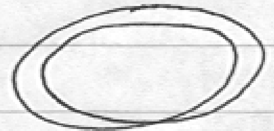
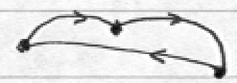
Lecture: Periodic orbits: How to get them.

(Graduate course, 09/01)

- 
flow

map

- "What makes periodic solutions so valuable is that they offer [...] the only opening through which we might try to penetrate into the fortress which has the reputation of being impregnable" Poincaré, 1892

flow:	$x \in p(T)$ iff	$f^{t+T}(x) = f^{t}(x) \quad \forall t$	closed curve
map:	$x \in p(n)$ iff	$f^{n}(x) = x$	set of points

- strategy: $g \equiv f^{n} - 1$ solve $g(x) = 0$, proceed by iteration...

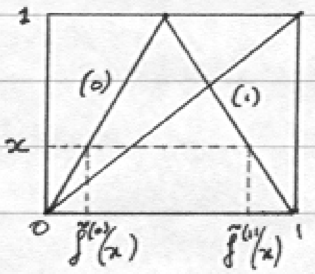
Search for periodic orbits of mappings

1. Iteration procedures for mapping

1.1. backwards iterations

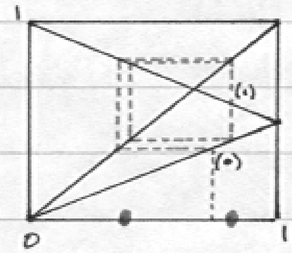
look at unstable periodic orbits $\rightarrow$ stable periodic cycles of "f^{-1}"

example:



tent map $f(x) = 1 - |1-2x|$ on $[0,1]$

two branches (0) and (1)



periodic cycle $\overline{01}$

convergence: $\varepsilon_{n+1} \sim \varepsilon_n (f^{-1})'(x_*) \sim \frac{1}{2^n}$ geometrical

$\varepsilon_n = x_n - x_*$

TBD

inverse

$g(x_n) = x_{n+1} = g(x_* + \varepsilon_n) = \underbrace{g(x_*)}_{x_*} + \varepsilon_n g'(x$

1.2. Newton-Raphson method

solve $g(x) = 0$ with initial guess x_0

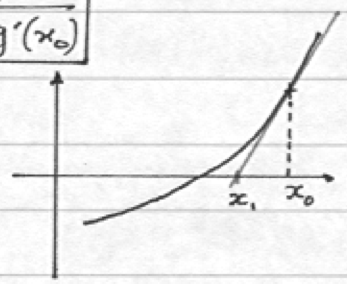
if good initial guess $x = x_0 + \delta x$ where δx small

$$\underbrace{g(x)}_{=0} = g(x_0) + g'(x_0)\,\delta x + O(\delta x^2)$$

consider a new approximation x_1 such that $g(x_0) + g'(x_0)(x_1 - x_0) = 0$

$$\boxed{x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}}$$

geometrically:



$x_* + \varepsilon_n$

convergence: $\varepsilon_n = x_n - x_*$ $\qquad$ $\varepsilon_{n+1} = \varepsilon_n - \dfrac{g(x_n)}{g'(x_n)}$

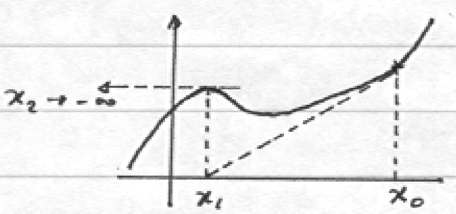
$$\varepsilon_{n+1} = \varepsilon_n - \frac{\cancel{g(x_*)} + \varepsilon_n g'(x_*) + \frac{\varepsilon_n^2}{2} g''(x_*) + \cdots}{g'(x_*) + \varepsilon_n g''(x_*) + \cdots}$$

$$\approx \varepsilon_n - \varepsilon_n \left(1 + \frac{\varepsilon_n}{2} \frac{g''(x_*)}{g'(x_*)}\right)\left(1 - \varepsilon_n \frac{g''(x_*)}{g'(x_*)}\right)$$

$$\varepsilon_{n+1} \approx \frac{\varepsilon_n^2}{2} \frac{g''(x_*)}{g'(x_*)} \sim \varepsilon_n^2$$

