28

Heat

The conservation of energy is one of the most fundamental laws of physics. It first arose as an almost trivial consequence of Newton's equations of motion but was later expanded into the present all-embracing notion during the development of thermodynamics by Carnot, Joule, Kelvin, Clausius, and others in the middle of the nineteenth century. Thermodynamics has turned out to be one of the most durable physical theories ever constructed. Although the basic laws are extremely simple, the fundamental concepts are abstract and often hard to connect with the complex reality of physics and chemistry. Paradoxically, thermodynamics is often perceived as very difficult at the first encounter.

The *First Law of Thermodynamics* states that there exists an abstract quantity, called *energy*, which is independent of the previous history of the system, and which does not change with time for an isolated system. Energy is abstract in the sense that we cannot afterwards tell how it was put into the system, whether by heat, by work, or by other means. The *Second Law of Thermodynamics* states that there is another abstract quantity, called *entropy*, which is likewise independent of the prehistory of the system, and which cannot diminish with time for a system in isolation. Entropy is perhaps best understood as a measure of *disorder*, informing us about the partition of the stored energy between the random motion of molecules that we call heat and other forms of energy more directly convertible to work. The inexorable growth of entropy with time parallels the intuitive notion that disorder spontaneously increases for a system in isolation.

In this chapter only the First Law of thermodynamics for continuous systems is discussed, completing thereby the analysis of laws of balance begun in chapter 17. For more complete presentations of continuum thermodynamics and the Second Law, see [18, 26, 10, 62, 65]. Convection, the dynamic interplay of flow and heat, will be treated in the following chapter.

28.1   Energy balance

The *First Law of Thermodynamics* postulates 1) that the energy $\mathcal{E}$ of a physical system is only a function of the instantaneous state, *i.e.* of the macroscopic fields which characterize the state of the system, and 2) that energy is conserved in a system that is isolated from its environment. For a non-isolated system, any change in energy can therefore be accounted for by the actions of the environment. In terms of the heat Q transferred to the system and the work W performed on the system, the First Law takes the form,

$$\boxed{\Delta\mathcal{E} = Q + W \ .}$$
(28-1)

Whereas the amounts of heat and work may both depend on the history of how the environment interacted with the system, their sum does not, because energy is only a function of the instantaneous state. If the environment can influence the system in other ways, for example through chemical reactions, further terms will appear on the right hand side.

First Law in continuum physics

Continuous systems are of course also physical systems, so the First Law must by default be valid in continuum physics, provided the possibility of material flow through the surface of the control volume is taken properly into account. In an arbitrary moving control volume, the First Law turns into an equation of balance for the material rate of change of the total energy,

$$\boxed{\frac{D\mathcal{E}}{Dt} = \dot{Q} + \dot{W} \ ,}$$
(28-2)

where $\dot{Q}$ is the rate at which the environment transfers heat to the system, and $\dot{W}$ is the rate of work performed on the system by the environment.

It is a basic assumption in continuum physics that energy is an extensive quantity, transported by material particles. The energy of a control volume is as any other extensive quantity an integral,

$$\mathcal{E} = \int_V \rho\,\epsilon\,dV \ ,$$
(28-3)

where the energy density $\rho\,\epsilon = d\mathcal{E}/dV$ is the product of the mass density ρ and the *specific energy* $\epsilon = d\mathcal{E}/dM$. The material time derivative of the energy then obeys Reynolds theorem (17-9) with ρ replaced by $\rho\,\epsilon$,

$$\frac{D\mathcal{E}}{Dt} = \frac{d\mathcal{E}}{dt} + \oint_S \rho\,\epsilon\,(\boldsymbol{v} - \boldsymbol{v}_S)\cdot d\boldsymbol{S} \ .$$
(28-4)

The last term represents the net outflow of energy from the control volume.

