Adjoint Operators

Action of matrices is most conveniently described by using their eigenvalues and eigenvectors. Same is true of operators, with eigenvectors → eigenfunctions:

$$\mathcal{L}\varphi_n(x) = \lambda_n \varphi_n(x)$$

The properties of $\varphi_n(x)$ depend on the properties of $\mathcal{L}$, in particular, on whether $\mathcal{L}$ is "symmetric".

$$\langle v|\mathcal{L}|u\rangle = \int_a^b v(x)\left\{P_0(x)u''(x) + P_1(x)u'(x) + P_2(x)u(x)\right\}dx$$

Integrating by parts:

① $$\int_a^b v(x)P_1(x)u'(x)\,dx = [vP_1u]_a^b - \int_a^b [vP_1]'u\,dx$$

② $$\int_a^b v(x)P_0(x)u''(x)\,dx = [vP_0u']_a^b - \int_a^b [vP_0]'u'\,dx$$

$$= [vP_0u']_a^b - [(vP_0)'u]_a^b + \int_a^b [vP_0]''u\,dx$$

so that

$$\langle v|\mathcal{L}|u\rangle = \int_a^b [(P_0v)''u - (P_1v)'u + (P_2v)u]\,dx$$
$$+ [vP_1u + vP_0u' - (vP_0)'u]_a^b$$

Factor out u inside the integral and note that

$$-(vP_0)'u = -vP_0'u - v'P_0u$$

such that

$$\langle v|\mathcal{L}u\rangle = \langle \mathcal{L}^+v|u\rangle + [vP_1u + vP_0u' - v'P_0u - vP_0'u]_a^b$$

where

$$\mathcal{L}^+v \equiv (P_0v)'' + (P_1v)' + P_2v$$ is an <u>adjoint</u> of $\mathcal{L}$.
(formal adjoint)

Self-adjoint operators

Under what conditions $\mathcal{L}^{+} = \mathcal{L}$?

$$\mathcal{L}u = P_0 u'' + P_1 u' + P_2 u$$

$$\mathcal{L}^{+}u = (P_0 u)'' - (P_1 u)' + P_2 u =$$

$$= P_0'' u + 2P_0' u' + P_0 u'' - P_1' u - P_1 u' + P_2 u$$

$$= P_0 u'' + (2P_0' - P_1)u' + (P_0'' - P_1' + P_2)u$$

Now, $\mathcal{L}u = \mathcal{L}^{+}u$ for arbitrary u, provided

$$\begin{cases} P_1 = 2P_0' - P_1 \\ P_0'' - P_1' = 0 \end{cases} \iff \boxed{P_1 = P_0'}$$

<u>Definition:</u> $\mathcal{L}$ is formally self-adjoint (Hermitian), if $\mathcal{L}^{+} = \mathcal{L}$.

What about the boundary value problem? For it to be self-adjoint not only $\mathcal{L}^{+} = \mathcal{L}$ is required, but bound. cond. have to be considered.

Remember,

$$\langle v | \mathcal{L}u \rangle = \langle \mathcal{L}^{+}v | u \rangle + \left[\cancel{vP_1u} + vP_0u' - v'P_0u \underbrace{- vP_0'u}_{-\cancel{vP_1u}} \right]_a^b$$

$$\Rightarrow \text{Need } \left[vP_0u' - v'P_0u \right]_a^b = 0.$$

If the b.c. on u are homogeneous, the entire BVP can be self-adjoint. For example, suppose

$$u(a) = u'(b) = 0$$

Then

$$[vP_0u' - v'P_0u]_a^b = v(b)P_0(b)\overset{0}{\cancel{u'(b)}} - v'(b)P_0(b)u(b) -$$
$$- v(a)P_0(a)u'(a) + v'(a)P_0(a)\overset{0}{\cancel{u(a)}}$$

So that the choice $v(a) = v'(b) = 0$ eliminates the remaining terms, and

$$\langle v | \mathcal{L}u \rangle = \langle \mathcal{L}^{+}v | u \rangle \quad \text{exactly.}$$