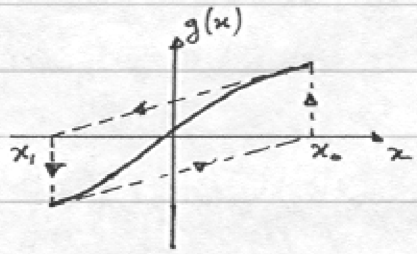
$$\varepsilon_n \sim \varepsilon_0^{2^n}$$ super-convergent algorithm

possible problems: i/ $g'(x_n) = 0 \rightarrow$ solution lost



solution: change slightly initial guess x_0

ii/ stuck into cycles



α - search for a fixed point of a 1D mapping $x' = f(x)$

consider $g(x) = f(x) - x$ and apply Newton's method

↳ iterate the map $\boxed{x_1 = x_0 - \dfrac{f(x_0) - x_0}{f'(x_0) - 1}}$

with a good initial guess... insights given by analysis of th

example: tent map $f(x) = 1 - |1-2x|$ $\quad f'(x) = 2\,\mathrm{sign}\,(1-2x)$

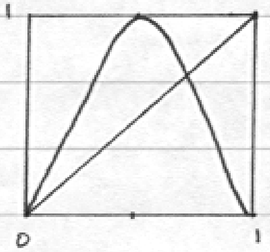
if $x_0 \in [\frac{1}{2}, 1]$, $\quad x_1 = \frac{2}{3}$ $\quad f'(x_1) = -2$

if $x_0 \in [0, \frac{1}{2}]$, $\quad x_1 = 0$ $\quad f'(x_1) = +2$

in one iteration, one ends up onto fixed point...

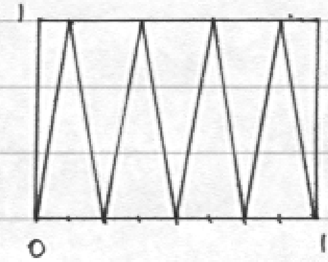
Ulam map : deformation of tent map, one expects very rapid conv
logistic

$f(x) = 4x(1-x)$



β - search for a n-cycle of a 1D mapping $x' = f(x)$

i/ consider $g(x) = f^n(x) - x$ and apply Newton's method



f^3

if n large, g has many wiggles 2^n ... not recommended metho

ii/ instead : multiple shooting method

looking at a set of n points such that $(x_1, x_2, \ldots, x_n)$ $\quad f(x_1) = x_2 \quad f(x_2) = x_3 \ldots$

$f(x_{n-1}) = x_n \quad f(x_n) = x_1$

consider the fonction $F(x) = \begin{pmatrix} x_1 - f(x_n) \\ x_2 - f(x_1) \\ \vdots \\ x_n - f(x_{n-1}) \end{pmatrix}$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$)

$x = (x_1, x_2, \dots, x_n)$

solve $F(x) = 0$ by Newton-Raphson method

▫ start with initial guess $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$

if $x^{(0)}$ good guess $\delta x = x - x^{(0)}$ small

$$F(x) = F(x^{(0)}) + \underbrace{J\,\delta x}_{\sum_j \left.\frac{\partial F_i}{\partial x_j}\right|_{x^{(0)}} \delta x_j} + O(\delta x^2)$$

consider a new approximation $x^{(1)}$ such that $0 = F(x^{(0)}) + J\,(x^{(1)} - x^{(0)})$

if J invertible : $\boxed{x^{(1)} = x^{(0)} - J^{-1} F(x^{(0)})}$

then iterate this map ▫

what's J? $J = \frac{dF}{dx} = \begin{pmatrix} 1 & & & -f'(x_n) \\ -f'(x_1) & 1 & 0 & \\ & -f'(x_2) & \ddots & \\ 0 & & \ddots & \\ & & -f'(x_{n-1}) & 1 \end{pmatrix}$

sparse matrix

$2n$ nonzero coeff over n^2

$$\text{determinant} = 1 - \underbrace{\prod_{k=1}^{n} f'(x_k)}$$

if $(x_1, \text{—}\, x_n)$ periodic orbit, $\prod_{k=1}^{n} f'(x_k)$ stabilit
of this orbit (3.