Rate of work

The environment can perform work on a system through long-range body forces such as gravity and electromagnetism, and through contact forces acting on the surface of the control volume. The rate of work of the contact forces is,

$$\dot{W} = \oint_S \boldsymbol{v} \cdot \boldsymbol{\sigma} \cdot d\boldsymbol{S} \ , \qquad (28\text{-}5)$$

where $\boldsymbol{\sigma} = \{\sigma_{ij}\}$ is the stress field. If external body forces, for example gravity, also act on the material in the control volume, their rate of work must be added to the right hand side. Alternatively, a static external gravitational field can be included as a potential energy term in the total energy (see section 17.10 on page 320), and the same is possible for external electrostatic forces. When all body forces are "internalized" as potential energy, we only need to account for the environment's work on the system through contact forces, as given by the above expression. We shall assume this to be the case in the following.

Rate of heat transfer

The everyday experience of handling a hot potato tells us that heat can be carried along with the movement of matter, but that possibility is already taken care of in the equation of energy balance (28-2) through the use of the material derivative, effectively turning the arbitrary control volume into a comoving volume. When we heat water to make tea, it becomes equally clear that heat can also be conducted through the solid bottom of the kettle. Heat transfer by conduction is not an advective transport phenomenon tied to the motion of material particles, but is rather a diffusion process by which the warmer material particles of the wall "infect" the colder particles of the water with heat.

Macroscopically we describe the flow of heat through a surface by a current density (vector) field $\boldsymbol{q}(\boldsymbol{x}, t)$, defined such that the amount of heat which flows through a surface element $d\boldsymbol{S}$ in the time interval δt is $\delta Q = \boldsymbol{q}\delta t \cdot d\boldsymbol{S}$. The total rate of heat transfer *into* the system thus becomes,

$$\dot{Q} = - \oint_S \boldsymbol{q} \cdot d\boldsymbol{S} + \int_V h \, dV \ , \qquad (28\text{-}6)$$

where we have also allowed for the possibility that heat may be produced at a rate $h(\boldsymbol{x}, t)$ per unit of volume by chemical, nuclear, or other processes.

> **Example 28.1.1 (Geothermal heat flow):** The Earth produces heat in the interior by radioactivity, and the average geothermal heat current is $q_c \approx 0.06$ W/m^2 for the continents and $q_s \approx 0.10$ W/m^2 for the seas [2, p. 192]. Using that the Earth's mean radius is $a = 6371$ km, and that the continents cover a fraction $f \approx 29\%$ of the surface, the total rate of heat flow out of the Earth is $\oint_S \boldsymbol{q} \cdot d\boldsymbol{S} = 4\pi a^2(fq_c + (1-f)q_s) \approx 45 \times 10^{12}$ W. For comparison, the present world power consumption is about 13×10^{12} W.

Internal energy

Separating out the kinetic energy $\mathcal{T}$ and the potential energy $\mathcal{V}$, we shall write the total energy as,

$$\boxed{\mathcal{E} = \mathcal{T} + \mathcal{V} + \mathcal{U} \ ,} \tag{28-7}$$

where $\mathcal{U}$ is the *internal energy*. Since energy is an extensive quantity, the internal energy must also be an integral over a density,

$$\mathcal{U} = \int_V \rho \, u \, dV \ , \tag{28-8}$$

where $u = d\mathcal{U}/dM$ is the *specific internal energy*. Using the usual expressions for the kinetic energy (17-72) and the potential energy (17-89) the specific total energy becomes

$$\boxed{\epsilon = \frac{1}{2}\boldsymbol{v}^2 + \Phi + u \ ,} \tag{28-9}$$

At this point we do not know the nature of the system and cannot be more explicit about the precise form of the specific internal energy.

Using energy balance (28-2) and mechanical energy balance (17-91) we obtain

$$\frac{D\mathcal{U}}{Dt} = \frac{D\mathcal{E}}{Dt} - \frac{D\mathcal{T}}{Dt} - \frac{D\mathcal{V}}{Dt} = \dot{W} + \dot{Q} - \widetilde{P} \ ,$$

where $\widetilde{P}$ is the reduced power, *i.e.* the total power (17-78) without the contribution due to the rate of work of gravity. Assuming as before that there are no other volume forces at play than static gravity, this reduces to,

$$\boxed{\frac{D\mathcal{U}}{Dt} = \dot{Q} + \int_V \sum_{ij} \sigma_{ij} \nabla_j v_i \, dV \ .} \tag{28-10}$$

This is the global equation of *internal energy balance*. Comparing with the expression for the total power (17-78), we conclude that the last term represents the rate at which kinetic energy is converted into internal energy by internal stresses.