Self-adjoint operators are of special interest for the same reason self-adjoint (Hermitian) matrices are:

1) They have special & useful properties
2) They often arise in physical problems

Any 2^{nd} order linear ODE can be transformed into a self-adjoint form by multiplying by $\sigma(x) = \exp\left[\int \frac{p_1 - p_0'}{p_0} dx\right]$:

$$\mathcal{L}u = \rho \Rightarrow \underbrace{\sigma\mathcal{L}}_{\text{new diff. operator, which is self-adjoint}} u = \sigma\rho$$

$$(\sigma\mathcal{L})^{+} = \sigma\mathcal{L} \Leftrightarrow \sigma p_1 = (\sigma p_0)' \Rightarrow \frac{\sigma'}{\sigma} = \frac{p_1 - p_0'}{p_0}$$

Example: Bessel's eq. is not formally self-adjoint:

$$x^2 u'' + x u' + (Ax^2 - B)u = 0$$

$$\rightarrow p_0' = (x^2)' = 2x \neq x = p_1$$

Multiplying by $\sigma = 1/x$ converts the ODE into self-adjoint form:

$$x u'' + u' + \left(Ax - \frac{B}{x}\right)u = 0$$

$$\rightarrow p_0' = (x)' = 1 = p_1$$

Alternative representation:

Any self-adjoint operator can be written in another form:

$$\mathcal{L}u = p_0 u'' + p_1 u' + p_2 u = p_0 u'' + p_0' u' + p_2 u = (p_0 u')' + p_2 u =$$

$$= \frac{d}{dx}\left(p_0(x) \frac{d}{dx} u(x)\right) + p_2(x) u(x)$$

Example: Bessel's eq. was originally obtained in the form

$$\rho\left(\partial_\rho(\rho\,\partial_\rho R) + (\rho K^2 - \tfrac{1}{\rho}\nu)R\right) = 0$$

– just multiply by ρ^{-1} to convert into alternative representation

Example: (Legendre's eq.)

$$(1-x^2)u'' - 2xu' + \ell(\ell+1)u = 0 \qquad (x = \cos\theta)$$

$$P_0' = (1-x^2)' = -2x = P_1 \Rightarrow \text{formally self-adjoint!}$$

Alternative form:

$$(1-x^2)u'' + (1-x^2)'u' + \ell(\ell+1)u =$$

$$= ((1-x^2)u')' + \ell(\ell+1)u = 0.$$

Sturm-Liouville (eigenvalue) problem

The eigenvalues of an arbitrary (not necessarily self-adjoint) operator $\tilde{\mathcal{L}}$ are defined via

$$\tilde{\mathcal{L}}\,\psi_n(x) = \lambda_n \psi_n(x)$$

If we construct a self-adjoint operator $\mathcal{L} = \sigma\tilde{\mathcal{L}}$, this e-value problem is equivalent to $\mathcal{L}\psi_n(x) = \lambda_n w(x)\psi_n(x)$

"weight function" $w = \sigma(x)$

Self-adjoint operators were defined with respect to a scalar product

$$\langle u|\mathcal{L}|v\rangle = \int_a^b u(x)\mathcal{L}v(x)\,dx$$

Non-self-adjoint operators $\tilde{\mathcal{L}} = \frac{1}{\sigma(x)}\mathcal{L}$ will satisfy

$\langle u|\tilde{\mathcal{L}}v\rangle = \langle \tilde{\mathcal{L}}^+u|v\rangle$, if we redefine scalar product:

$$\langle u|\tilde{\mathcal{L}}|v\rangle \equiv \int_a^b u^*(x)\tilde{\mathcal{L}}v(x)w(x)\,dx = \int_a^b u^*(x)\frac{1}{\cancel{\sigma(x)}}\mathcal{L}v(x)\cancel{w(x)}\,dx$$

$$= \int_a^b \mathcal{L}^+u^*(x)\cdot v(x)\,dx = \int_a^b \mathcal{L}^+u^*(x)\,v(x)\frac{w(x)}{\sigma(x)}\,dx =$$

$$= \int_a^b \left(\frac{1}{\sigma(x)}\mathcal{L}\right)^+u^*(x)\,v(x)\,w(x)\,dx = \langle\tilde{\mathcal{L}}^+u|v\rangle$$

$$\Rightarrow \boxed{\langle u|v\rangle = \int_a^b u^*(x)v(x)w(x)\,dx, \quad w(x) = \sigma(x)}$$

