Homework assignments are posted on the web on Tuesday and your solutions are due next Tuesday in class (stapled together and clearly labeled with your name and the homework number). You get bonus points for optional problems worked through.

1) Fermat's principle: According to Fermat's principle, the path taken by a ray of light between any two points makes stationary the travel time between those points. (In this problem you may assume that all light paths lie in a suitable plane.) A medium may be characterised optically by its refractive index n, such that the speed of light in the medium is c/n.

a) Use Fermat's principle to show that light propagates along straight lines in homogeneous media (*i.e.*, media in which n is independent of position).

b) Consider the propagation of light from one semi-infinite homogeneous medium of refractive index n_1 to another with refractive index n_2. By examining paths that need not be differentiable at the interface establish Snell's law.

c) A planar light ray propagates in an inhomogeneous medium with refractive index $n(r)$, where r is the distance to a fixed centre, $\mathcal{O}$. Let r and ϕ be plane polar coordinates. Use Fermat's principle to find the first-order nonlinear differential equation obeyed by the light path. Choose the constant of integration to be the distance a from $\mathcal{O}$ at which $dr/d\phi = 0$.

d) If $n(r) = \sqrt{1 + \alpha^2/r^2}$ and the ray starts far from $\mathcal{O}$, find the ultimate angle through which the ray is refracted if its minimum distance from $\mathcal{O}$ is a. (A discussion of apsidal angles may be useful here.) Sketch the path of the ray.
[Hint: you may find helpful the substitution $r = a/\sin\psi$.]

3) Calculus of variations – straight lines:

a) By using the calculus of variations with fixed end-points, show that the shortest path between two fixed points in a plane is a straight line.

b) By using the calculus of variations with fixed end-points, for the case of several dependent variables, show that the shortest path between two points in three dimensional space is given by a straight line.

c) Reconsider the derivation of the Euler equation for a single dependent variable, but now allow variations that do not necessarily vanish at the end points. By paying particular attention to the boundary terms, show that the shortest path between a straight line and a point (in the plane containing that line and point) is the perpendicular from the point to the line.

[Note: For the purposes of this question you need only establish that such distances are *stationary* with respect to small variations, and not necessarily (even local) minima.]

4) Scalar wave equation: The functional $\mathcal{S}$ has, as its argument, a single function u of the two independent variables x and t:

$$\mathcal{S}[u] = \frac{1}{2}\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left\{\bar{\rho} u_t(x,t)^2 - \bar{\kappa} u_x(x,t)^2\right\},$$

where $\bar{\rho}$ and $\bar{\kappa}$ are constants.

a) Find the condition on u that makes $\mathcal{S}$ stationary with respect to variations that vanish at all times at the boundary points x_1 and x_2, and at all points at the initial and final times t_1 and t_2.

[Note: $u_t(x,t) \equiv \partial u/\partial t$ and $u_x(x,t) \equiv \partial u/\partial x$.]

You have just implemented Hamilton's principle to obtain the equation of motion for transverse displacements of a stretched string of line density $\bar{\rho}$ and tension $\bar{\kappa}$, *i.e.*, the scalar one-dimensional wave equation. $\mathcal{L} = \frac{1}{2}\left\{\bar{\rho} u_t(x,t)^2 - \bar{\kappa} u_x(x,t)^2\right\}$ is known as the Lagrange density; $L = \int_{x_1}^{x_2} dx\, \mathcal{L}$ is known as the Lagrangian; and $\mathcal{S} = \int_{t_1}^{t_2} dt\, L$ is known as the action.

b) Repeat part (a), but now supposing that the line density varies with position, *i.e.*, $\bar{\rho} \to \rho(x)$, and that the Lagrange density has also acquired the additional term $g\rho(x)\, u(x,t)$. State a possible physical origin for such a term.

c) Show that the vector wave equation follows from the stationarity of the functional

$$\mathcal{W}[\mathbf{u}] = \frac{1}{2}\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx \left\{\bar{\rho}|\mathbf{u}_t|(x,t)^2 - \bar{\kappa}|\mathbf{u}_x|(x,t)^2\right\},$$

where $\mathbf{u}(x,t)$ is a vector function of x and t.