The surface integral in $\dot{Q}$ is transformed into a volume integral by means of Gauss' theorem, and making use of an equation analogous to (17-13), it follows that the specific internal energy must obey the local equation,

$$\boxed{\rho \frac{Du}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{q} + h + \sum_{ij} \sigma_{ij} \nabla_j v_i \ .} \tag{28-11}$$

Given explicit expressions for the specific internal energy u, the current of heat conduction $\boldsymbol{q}$, the rate of heat production per unit of volume h, and the internal stress field $\boldsymbol{\sigma}$, this equation becomes a dynamic equation for the temperature field, called the *heat equation*. In the following sections we shall write down this general equation under various simplifying assumptions.

28.2 Heat equation for isotropic matter at rest

The Second Law of Thermodynamics implies that in an isolated system heat will spontaneously stream from regions of higher to regions of lower temperatures (and conversely: this general rule is equivalent to the Second Law). We expect for this reason that the heat flow q is locally related to the gradient of the temperature field $\boldsymbol{\nabla} T$. In isotropic matter, the simplest local relationship is *Fourier's law of heat conduction*,

$$\boxed{q = -k\boldsymbol{\nabla} T \ ,} \tag{28-12}$$

where the sign has been chosen so that heat is conducted from hot to cold. The positive constant k is called the *thermal conductivity* of the material, and is measured in units of watts per kelvin per meter. For water it is $k \approx 0.6$ W/Km and for air $k \approx 0.025$ W/Km.

Jean Baptiste Joseph Fourier (1768–1830). *French scientist who made fundamental contributions to mathematics (Fourier series) and to the theory of heat.*

In the following we shall always assume Fourier's law to be valid.

> **Example 28.2.1 (Geothermal gradient):** The average continental geothermal heat flow is $q_c \approx 0.06$ W/m^2 [2]. Taking the thermal conductivity of bedrock to be $k \approx 2$ W/Km [3], the average geothermal temperature gradient in the upper crust becomes $|\boldsymbol{\nabla} T| \approx q_c/k \approx 0.03$ K/m, or 30 kelvin per kilometer. It must be emphasized that the geothermal gradient varies strongly from place to place because of variations in the composition of the upper crust.

Fourier's equation

The simplest of all materials is an incompressible homogeneous isotropic fluid (or solid) at rest with constant mass density field, $\rho = \rho_0$. In such a material there is only one thermodynamic variable, which may be taken to be the temperature field $T = T(\boldsymbol{x}, t)$, and any other local thermodynamic quantity, for example the specific energy $u = u(T)$, becomes a local function of the temperature. We shall for simplicity assume that the specific energy is linear in the temperature, $u = c_0 T$, or at least linear in a certain temperature range. The constant c_0 is the *specific heat capacity* of the material, defined as the amount of heat necessary to raise a unit of mass by one unit of temperature[1].

Inserting $u = c_0 T$ and $\boldsymbol{v} = \boldsymbol{0}$ into the equation of local internal energy balance (28-11), and using Fourier's law of heat conduction (28-12), we obtain the heat equation for isotropic matter at rest,

$$\rho_0 c_0 \frac{\partial T}{\partial t} = k\boldsymbol{\nabla}^2 T + h \ . \tag{28-13}$$

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[1] There is a small ambiguity of language when we use the verb "heat" to mean both to "raise the temperature" and to "add heat". It is perfectly possible to add heat to a system without raising its temperature, for example when boiling water. Conversely, the temperature may rise without heat transfer when you pump your bicycle.

Dividing by $\rho_0 c_0$ it takes for $h = 0$ the form of a standard diffusion equation, called *Fourier's equation*

$$\boxed{\frac{\partial T}{\partial t} = \kappa \mathbf{\nabla}^2 T \, ,}$$
(28-14)

with *heat diffusivity*,

$$\kappa = \frac{k}{\rho_0 c_0} \, .$$
(28-15)

For water we have $\kappa \approx 1.4 \times 10^{-7}$ m^2/s which is about 6 times smaller than the momentum diffusivity (kinematic viscosity) ν. Somewhat surprisingly, the heat diffusivity of air turns out to be 140 times larger than for water (see page 588).