if periodic orbit is marginal, this method should fail.

inverse: $\beta_{k,m} = \prod_{l=k}^{m-1} f'(x_l)$ $\qquad \beta_{k,m} = 1$ for $k \geqslant m$

$y = J\mathbf{x}$

$$x_m = \sum_{k=1}^{m-1} \beta_{k,m} y_k + y_m + \frac{f'(x_n)\,\beta_{1,m}}{1 - \beta_{1,n+1}} \left[y_n + \sum_{k=1}^{n-1} \beta_{k,n} y_k \right]$$

▫ check : $y_{m+1} = -f'(x_m)\,\mathbf{x}_m + x_{m+1}$ for $m = 1, \dots, n-1$

$$-f'(x_m)\left[\sum_{k=1}^{m-1}\beta_{k,m}y_k + y_m\right] - \cancel{\frac{f'(x_n)\,\overbrace{f'(x_m)\,\beta_{1,m}}^{\beta_{1,m+1}}}{1-\beta_{1,n+1}}\left[y_n + \sum_{k=1}^{n-1}\beta_{k,n}y_k\right]}$$

$$+\sum_{k=1}^{m}\beta_{k,m}y_k + y_{m+1} + \cancel{\frac{f'(x_n)\,\beta_{1,m+1}}{1-\beta_{1,n+1}}\left[y_n + \sum_{k=1}^{n-1}\beta_{k,n}y_k\right]}$$

$$= y_{m+1}$$

ii/ $y_1 = -f'(x_n)\,x_n + x_1$ □

γ. search for fixed points, periodic cycles of d-dimensional mappings

$$x' = f(x) \quad \text{where } x = (x_1, \dots, x_d)$$

i/ consider $F(x) = x - f(x)$, solve $F(x)=0$ by Newton-Raphson met

$$\frac{dF}{dx}(x'-x) = f(x) - x$$

$$\underbrace{\frac{dF}{dx}}_{} = \left(\mathbb{1} - \frac{df}{dx}\right)_{ij} = \delta_{ij} - \frac{\partial f_i}{\partial x_j}$$

main difference

need a routine to inverse matrix (if $d>3$ ...)

not necessary sparse

if invertible, iterate the map $\boxed{x' = x + \left(\mathbb{1} - \frac{df}{dx}\right)^{-1}(f(x) - x)}$

ii/ search for n-cycle: consider $F(x^{(1)}, \dots, x^{(n)}) = \begin{pmatrix} x^{(1)} - f(x^{(n)}) \\ x^{(2)} - f(x^{(1)}) \\ \vdots \\ x^{(n)} - f(x^{(n-1)}) \end{pmatrix}$

where $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots, x_d^{(k)})$

solve $F(x^{(1)}, \dots, x^{(n)}) = 0$ in $\mathbb{R}^{d \times n}$

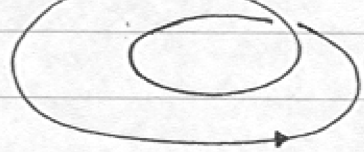
one needs to invert $\frac{dF}{dx} = \begin{pmatrix} \mathbb{1}_d & 0 & & -Df(x^{(n)}) \\ -Df(x^{(1)}) & \mathbb{1}_d & & 0 \\ & \ddots & \ddots & \\ 0 & & -Df(x^{(n-1)}) & \mathbb{1}_d \end{pmatrix}$ $nd \times nd$ matrix

$$[Df(x)]_{ij} = \frac{\partial f_i}{\partial x_j} \quad d \times d \text{ matrix}$$

