mathematical methods - week 9

Fourier transform

Georgia Tech PHYS-6124

Homework HW #9 due Monday, October 21, 2019

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 9.2 *d-dimensional Gaussian integrals* 5 points
Exercise 9.3 *Convolution of Gaussians* 5 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

edited October 18, 2019

Week 9 syllabus

Wednesday, October 16, 2019

Wed Arfken and Weber [1] (click here) Chapter 14. Fourier Series.
Farazmand notes on Fourier transforms.

Fri Grigoriev notes
4. Integral transforms, 4.3-4.4 square wave, Gibbs phenomenon;
5. Fourier transform: 5.1-5.6 inverse, Parseval's identity, ..., examples

Optional reading

- Stone and Goldbart [4] (click here) Appendix B
- Roger Penrose [3] chapter on Fourier transforms is sophisticated, but too pretty to pass up.

Question 9.1. Henriette Roux asks
Q What percentage score on problem sets is a passing grade?
A That might still change, but currently it looks like 60% is good enough to pass the course. 70% for C, 80% for B, 90% for A. Very roughly - will alert you if this changes.

Question 9.2. Henriette Roux find course notes confusing
Q Couldn't you use one single, definitive text for methods taught in the course?
A It's a grad school, so it is research focused - I myself am (re)learning the topics that we are going through the course, using various sources. My emphasis in this course is on understanding and meaning, not on getting all signs and 2π's right, and I find reading about the topic from several perspectives helpful. But if you really find one book more comfortable, nearly all topics are covered in Arfken & Weber [1].

9.1 A bit of noise

Fourier invented Fourier transforms to describe the diffusion of heat. How does that come about?

Consider a noisy *discrete time* trajectory

$$x_{n+1} = x_n + \xi_n \,, \qquad x_0 = 0 \,, \tag{9.1}$$

where x_n is a d-dimensional state vector at time n, $x_{n,j}$ is its jth component, and ξ_n is a noisy kick at time n, with the prescribed probability distribution of zero mean and the covariance matrix (diffusion tensor) Δ,

$$\langle \xi_{n,j} \rangle = 0 \,, \qquad \langle \xi_{n,i}\, \xi_{m,j}^T \rangle = \Delta_{ij}\, \delta_{nm} \,, \tag{9.2}$$

where $\langle \cdots \rangle$ stands for average over many realizations of the noise. Each 'Langevin' trajectory $(x_0, x_1, x_2, \cdots)$ is an example of a Brownian motion, or diffusion.

In the Fokker-Planck description individual noisy trajectories (9.1) are replaced by the evolution of a density of noisy trajectories, with the action of discrete one-time step *Fokker-Planck operator* on the density distribution ρ at time n,

$$\rho_{n+1}(y) = [\mathcal{L}\rho_n](y) \;=\; \int dx\, \mathcal{L}(y,x)\, \rho_n(x)\,, \tag{9.3}$$

given by a normalized Gaussian (work through exercise 9.2)

$$\mathcal{L}(y,x) = \frac{1}{N}\, e^{-\frac{1}{2}(y-x)^T \frac{1}{\Delta}(y-x)}\,, \quad N = (2\pi)^{d/2}\sqrt{\det(\Delta)}\,, \tag{9.4}$$

which smears out the initial density ρ_n diffusively by noise of covariance (9.2). The covariance Δ is a symmetric $[d{\times}d]$ matrix which can be diagonalized by an orthogonal transformation, and rotated into an ellipsoid with d orthogonal axes, of different widths (covariances) along each axis. You can visualise the Fokker-Planck operator (9.3) as taking a δ-function concentrated initial distribution centered on $x = 0$, and smearing it into a cigar shaped noise cloud.

As $\mathcal{L}(y,x) = \mathcal{L}(y-x)$, the Fokker-Planck operator acts on the initial distribution as a *convolution*,

$$[\mathcal{L}\rho_n](y) \;=\; [\mathcal{L} * \rho_n](y) \;=\; \int dx\, \mathcal{L}(y-x)\, \rho_n(x)$$

Consider the action of the Fokker-Planck operator on a normalized, cigar-shaped Gaussian density distribution

$$\rho_n(x) = \frac{1}{N_n} e^{-\frac{1}{2}x^T \frac{1}{\Delta_n}\, x}\,, \qquad N_n = (2\pi)^{d/2}\sqrt{\det(\Delta_n)}\,. \tag{9.5}$$

That is also a cigar, but in general of a different shape and orientation than the Fokker-Planck operator (9.4). As you can check by working out exercise 9.3, a convolution of a Gaussian with a Gaussian is again a Gaussian, so the Fokker-Planck operator maps the Gaussian $\rho_n(x_n)$ into the Gaussian

$$\rho_{n+1}(x) = \frac{1}{N_{n+1}} e^{-\frac{1}{2}x^T \frac{1}{\Delta_n+\Delta}\, x}\,, \qquad N_{n+1} = (2\pi)^{d/2}\sqrt{\det(\Delta_n+\Delta)} \tag{9.6}$$

one time step later.