> Why is it then that we use air for insulation in thermoglass windows, bed covers, and winter coats — rather than sleeping and walking in wetsuits? The explanation is that although heat diffuses about 140 times quicker in air than in water, the actual heat current (28-12) is for a given temperature gradient not determined by the diffusivity but by the thermal conductivity, k, which is about 25 times larger in water than in air. Consequently, you loose less heat in a fur coat than in a wetsuit even if the cold penetrates the fur coat much faster. The role of a fur coat or wetsuit is mainly to prevent advection of heat by air or water currents which will rapidly remove the warm fluid adjacent to your skin, increasing thereby the temperature gradient at the skin and thus the heat flow from your body.
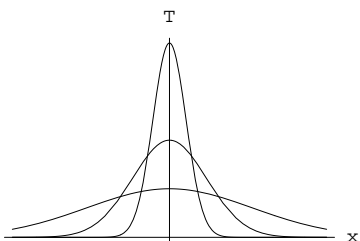
Heat diffusion time

The analogy between Fourier's equation (28-14) and the planar momentum diffusion equation (18-5) on page 331 allows us immediately to take over the solution (18-12) and adapt it to a uniform planar Gaussian temperature distribution in the yz-plane. Let the planar temperature field initially at $t = 0$ be Gaussian, $T = T_0 + \Theta \exp(-x^2/a^2)$, where Θ is the temperature excess at $x = 0$. At a later time the temperature field will then be,

$$T(x,t) = T_0 + \Theta \frac{a}{\sqrt{a^2 + 4\kappa t}} \exp\left(-\frac{x^2}{a^2 + 4\kappa t}\right) \, .$$
(28-16)

Heat diffusion makes a Gaussian temperature distribution widen and diminish in height as time goes by.

It may of course be verified by direct insertion that this solution indeed satisfies Fourier's equation. The temperature distribution thus remains Gaussian at all times with a width that grows like $\sqrt{a^2 + 4\kappa t}$ with time. Since the environment transfers no heat to the system and performs no work on it (because the material is everywhere at rest), the total internal energy in the heated material must remain constant while it spreads away from the central region (problem 29.5).

At large times, $t \gg a^2/4\kappa$, the Gaussian factor $\exp(-x^2/4\kappa t)$ becomes universal for all bounded temperature distributions. The strong fall-off of the Gaussian makes it appear as if the temperature excess $\Delta T = T - T_0$ expands with a well-defined front. At the distance $x = 2\sqrt{\kappa t}$ the Gaussian has fallen to $e^{-1} \approx 37\%$

of its central value. Depending on the application it may be more convenient to choose a more conservative front, for example $x = 3.5\sqrt{\kappa t}$ where the Gaussian has fallen to 5% of its central value.

Conversely, one may also characterize heat diffusion by the characteristic time it takes for the front to reach a distance $x = L \gg a$. At the 37% level, this time becomes,

$$t \approx \frac{L^2}{4\kappa} , \qquad (28\text{-}17)$$

For a more conservative definition of the front, a correspondingly smaller estimate may be used, for example $t \approx L^2/12\kappa$ for the 5% level. Such estimates are in fact very general and may be used to get an idea of the heat diffusion time in any system.

> **Example 28.2.2 (Hot porridge):** Everybody has painfully learnt early in life that the center of a bowl of hot porridge may remain hot for a long time, even as the periphery grows cold. Taking the heat diffusivity of porridge equal to that of water, the time it takes for a spherical ball of hot porridge of radius $a \approx 5$ cm to reach room temperature is estimated from (28-17) to be 4500 seconds or about an hour and a quarter. Parents have found a solution to this problem (see example 28.3.1 below).