$\mathbb{1}_d$ identity matrix $(d \times d)$

$\hookrightarrow$ iterate $x' = x + \left(\frac{dF}{dx}\right)^{-1} F(x)$ where $x = (x^{(1)}, \dots, x^{(n)})$ in $\mathbb{R}^{nd}$

2. Search of periodic orbits for flows

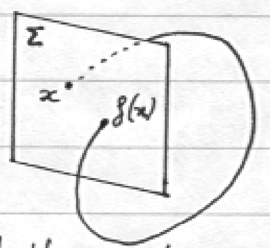


$\dot{x} = v(x)$

2.1 search for a fixed point of the Poincaré map

given Σ a surface of $\mathcal{M}$

given $x \in \Sigma$



integrate the flow until the orbit crosses the surface Σ again

$x' = f^t(x) \equiv f(x)$ $\qquad f$: Poincaré map

t: time of integration

Apply Newton-Raphson's method:

given $x' \in \Sigma$ close to x

$$f(x') = f(x) + \underset{\downarrow \atop \frac{df^t}{dx}}{\mathbb{J}}\,(x'-x) + o(\|x'-x\|^2)$$

in principle: $x' = f(x) + \mathbb{J}\,(x'-x)$ (if good initial guess)

$$\hookrightarrow \underline{(\mathbb{1} - \mathbb{J}^t)(x'-x) = f(x) - x}$$

<u>But</u>: i/ $f(x')$ not necessary on the surface Σ (Poincaré surface)

ii/ $\mathbb{1} - \mathbb{J}$ is degenerate (has 0 as an eigenvalue)

$\mathbb{J}(x_p)$ has 1 as eigenvalue, with eigenvector in the direction of the flow

□ consider a periodic solution x_p of $\dot{x} = v(x)$

. variation around it: $x(t) = x_p(t) + u(t)$

neglecting $O(\|u\|^2)$ $\quad \dot{u} = \underbrace{\left.\frac{dv}{dx}\right|_{x_p}}_{\equiv A(t)} u \quad (*)$

$\equiv A(t)$ where $A(t+T) = A($

. claim: any solution of $(*)$ satisfies $u(T) = \mathbb{J}^T u(0)$

where $\mathbb{J}^t$ solution of $\dot{\mathbb{J}} = A\mathbb{J}$, $\mathbb{J}^0 = \mathbb{1}$

proof: $\tilde{u}(t) = \mathbb{J}^t u(0)$ satisfies $\dot{\tilde{u}} = \dot{\mathbb{J}} u(0) = A\mathbb{J}u(0) = A\tilde{}$

$\tilde{u}$ satisfies $(*)$ and $\tilde{u}(0) = u(0) \Rightarrow \tilde{u}(t) = u(t)$

$t = T \qquad u(T) = \mathbb{J}^t u(0)$

. $\dot{x}_p$ solution of $(*)$: $\quad \dot{x}_p = v(x_p)$ $\quad \Big\} \frac{d}{dt}$

since $\ddot{x}_p = \left.\frac{dv}{dx}\right|_{x_p} \dot{x}_p$

thus $\dot{x}_p(T) = \mathbb{J}^T \dot{x}_p(0)$

$\quad\; \| $

$\dot{x}_p(0)$

therefore: $v = \dot{x}_p(0)$ eigenvector of $\mathbb{J}^T$ with eigenvalue 1

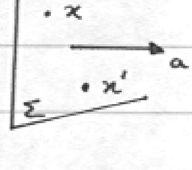
$\downarrow$

direction along the flow

□

Patch to 1/ and 2/ : add a small vector along the flow

ask that $x' \in \Sigma$ | assume that $\Sigma: (x^*-x).a$



$$\begin{cases} x' = f(x) + \mathbb{J}(x'-x) + \underline{v(x)\,\delta T} \\ (x'-x).a = 0 \end{cases}$$

↳ call it - ...