——**Optional problems**——

2) Euler equations for several dependent variables: Suppose that the functional $\mathcal{A}$ depends on n independent functions $\{y_i(x)\}_{i=1}^{n}$ of a single independent variable x, with $y_i' \equiv dy_i/dx$:

$$\mathcal{A}[\{y_i\}] = \int_{x_1}^{x_2} dx\, f(\{y_i\}, \{y_i'\}; x).$$

a) By considering independent variations $\{\eta_i(x)\}_{i=1}^{n}$ of the functions $\{y_i(x)\}$, variations that vanish at the end-points, establish the set of Euler equations.

b) Show that if f does not depend explicitly on x then the so-called *first integral*

$$-f + \sum_{i=1}^{n} y_i' \frac{\partial f}{\partial y_i'}$$

is constant (*i.e.*, independent of x, provided that $\{y_i(x)\}$ satisfy the Euler equations).

c) Find the Euler equations for the following functionals, and compare them with equations that you have seen before:

   i) $\mathcal{S}[x] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt\, \frac{1}{2}m\dot{x}(t)^2$;

   ii) $\mathcal{S}[x] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt\, \left\{\frac{1}{2}m\dot{x}(t)^2 - U\big(x(t)\big)\right\}$;

   iii) $\mathcal{S}[\{r_j\}_{j=1}^3] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt\, \frac{1}{2}m\sum_{k=1}^3 \dot{r}_k(t)^2$;

   iv) $\mathcal{S}[\{r_j\}_{j=1}^3] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt\, \left\{\frac{1}{2}m\sum_{k=1}^3 \dot{r}_k(t)^2 - U\big(\{r_j(t)\}\big)\right\}$.

5) Nonlocal functionals – optional: Consider the functional

$$\mathcal{J}[w] = \frac{1}{2}\left\{\int_{x_1}^{x_2} dx\, \alpha(x)\, w(x)\right\}^2,$$

where $\alpha(x)$ is some prescribed function.

a) Is this functional *local*?

b) Starting from first principles, establish the condition on w that guarantees that $\mathcal{J}$ is stationary.

c) Was it necessary to impose boundary conditions on the allowed variations? Why?

d) Repeat parts (b) and (c) for the functional

$$\mathcal{J}[w] = \frac{1}{2}\left\{\int_{x_1}^{x_2} dx\, \alpha(x)\, \frac{dw}{dx}\right\}^2.$$

e) Is the following functional local:

$$\mathcal{J}[w] = \int_{x_1}^{x_2} dx \int_{x_1}^{x_2} dy \left\{\frac{w(x) - w(y)}{x - y}\right\}^2 ?$$

6) Drumhead statics – optional: The shape of a distorted drumhead is described by the function $h(x, y)$, which gives the height to which the point (x, y) of the undistorted drumhead is displaced.

a) Show that, for small distortions, the increase in the area of the drumhead that occurs upon distortion is given by the functional

$$\mathcal{A}[h] = \frac{1}{2}\int dx\, dy\, |\nabla h|^2,$$

where $\nabla \equiv \mathbf{e}_x\partial_x + \mathbf{e}_y\partial_y$, and the integral is taken over the area of the undistorted drumhead.

b) Show that if h satisfies the two-dimensional Laplace equation then $\mathcal{A}$ is stationary with respect to variations that vanish at the boundary

c) What is additionally required of h, if $\mathcal{A}$ is to be stationary with respect to the wider class of variations, namely those that do not necessarily vanish at the boundary?

7) The brachistochrone – optional: In this question we will use the calculus of variations to solve a famous old problem, the brachistochrone (*i.e.*, least time) problem, first posed and solved by Johannes Bernoulli in 1696.

Let A and B be two points in a vertical plane, with A higher than B. Suppose A and B are connected by a smooth wire, which describes some curve in the vertical plane, and that the downward gravitational acceleration is g (a constant). A bead starts from rest at the point A and is allowed to slide to the point B. $\mathcal{T}$, the time taken to slide from A to B, depends on the curve of the wire. By following the steps outlined below, find the curve that makes $\mathcal{T}$ stationary.

a) Let y be the depth below A and let x be the horizontal displacement from A. Then A is the point $x = 0$, $y = 0$ and B is the point $x = a$, $y = b$. Construct a functional, $\mathcal{T}$, which depends on g and on the curve of the wire, $y(x)$, which gives the time taken to reach B. Write $\mathcal{T}$ in the form

$$\mathcal{T}[y] = \frac{1}{\sqrt{2g}} \int_0^a dx\, f(y, y')$$

and write down the appropriate function f, where y' denotes dy/dx.

b) Derive the Euler-Lagrange equation for the curve $y(x)$ that makes $\mathcal{T}$ stationary. Of what order is this differential equation?

c) The solution to this problem is simplified by recognising that the function f does not depend on x. To see the simplification, consider a general functional of the form

$$\mathcal{R}[y] = \int_0^a dx\ h(y, y').$$

Write down the Euler-Lagrange equation for this problem, in terms of the function $h(y, y')$. Show that, because $\partial h/\partial x = 0$, we can immediately integrate the Euler-Lagrange equation to obtain the so-called first integral

$$h - y' \frac{\partial h}{\partial y'} = -\epsilon,$$

where ϵ is a constant.