Planar heat wave

Similarly, we may take over the plane wave solution (18-14). Suppose the temperature is forced to oscillate sinusoidally in the plane $z = 0$ with period τ and amplitude Θ around the mean temperature T_0, such that $T = T_0 + \Theta \cos(2\pi t/\tau)$ at $z = 0$. Recasting the solution (18-14) in terms of these variables, with $\omega \to 2\pi/\tau$ and $k \to 1/d$, it becomes a damped heat wave in which the temperature at depth z is,

$$T = T_0 + \Theta\, e^{-z/d} \cos\left(2\pi\frac{t}{\tau} - \frac{z}{d}\right) , \qquad d = \sqrt{\frac{\kappa\tau}{\pi}} , \qquad (28\text{-}18)$$

where d is the penetration depth. The corresponding wavelength is $\lambda = 2\pi d$, and at a depth of one wavelength the damping factor is $e^{-2\pi} = 1.8 \times 10^{-3}$.

> **Example 28.2.3 (Annual soil temperature variation):** The surface temperature of soil follows the annual variations in atmospheric temperature with period $\tau = 1$ year $= 3.2 \times 10^7$ s. The thermal conductivity of soil consisting of sand, stones, and clay, is probably not unlike that of water with, say, $k \approx 0.5$ W/Km, density $\rho \approx 2$ g/cm^3, and specific heat capacity about $c_0 \approx 2$ J/g K, implying a heat diffusivity of around $\kappa \approx 1.3 \times 10^{-7}$ m^2/s, which is nearly identical to that of water. The soil penetration depth becomes $d = 1.1$ m, meaning that the amplitude of the temperature variations at this depth has fallen to $e^{-1} = 37\%$ of the surface amplitude (see fig. 28.1). Since the wave length is $2\pi \approx 6$ times the penetration depth, the surface temperatures will be delayed by about 2 months at $z = d$. Denmark
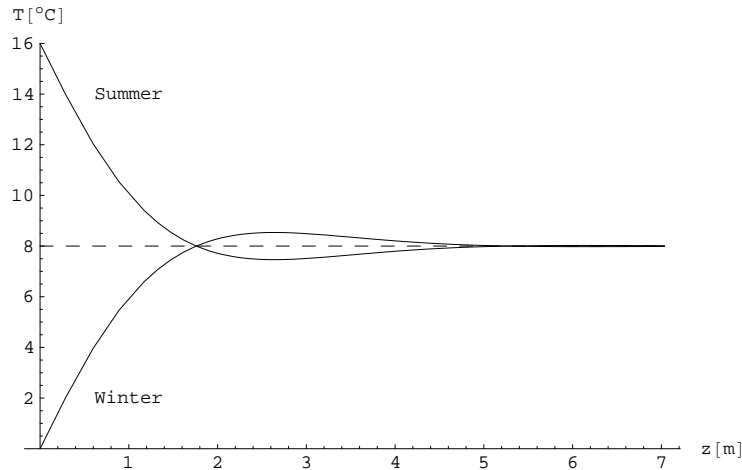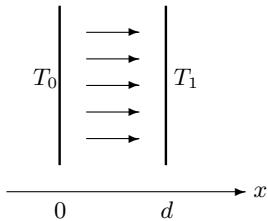
Figure 28.1: *Calculated ground penetration of mean summer and winter temperatures in Denmark as function of depth z (example 28.2.3). It takes nearly two months to penetrate one meter of depth, and beyond $z = 5$ m there is no yearly variation.*

has for example a temperate Northern climate with mean temperature $T_0 \approx 8°$C and a yearly variation of $\Theta \approx 8°$C. Freezing temperatures occur often in winter, even if the average barely gets below $0°$C, but a freezing spell will have to last more than 2 months to penetrate to a depth of one meter, and that is highly improbable. Frost-free depth is accordingly defined to be 90 cm for water mains and foundations of houses.

Steady planar heat flow

Suppose a slab of isotropic material at rest is enclosed between two infinitely extended flat plates held at different temperatures, T_0 and $T_1 = T_0 + \Theta$. The planar symmetry suggests that the temperature field only depends on the transverse coordinate, $T = T(x)$. Inserting this into Fourier's equation we obtain $\nabla_x^2 T = 0$ which has the solution

$$T(x) = T_0 + \Theta \frac{x}{d} , \tag{28-19}$$

where d is the distance between the plates. Although heat conduction is necessary for establishing a flow of heat, the final distribution does not depend on the value of the coefficient of thermal conductivity. As we shall now see, thermal conductivity determines instead the time scale for the appearance of a steady flow pattern. This is completely analogous to velocity-driven planar flow between moving plates which does not depend on the viscosity (section 18.2 on page 331), although viscosity does determine the time scale for the flow to settle down into a steady pattern.