In other words, covariances Δ_n add up. This is the d-dimensional statement of the familiar fact that cumulative error squared is the sum of squares of individual errors. When individual errors are small, and you are adding up a sequence of them in time, you get Brownian motion. If the individual errors are small and added independently to a solution of deterministic equations (so-called ‘drift’), you get the Langevin and the Fokker-Planck equations.

References

[1] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists: A Comprehensive Guide*, 6th ed. (Academic, New York, 2005).

[2] N. Bleistein and R. A. Handelsman, *Asymptotic Expansions of Integrals* (Dover, New York, 1986).

[3] R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe* (A. A. Knopf, New York, 2005).

[4] M. Stone and P. Goldbart, *Mathematics for Physics: A Guided Tour for Graduate Students* (Cambridge Univ. Press, Cambridge, 2009).

Exercises

9.1. **Who ordered $\sqrt{\pi}$?** Derive the Gaussian integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2a}} = \sqrt{a}\,, \qquad a > 0\,.$$

assuming only that you know to integrate the exponential function e^{-x}. Hint, hint: x^2 is a radius-squared of something. π is related to the area or circumference of something.

9.2. **d-dimensional Gaussian integrals.** Show that the Gaussian integral in d-dimensions is given by

$$\begin{aligned} Z[J] &= \int d^d x\, e^{-\frac{1}{2}x^\top \cdot M^{-1}\cdot x + x^\top \cdot J} \\ &= (2\pi)^{d/2} |\det M|^{\frac{1}{2}}\, e^{\frac{1}{2}J^\top \cdot M \cdot J}\,, \end{aligned} \tag{9.7}$$

where M is a real positive definite $[d \times d]$ matrix, i.e., a matrix with strictly positive eigenvalues, x and J are d-dimensional vectors, and $(\cdots)^\top$ denotes the transpose.

This integral you will see over and over in statistical mechanics and quantum field theory: it's called 'free field theory', 'Gaussian model', 'Wick expansion', etc.. This is the starting, 'propagator' term in any perturbation expansion.

Here we require that the real symmetric matrix M in the exponent is strictly positive definite, otherwise the integral is infinite. Negative eigenvalues can be accommodated by taking a contour in the complex plane [2], see exercise 14.1 *Fresnel integral*. Zero eigenvalues require stationary phase approximations that go beyond the Gaussian saddle point approximation, typically to the Airy-function type stationary points, see exercise 7.2 *Airy function for large arguments*.

9.3. **Convolution of Gaussians.**

(a) Show that the Fourier transform of the convolution

$$[f * g](x) = \int d^d y\, f(x-y) g(y)$$

corresponds to the product of the Fourier transforms

$$[f * g](x) = \frac{1}{(2\pi)^d}\int d^d k\, F(k) G(k) e^{-ik\cdot x}\,, \tag{9.8}$$

where

$$F(k) = \int \frac{d^d x}{(2\pi)^{d/2}}\, f(x)\, e^{-ik\cdot x}\,, \quad G(k) = \int \frac{d^d x}{(2\pi)^{d/2}}\, g(x)\, e^{-ik\cdot x}\,.$$

(b) Consider two normalized Gaussians

$$\begin{aligned} f(x) &= \frac{1}{N_1}\, e^{-\frac{1}{2}x^\top\cdot\frac{1}{\Delta_1}\cdot x}\,, \quad N_1 = \sqrt{\det(2\pi\Delta_1)} \\ g(x) &= \frac{1}{N_2}\, e^{-\frac{1}{2}x^\top\cdot\frac{1}{\Delta_2}\cdot x}\,, \quad N_2 = \sqrt{\det(2\pi\Delta_2)} \\ 1 &= \int d^d k\, f(x) = \int d^d k\, g(x)\,. \end{aligned}$$

Evaluate their Fourier transforms

$$F(k) = \frac{1}{(2\pi)^{d/2}} e^{\frac{1}{2}k^\top\cdot\Delta_1\cdot k}\,, \qquad G(k) = \frac{1}{(2\pi)^{d/2}} e^{\frac{1}{2}k^\top\cdot\Delta_2\cdot k}\,.$$

Show that the convolution of two normalized Gaussians is a normalized Gaussian

$$[f * g](x) = \frac{(2\pi)^{-d/2}}{\sqrt{\det(\Delta_1+\Delta_2)}}\; e^{-\frac{1}{2}x^\top\cdot\frac{1}{\Delta_1+\Delta_2}\cdot x}\,.$$

In other words, covariances Δ_j add up. This is the d-dimenional statement of the familiar fact that cumulative error squared is the sum of squares of individual errors. When individual errors are small, and you are adding up a sequence of them in time, you get Brownian motion. If the individual errors are small and added independently to a solution of a deterministic equation, you get Langevin and Fokker-Planck equations.