If $\mathcal{L}$ is self-adjoint and $w(x)=1$ (or $w(x)=\sigma(x)$), then

1) eigenvalues λ_n are real:

$$\langle \psi_n | \mathcal{L}\psi_n \rangle = \langle \psi_n | \lambda_n w \psi_n \rangle = \lambda_n \langle \psi_n | w \psi_n \rangle = \lambda_n \int_a^b \psi_n^* \psi_n w \, dx$$

$$\parallel$$

$$\langle \mathcal{L}^+ \psi_n | \psi_n \rangle = \langle \lambda_n^* \underset{\uparrow \text{ real!}}{w} \psi_n | \psi_n \rangle = \lambda_n^* \int_a^b \psi_n \psi_n w \, dx \Rightarrow \lambda_n^* = \lambda_n$$

2) eigenfunctions are orthogonal

a) $\lambda_n \neq \lambda_m$:

$$\langle \psi_m | \mathcal{L}\psi_n \rangle = \lambda_n \int_a^b \psi_m^* \psi_n w \, dx$$

$$\parallel$$

$$\langle \mathcal{L}^+ \psi_m | \psi_n \rangle = \lambda_m^* \int_a^b \psi_m^* \psi_n w \, dx = \lambda_m \int_a^b \psi_m^* \psi_n w \, dx$$

b) $\lambda_m = \lambda_n$, $m \neq n$ can find linear combinations of ψ_m, ψ_n which are orthogonal.

3) the full set of eigenfunctions is complete, so that for "any" $f(x)$ we can find coeffs. c_n, such that

$$f(x) = \sum_n c_n \psi_n(x)$$

<u>Note:</u> This is just an ∞-dim. analog of the fact that an $N \times N$ Hermitian matrix has N eigenvectors spanning $\mathbb{R}^N$.

Remember the Gram-Schmidt orthogonalization procedure we have developed for matrices? It works here just as well!

Let us find the coefficients c_n:

$$\int_a^b \psi_m(x) f(x) w(x) dx = \int_a^b \psi_m(x) \sum_n c_n \psi_n(x) w(x) dx =$$

$$= \sum_n c_n \underbrace{\int_a^b \psi_m(x) \psi_n(x) w(x) dx}_{\delta_{nm} \text{ (if } \psi_n, \psi_m \text{ are also normalized)}} = c_m$$

$$\boxed{c_m = \int_a^b \psi_m(x) f(x) w(x) dx}$$ ← These are often called generalized Fourier coefficients of $f(x)$

Many similar properties, such as Parseval's identity:

$$\int_a^b f(x) g(x) w(x) dx = \int_a^b \sum_n F_n \psi_n(x) \sum_m G_m \psi_m(x) w(x) dx =$$

$$= \sum_{n,m} F_n G_m \int_a^b \psi_n(x) \psi_m(x) w(x) dx = \sum_n F_n G_n$$

$$\Rightarrow \int_a^b |f(x)|^2 w(x) dx = \sum_n |F_n|^2$$

Completeness relation (for eigenfunctions of self-adjoint operators)

For any $f(x)$:
$$f(x) = \sum_n F_n \psi_n(x) = \sum_n \int_a \psi_n(y) f(y) dy \, \psi_n(x) =$$

$$= \int_a^b \sum_n \psi_n(y) \psi_n(x) f(y) dy = \int_a^b \delta(x-y) f(y) dy$$

$$\Rightarrow \boxed{\delta(x-y) = \sum_n \psi_n(x) \psi_n(y)}$$