Steady heat flow between parallel plates at different temperatures. The plates continue far above and below the section shown here.

Steady heat production

If heat is produced at a constant rate h per unit of volume and if the environment has constant properties, an equilibrium will eventually be attained in which the heat produced in a volume equals the amount of heat that leaves the volume. The temperature field will then become time independent and obey the steady heat equation (28-13),

$$k\boldsymbol{\nabla}^2 T + h = 0 \ . \tag{28-20}$$

Given the heat production and suitable boundary conditions this equation determines the temperature distribution. Often h will itself vary with temperature because the heat producing processes are temperature dependent.

Suppose, for example, that heat is produced at a fixed rate $h(\boldsymbol{x}, t) = h_0$ inside a sphere of radius a, making the total heat rate $\dot{Q} = \frac{4}{3}\pi a^3 h_0$. The spherical solution to the steady heat equation (28-20) then becomes,

$$T = T_s + \frac{h_0}{6k}\left(a^2 - r^2\right) \ , \tag{28-21}$$

where T_s is the (constant) surface temperature. The temperature is (naturally) highest in the center for $h_0 > 0$. Averaging over the sphere, the mean temperature becomes (see problem 28.1),

$$\langle T \rangle = T_s + \frac{h_0}{15k}a^2 \ . \tag{28-22}$$

The average temperature excess in the sphere is thus merely 2/5 of the temperature difference between the center and the surface.

> **Example 28.2.4 (Geothermal heat production):** The total geothermal heat output of the Earth is $\dot{Q} \approx 45 \times 10^{12}$ W (see example 28.1.1), implying an average (radioactive) heat production rate $h_0 \approx 4 \times 10^{-8}$ W/m^3. If the Earth were made from uniform material with $k \approx 2$ W/Km, the central temperature excess would be $T_c - T_s \approx 140,000$ K. This estimate fails miserably in comparison with geophysical models which place the central temperature at $4,000 - 6,000$ K, demonstrating that the Earth is a highly non-uniform object with respect to heat transfer. In the liquid mantle rapid convection rather than slow conduction is the dominant mechanism of heat transfer, although thermal conductivity also increases with depth.

> **Example 28.2.5 (Human skin temperature):** The average heat output from a human being (used for example for dimensioning cooling systems for concert halls) is $\dot{Q} \approx 100$ W. With a typical mass of $M \approx 70$ kg and density equal to that of water $\rho_0 \approx 1000$ kg/m^3 the volume becomes $V = 0.07$ m^3, implying an average heat production density rate $h_0 \approx 1.4$ kW/m^3. In the "spherical approximation" a human body of this mass would have radius $a \approx 0.25$ m, and using (28-22) with $k = 0.6$ W/Km we obtain the difference between the average temperature and the skin temperature $\langle T \rangle - T_s \approx 10$ K. Taking $\langle T \rangle = 37°$C, the skin temperature is predicted to be $27°$C. The conclusion is that a naked human being should be able to survive "indefinitely" in water of this temperature, a result which agrees decently with common experience (see also problem 28.2).

28.3   Heat equation for fluids in motion

The dominant effect of the velocity of a moving fluid is advective transport of internal energy, expressed through the material derivative on the left hand side of the equation of local internal energy balance (28-11). The last term on the right hand side also depends on the velocity but it only plays a role for compressible or viscous fluids. Even then, dissipative heating will only be important under extreme circumstances, for example at the leading edges of a supersonic aircraft or a spacecraft reentering the atmosphere.

Incompressible inviscid fluid

For an incompressible inviscid fluid the heat equation takes (in the absence of heat production) almost the same form as Fourier's equation (28-14),

$$\boxed{\frac{\partial T}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})T = \kappa \boldsymbol{\nabla}^2 T \; .}$$

(28-23)

The extra advective term on the left hand side expresses that heat moves along with the flow of matter, whereas the right hand side expresses that it also streams down temperature gradients as for a fluid at rest. We shall see below that this heat equation can essentially always be maintained when the fluid is nearly inviscid and the flow speed is everywhere small compared to the velocity of sound.