d) Derive the first integral for the brachistochrone problem. Of what order is this differential equation? What information have we not yet used, which, at a later stage, will allow us to fix the constant of integration?

e) A cycloid is a curve generated by a point on the radius of a wheel of radius α rolling under the x-axis and is specified parametrically by the equations

$$\begin{aligned} x(\theta) &= \alpha(\theta - \sin\theta) \\ y(\theta) &= \alpha(1 - \cos\theta). \end{aligned}$$

Show that this curve solves the brachistochrone problem. What information would you use to fix the constant α?

f) Discuss briefly, without any calculations, how you might set about establishing that the cycloid not only makes $\mathcal{T}$ stationary, but actually minimises it.

g) Solve the modified brachistochrone problem, in which the potential energy $-mgy$ is replaced by $-\frac{1}{2}m\omega^2 y^2$?

8) Hypothetical soap film – optional: Imagine that space is four-dimensional. Then, in the analogue of the soap-film problem, a *three*-dimensional soap film connects the *two*-dimensional surfaces of a pair of non-concentric coaxial spheres. Suppose that the film adopts a profile $\rho(x)$ that is axially symmetric. Then the area functional $\mathcal{A}$ is given by

$$\mathcal{A} = \int_0^a dx\, 4\pi\rho^2 \sqrt{1+\rho'^2},$$

where $\rho' \equiv d\rho/dx$. Construct and solve the *first order* ordinary differential equation, which the profile must satisfy if the area is to be stationary. Express your answer in terms of two constants of integration.
[Note: The answer can be expressed in terms of Jabobi's elliptic functions.]

9) Fermat principle in higher dimensions – optional: Suppose that the velocity of rays in a sphere varies with depth, and is given by $c(r) = \alpha - \beta r^2$, where α and β are constants.

a) Show that that paths of stationary time are circles.

b) A ray enters the sphere at an angle ϵ to the surface. Determine the polar coordinates of the deepest point of the path, and find the time to reach it.

c) Is it possible to find a speed function $c(r)$ such that the stationary paths would be circles in a d-dimensional sphere?

[Note: I haven't tried this problem yet!]

10) Lagrangian for charged particle motion – optional: In this question we will obtain the equation of motion for a particle of mass m and charge e moving in an electromagnetic field prescribed by the scalar potential $\phi(\mathbf{r}, t)$ and the vector potential $\mathbf{A}(\mathbf{r}, t)$. The electric and magnetic fields, $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$, can be obtained from these potentials through

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}, \\ \mathbf{B}(\mathbf{r}, t) &= \nabla\times\mathbf{A}.\end{aligned}$$

The Lagrangian for this system L is given by

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2}m|\dot{\mathbf{q}}|^2 + e\,\dot{\mathbf{q}}\cdot\mathbf{A}(\mathbf{q}, t) - e\phi(\mathbf{q}, t),$$

where $\mathbf{q}$ and $\dot{\mathbf{q}}$ ($\equiv d\mathbf{q}/dt$) respectively denote the Cartesian position and velocity vectors.

a) Derive the momentum canonically conjugate to $\mathbf{q}$.

b) Construct the Euler-Lagrange equation for $\mathbf{q}$.

c) Show that the Euler-Lagrange equation is equivalent to the Lorentz equation

$$m\ddot{\mathbf{q}} = e\mathbf{E}(\mathbf{q}, t) + e\dot{\mathbf{q}}\times\mathbf{B}(\mathbf{q}, t).$$

d) Construct the Hamiltonian $\mathcal{H}$ in terms of ϕ and $\mathbf{A}$.

Now suppose that $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ are replaced by $\phi'(\mathbf{r}, t)$ and $\mathbf{A}'(\mathbf{r}, t)$, where

$$\begin{aligned}\phi'(\mathbf{r}, t) &= \phi(\mathbf{r}, t) - \frac{\partial}{\partial t}\chi(\mathbf{r}, t), \\ \mathbf{A}'(\mathbf{r}, t) &= \mathbf{A}(\mathbf{r}, t) + \nabla\chi(\mathbf{r}, t),\end{aligned}$$

and where $\chi(\mathbf{r}, t)$ is an arbitrary function of $\mathbf{r}$ and t. Such a transformation is known as a gauge transformation.

e) What effect does this a gauge transformation have on the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$?

f) Show that the change in L resulting from this change in $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ can be written as a total time-derivative. What effect does this change have on the action? What effect does it have on the equation of motion for $\mathbf{q}$?