↳ advantage: linear $\rightarrow$ matrix formulation of the algorithm

$$\underbrace{\begin{pmatrix} \mathbb{1}-\mathbb{J} & v(x) \\ a & 0 \end{pmatrix}}_{(d+1)\times(d+1)\text{ matrix}} \begin{pmatrix} x'-x \\ \delta T \end{pmatrix} = \begin{pmatrix} f(x)-x \\ 0 \end{pmatrix}$$

Does it really do the trick?

example: 3 dimensional flow with Poincaré surface of section $z=0$ ((x,y) plane)

assume that the flow is crossing Σ perpendicularly

$v = (0, 0, v_z)$

then $\mathbb{J}$ writes like that:

$$\mathbb{J} = \begin{pmatrix} J_{11} & J_{12} & 0 \\ J_{21} & J_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ eigenvector}$$

$$\begin{pmatrix} 1-J_{11} & -J_{12} & 0 & 0 \\ -J_{21} & 1-J_{22} & 0 & 0 \\ 0 & 0 & 0 & v_z \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

matrix with determinant $-\underbrace{m_{11}}_{\text{minor}}(\mathbb{1}-\mathbb{J})\, v_z \neq 0$ a priori

2.2. search for n cycles $\qquad f$: Poincaré map $\qquad \Sigma : (x-x_0).a = 0$

consider $F(\{x\}) = \begin{pmatrix} x_1 - f(x_n) \\ x_2 - f(x_1) \\ \vdots \\ x_n - f(x_{n-1}) \end{pmatrix}$ where $(x_1, x_2, \dots, x_n) \in \Sigma^n$

$$F(\{x'\}) = F(\{x\}) + DF(\{x'\} - \{x\}) + \dots$$

$\underline{x} = (x_1, \dots, x_n)$ $\qquad$ / add n small vectors along the flow

// add n conditions : $\underline{x}' \in \Sigma^n$

has to be in Σ^n

$$\underbrace{F(\underline{x}')}_{=0} = F(\underline{x}) + DF\,(\underline{x}' - \underline{x}) + \begin{pmatrix} \delta t_1 v_1 \\ \delta t_2 v_2 \\ \vdots \\ \delta t_n v_n \end{pmatrix}$$

$\Sigma : (x - x_0)\cdot a = 0$ $\rightarrow$ matrix formulation

$$DF = \begin{pmatrix} \mathbb{1}_d & & & -J_n & v_1 & & \\ -J_1 & \mathbb{1}_d & & & & \ddots & \\ & \ddots & \ddots & & & & \\ & & -J_{n-1} & \mathbb{1}_d & & & v_n \\ a & & & & & & \\ & \ddots & & & & 0 & \\ & & & a & & & \end{pmatrix}$$

dimension : $(nd + n) \times (nd + n)$ to invert ...

3_ other methods

The application of Newton - Raphson method requires a good initial guess.
Sometimes this is not possible (one needs good hints on the dynamics like e.g. symbolic code).

The search for new algorithms to find periodic orbits still goes on ...

α_ [PRE 57, 2739 (1998)]

given a d-dimensional dissipative map $x' = f(x)$
look for a fixed point x_* (unstable)

idea: construct a new ~~dy~~ map such that x_* is a stable fixed point

$$x' = x + \underset{\substack{\uparrow \\ d\times d \text{ invertible matrix}}}{\Lambda}\left(f(x) - x \right) \equiv \tilde{f}_\Lambda(x)$$

