Georgia Tech PHYS 6124
Mathematical Methods of Physics I

Instructor: Predrag Cvitanović
Fall semester 2011

Homework Set #3

due Tue, Sept 13 2011, in class

(with **solutions**, September 12, 2011)

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

[problems from Stone and Goldbart]

Exercise 2.20 Test functions and distributions

Let $f(x)$ be a continuous function. Show that $f(x)\delta(x) = f(0)\delta(x)$. Deduce that

$$\frac{d}{dx}[f(x)\delta(x)] = f(0)\delta'(x)\,.$$

If $f(x)$ were differentiable we might also have used the product rule to conclude that

$$\frac{d}{dx}[f(x)\delta(x)] = f'(0)\delta(x) + f(0)\delta'(x)\,.$$

Show, by integrating both against a test function, that the two expressions for the derivative of $f(x)\delta(x)$ are equivalent.

Exercise 2.21 Let $\phi(x)$ be a test function...

Using the definition of the principal part integrals, show that

$$\frac{d}{dt}\left\{P\int_{-\infty}^{\infty}\frac{\phi(x)}{(x-t)}dx\right\} = P\int_{-\infty}^{\infty}\frac{\phi(x)-\phi(t)}{(x-t)^2}dx$$

in two different ways:

1. Fix the value of the cutoff ϵ. Differentiate the resulting ϵ-regulated integral, taking care to include the terms arising from the t dependence of the limits at $x = t \pm \epsilon$.

2. First make a change of variables $y = x - t$, so that the singularity is fixed at $y = 0$. Now differentiate with respect to t. Next integrate by parts to take the derivative off ϕ and onto the singular factor. (Take care to include the boundary contributions.) Finally change back to the original x, t variables.

Both methods should give the same result!

ChaosBook Exercise 16.1 (a) Integrating over Dirac delta functions

Check the delta function integrals in 1 dimension,

$$\int dx\,\delta(h(x)) = \sum_{\{x:h(x)=0\}} \frac{1}{|h'(x)|}\,, \tag{1}$$

and in d dimensions, $h : \mathbb{R}^d \to \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} dx\,\delta(h(x)) \quad = \quad \sum_j \int_{\mathcal{M}_j} dx\,\delta(h(x)) \;=\; \sum_{\{x:h(x)=0\}} \frac{1}{\left|\det \frac{\partial h(x)}{\partial x}\right|}\,. \tag{2}$$

where $\mathcal{M}_j$ are arbitrarily small regions enclosing the zeros x_j (with x_j not on the boundary $\partial\mathcal{M}_j$). For a refresher on Jacobian determinants, read, for example, Stone and Goldbart Sect. 12.2.2.

Solution

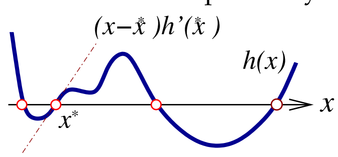
As long as the zero is not smack on the border of $\partial\mathcal{M}$, integrating Dirac delta functions is easy: $\int_{\mathcal{M}} dx\,\delta(x) = 1$ if $0 \in \mathcal{M}$, zero otherwise. The integral over a 1-dimensional Dirac delta function picks up the Jacobian of its argument evaluated at all of its zeros:

use gnu-plotted DiracGauss.eps here, once fixed

$$\int dx\,\delta(h(x)) = \sum_{\{x:h(x)=0\}} \frac{1}{|h'(x)|}\,, \tag{3}$$

and in d dimensions the denominator is replaced by

$$\int dx\,\delta(h(x)) \quad = \tag{4}$$



$$= \sum_j \int_{\mathcal{M}_j} dx\,\delta(h(x)) \;=\; \sum_{\{x:h(x)=0\}} \frac{1}{\left|\det \frac{\partial h(x)}{\partial x}\right|}\,.$$

(a) Whenever $h(x)$ crosses 0 with a nonzero velocity ($\det\partial_x h(x) \neq 0$), the delta function contributes to the integral. Let $x_0 \in h^{-1}(0)$. Consider a small neighborhood V_0 of x_0 so that $h : V_0 \to V_0$ is a one-to-one map, with the inverse function $x = x(h)$. By changing variable from x to h, we have

$$\begin{aligned}\int_{V_0} dx\, \delta(h(x)) &= \int_{h(V_0)} dh\, |\det\partial_h x|\, \delta(h) = \int_{h(V_0)} dh\, \frac{1}{|\det\partial_x h|}\delta(h) \\ &= \frac{1}{|\det\partial_x h|_{h=0}}\,.\end{aligned}$$

Here, the absolute value $|\cdot|$ is taken because delta function is always positive and we keep the orientation of the volume when the change of variables is made. Therefore all the contributions from each point in $h^{-1}(0)$ add up to the integral

$$\int_{\mathbb{R}^d} dx\, \delta(h(x)) = \sum_{x\in h^{-1}(0)} \frac{1}{|\det\partial_x h|}\,.$$

Note that if $\det\partial_x h = 0$, then the delta function integral is not well defined.

Optional problems

ChaosBook Exercise 16.1 (b) Integrating over $\delta(x^2)$

The delta function can be approximated by a sequence of Gaussians

$$\int dx\, \delta(x) f(x) = \lim_{\sigma \to 0} \int dx\, \frac{e^{-\frac{x^2}{2\sigma}}}{\sqrt{2\pi\sigma}} f(x)\,.$$

Use this approximation to see whether the formal expression

$$\int_{\mathbb{R}} dx\, \delta(x^2)$$

makes sense.

