.10), the spectral determinant for the Perron-Frobenius operator takes form (17.12)*

$$\det(1-z\mathcal{L}) = \prod_{k=0}^{\infty}\frac{1}{\zeta_k}, \qquad \frac{1}{\zeta_k} = \prod_p\left(1-\frac{z^{n_p}}{|\Lambda_p|\Lambda_p^k}\right).$$

The mapping is piecewise linear, so the form of the topological zeta function worked out in (13.16) already suggests the form of the answer. The alphabet {1,2} is unrestricted binary, so the dynamical zeta functions receive contributions only from the two fixed points, with all other cycle contributions cancelled exactly. The $1/\zeta_0$ is the spectral determinant for the transfer operator like the one in (15.19) with the $T_{00}=0$, and in general

$$\begin{aligned}\frac{1}{\zeta_k} &= \left(1-\frac{z}{|\Lambda_1|\Lambda_1^k}\right)\left(1-\frac{z^2}{|\Lambda_2|\Lambda_2^k}\right)\left(1-\frac{z^3}{|\Lambda_{12}|\Lambda_{12}^k}\right)\cdots \\ &= 1-(-1)^k\left(\frac{z}{\Lambda^{k+1}}+\frac{z^2}{\Lambda^{2k+2}}\right).\end{aligned} \tag{S.45}$$

The factor $(-1)^k$ arises because both stabilities Λ_1 and Λ_2 include a factor $-\Lambda$ from the right branch of the map.

Solution 13.11: Whence Möbius function? *Written out $f(n)$ line-by-line for a few values of n, (13.38) yields*

$$\begin{aligned}
f(1) &= g(1) \\
f(2) &= g(2) + g(1) \\
f(3) &= g(3) + g(1) \\
f(4) &= g(4) + g(2) + g(1) \\
&\cdots \\
f(6) &= g(6) + g(3) + g(2) + g(1) \\
&\cdots
\end{aligned} \tag{S.46}$$

Now invert recursively this infinite tower of equations to obtain

$$\begin{aligned}
g(1) &= f(1) \\
g(2) &= f(2) - f(1) \\
g(3) &= f(3) - f(1) \\
g(4) &= f(4) - [f(2) - f(1)] - f(1) = f(4) - f(2) \\
&\cdots \\
g(6) &= f(6) - [f(3) - f(1)] - [f(2) - f(1)] - f(1) \\
&\cdots
\end{aligned}$$

We see that $f(n)$ contributes with factor -1 if n prime, and not at all if n contains a prime factor to a higher power. This is precisely the raison d'etre for the Möbius function, with whose help the inverse of (13.38) can be written as the Möbius inversion formula [29.29] (13.39).