(in practice: $\Lambda = \overset{\text{constant} > 0}{\mu} \times \underset{\text{involution}}{C}$)

aim: stabilize x_* by a suitable choice of Λ

consider the stability matrix:

$$\tilde{A} = \frac{d\tilde{f}}{dx} = \underbrace{\mathbb{1} + \Lambda(A - \mathbb{1})}_{\text{with eigenvalues } |\cdot| < 1}$$

assume A diagonalizable: $\exists P$ s.t. $P^{-1} A P = \Delta$

$$P^{-1} \tilde{A} P = \mathbb{1} + P^{-1} \Lambda P \underbrace{P^{-1}(A - \mathbb{1}) P}_{\Delta - 1}$$

$$\hookrightarrow P^{-1} \Lambda P = \mu C_\Delta = \mu \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_d \end{pmatrix}$$

$\varepsilon_k = 0, \pm 1$.

$$\tilde{\lambda} = 1 + \mu \underbrace{\varepsilon_k (\lambda - 1)}_{\text{want it negative}}$$

if $Re(\lambda) < 1$, $\varepsilon_k = +1$

if $Re(\lambda) > 1$, $\varepsilon_k = -1$

$Re(\varepsilon(\lambda - 1)) < 0$ $\quad Re(\cdot) < 0$

μ small s.t. $|\tilde{\lambda}| < 1$.

$\tilde{\lambda}^2 = (1 + \mu \varepsilon_k (Re(\lambda) - 1))^2 + \mu^2 Im(\lambda)^2$ seul terme negatif: $2\mu\varepsilon_k(Re(\lambda) - 1)$

β. unstable periodic orbits from time series measurments

. deterministic dynamics but you don't know $f(x)$

$\{x_i\}_{i=1,\dots,N}$ $\qquad x_i \in \mathbb{R}^d$

remark: if N sufficiently large, the time series will visit the neighborhood of an arbitrary n-cycle ϕ

↳ scan for pairs of points separated by n time steps
(close)

i.e. search for indices i such that $\| x_i - x_{i+n} \| \leqslant \varepsilon_1$

given two indices i and j satisfying this condition

↳ two cycles $\Theta_1 = (x_i, x_{i+1}, \dots, x_{i+n-1}) \equiv (x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)})$

$\Theta_2 = (x_j, x_{j+1}, \dots, x_{j+n-1}) \equiv (x_1^{(2)}, x_2^{(2)}, \dots, x_n^{(2)})$

same orbit (up to a permutation)?

iff $d(\Theta_1, \Theta_2) < \varepsilon_2$ $\qquad$ e.g. $d(\Theta_1,\Theta_2) = \sum_{i=1}^{n} \min_{j=1,\dots,n} \| x_i^{(1)} - x_j^{(2)} \|$

then: average the positions of the determined same orbits.

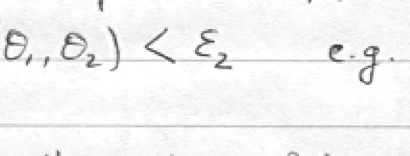
[PRL 58, 2387 (1987)] $\qquad$ However: . requires very long time series

. not very precise.

γ. relaxation method $\qquad$ [11.6.1]

or [PRL 63, 819 (1989)]

δ. orbit length extremization method for billiards



2 dimensional

use variational method on length $L(s)$ to find periodic cycles

↳ on n bounces

remark: multiple shooting method $dn \times dn$ matrices here: n coordinates (only s_i)

$s = (s_1, \dots, s_n)$ find the roots of $\partial_i L(s) = 0$ by Newton's method

$$\partial_i L(s_0 + \delta s) = \partial_i L(s_0) + \sum_j \partial_i \partial_j L(s_0)\, \delta s_j + \dots$$

good initial guess: δs small

↳ iterate the map $s' = s - J^{-1} D L(s)$

where $J_{ij} = \partial_i \partial_j L(s)$

$DL_i = \partial_i L(s)$

γ_ relaxation method unstable p-cycle of maps $\longrightarrow$ stable equilibrium of flow

$$\text{solve} \begin{cases} \dfrac{dx_n}{dt} = s_n F_n \\ s_n = \pm 1 \\ x_{p+1} = x_1 \end{cases} \qquad F_n = -\frac{\partial V}{\partial x_n} \qquad n = 1, \dots, p$$