Solution

(b) The formal expression can be written as the limit

$$F := \int_{\mathbb{R}} dx\, \delta(x^2) = \lim_{\sigma \to 0} \int_{\mathbb{R}} dx\, \frac{e^{-\frac{x^4}{2\sigma}}}{\sqrt{2\pi\sigma}}\,,$$

by invoking the approximation given in the exercise. The change of variable $y = x^2/\sqrt{\sigma}$ gives

$$F = \lim_{\sigma \to 0} \sigma^{-3/4} \int_{\mathbb{R}^+} dy\, \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi y}} = \infty\,,$$

where $\mathbb{R}^+$ represents the positive part of the real axis. So, the formal expression does not make sense. The zero derivative of x^2 at $x = 0$ invalidates the expression in (a).

ChaosBook Exercise 16.2 Derivatives of Dirac delta functions

Consider $\delta^{(k)}(x) = \frac{\partial^k}{\partial x^k} \delta(x)$.
Using integration by parts, determine the value of

$$\int_{\mathbb{R}} dx\, \delta'(y) \quad , \qquad \text{where } y = f(x) - x \tag{5}$$

$$\int dx\, \delta^{(2)}(y) = \sum_{\{x: y(x)=0\}} \frac{1}{|y'|} \left\{ 3\frac{(y'')^2}{(y')^4} - \frac{y'''}{(y')^3} \right\} \tag{6}$$

$$\int dx\, b(x) \delta^{(2)}(y) = \sum_{\{x: y(x)=0\}} \frac{1}{|y'|} \left\{ \frac{b''}{(y')^2} - \frac{b' y''}{(y')^3} + b\left(3\frac{(y'')^2}{(y')^4} - \frac{y'''}{(y')^3} \right) \right\}. \tag{7}$$

Solution

We do this problem by direct evaluation. Denote by Ω_y a small neighborhood of y.

(a)

$$\begin{aligned}\int_{\mathbb{R}} dx\, \delta'(y) &= \sum_{x\in y^{-1}(0)} \int_{\Omega_y} dy \left|\frac{dx}{dy}\right| \delta'(y) \\ &= \sum_{x\in y^{-1}(0)} \frac{\delta(y)}{|y'|}\bigg|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \frac{\delta(y)}{y'^2}(-y'')\frac{1}{y'} \\ &= \sum_{x\in y^{-1}(0)} \frac{y''}{|y'|y'^2}\,.\end{aligned}$$

(b)

$$\begin{aligned}\int_{\mathbb{R}} dx\, \delta^{(2)}(y) &= \sum_{x\in y^{-1}(0)} \int_{\Omega_y} dy\, \frac{\delta^{(2)}(y)}{y'} \\ &= \sum_{x\in y^{-1}(0)} \frac{\delta'(y)}{|y'|}\bigg|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \frac{\delta'(y)}{y'^2}(-y'')\frac{1}{y'} \\ &= \sum_{x\in y^{-1}(0)} \frac{y''\delta(y)}{|y'|y'^2}\bigg|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \delta(y)\frac{d}{dx}(\frac{y''}{y'^3})\frac{1}{y'} \\ &= -\sum_{x\in y^{-1}(0)} \int_{\Omega_y} dy\, \delta(y)\left(\frac{y'''}{y'^3} - 3\frac{y''^2}{y'^4}\right)\frac{1}{y'} \\ &= \sum_{x\in y^{-1}(0)} \left(3\frac{y''^2}{y'^4} - \frac{y'''}{y'^3}\right)\frac{1}{|y'|}\,.\end{aligned}$$

(c)

$$\begin{aligned}\int_{\mathbb{R}} dx\, b(x)\delta^{(2)}(y) &= \sum_{x\in y^{-1}(0)} \int_{\Omega_y} dy\, b(x)\frac{\delta^{(2)}(y)}{y'} \\ &= \sum_{x\in y^{-1}(0)} \frac{b(x)\delta'(y)}{|y'|}\bigg|_{-\epsilon}^{\epsilon} - \int_{\Omega_y} dy\, \delta'(y)\frac{d}{dx}(\frac{b}{y'})\frac{1}{y'} \\ &= \sum_{x\in y^{-1}(0)} -\delta(y)\frac{d}{dx}(\frac{b}{y'})\frac{1}{y'}\bigg|_{-\epsilon}^{\epsilon} + \int_{\Omega_y} dy\, \delta(y)\frac{d}{dx}(\frac{d}{dx}(\frac{b}{y'})\frac{1}{y'})\frac{1}{y'} \\ &= \sum_{x\in y^{-1}(0)} \frac{1}{|y'|}\frac{d}{dx}(\frac{b'}{y'^2} - \frac{by''}{y'^3})) \\ &= \sum_{x\in y^{-1}(0)} \frac{1}{|y'|}\left[\frac{b''}{y'^2} - \frac{b'y''}{y'^3} - 2\frac{b'y''}{y'^3} + b(3\frac{y''^2}{y'^4} - \frac{y'''}{y'^3})\right]\end{aligned}$$

$$= \sum_{x \in y^{-1}(0)} \frac{1}{|y'|} \left[\frac{b''}{y'^2} - 3\frac{b'y''}{y'^3} + b(3\frac{y''^2}{y'^4} - \frac{y'''}{y'^3}) \right] .$$