The Péclet number

The time it takes for heat to be advected by the flow of fluid with velocity U through a distance L is $t_{\mathsf{adv}} \sim L/U$ whereas the time it takes for it to diffuse through the same distance is $t_{\mathsf{diff}} \sim L^2/\kappa$, disregarding all purely numeric factors. The ratio of the time scales for diffusion and flow is an important dimensionless quantity, called the *Péclet number*, $\mathsf{Pe} \sim t_{\mathsf{diff}}/t_{\mathsf{adv}} \sim UL/\kappa$.

   More formally, and in analogy with the Reynolds number, the Péclet number is defined to be the ratio of the advective to diffusive terms in the heat equation,

Jean-Claude Eugene Péclet (1793–1857). *French physicist. One of the first scholars of Ecole Normale (Paris); known for his clarity of style, sharp-minded views and well-performed experiments.*

$$\mathsf{Pe} = \frac{|\rho_0 c_0 (\boldsymbol{v} \cdot \boldsymbol{\nabla})T|}{|k \boldsymbol{\nabla}^2 T|} \approx \frac{UL}{\kappa} \; .$$

(28-24)

The value of the Péclet number determines whether heat flow in a fluid is dominated by advection or by conduction. For large Péclet number the right hand side of the heat equation (28-23) can be disregarded, and the heat equation now simply says that the temperature is constant around any comoving material particle. The smallness of the heat diffusivity in most materials usually makes the Péclet number quite large for flows of ordinary dimensions and flow speeds. For small Péclet number, the advective term can be disregarded, and the heat equation reverts to that of a fluid at rest.

Example 28.3.1 (Cooling hot porridge): The parental solution to the problem of the hot porridge (see example 28.2.2) is to stir it a few times with a spoon. Stirring sets the porridge into motion and the heat (or rather the internal energy) is carried along with the porridge by advection. Mixing the hot and cold regions increases the local temperature gradients, resulting in a large diffusive heat flow which quickly evens out the temperature differences and makes the porridge eatable. During stirring, layers of hot and cold porridge are interwoven with each other, lowering the typical distance scale to perhaps $L = 1$ cm. Taking $U = 10$ cm/s and the diffusivity to be approximatively that of water, $\kappa \approx 1.4 \times 10^{-7}$ m^2/s, we find a Péclet number of Pe $\approx 7,000$, showing that heat conduction can safely be ignored while you stir. Diffusive cooling due to the larger temperature gradients must mainly take place when you stop stirring after a few seconds. The heat diffusion time (28-17) in the stirred porridge is now estimated to be of the order of minutes.

Compressible ideal gas

In section 17.11 on page 322 we calculated the specific internal energy (17-97) of a compressible ideal gas with adiabatic index γ. As for incompressible fluid, the specific energy density was found to be linear in the absolute temperature[2],

$$u = c_v T \ , \qquad\qquad c_v = \frac{1}{\gamma - 1} \frac{R}{M_{\mathsf{mol}}} \ , \qquad (28\text{-}25)$$

where the constant c_v is *the specific heat at constant density (i.e. volume)*. It represents — as the first equation shows — the amount of *energy* that is necessary to raise a unit of mass of the gas by one unit of temperature. If the density (volume) is held constant, so that no work is performed on the gas, this is the same as the amount of *heat* that must be transferred to a unit mass of the gas to raise its temperature by one unit.

If on the other hand the pressure is held constant, the amount of *heat* that must be transferred to a unit mass of the gas to raise its temperature by one unit is instead (see for example [6]),

$$c_p = c_v + \frac{R}{M_{\mathsf{mol}}} = \frac{\gamma}{\gamma - 1} \frac{R}{M_{\mathsf{mol}}} \ , \qquad (28\text{-}26)$$

called the *specific heat at constant pressure* or the *isobaric specific heat*.

Disregarding viscous friction and heat production, local internal energy balance (28-11) becomes,

$$\boxed{\rho c_v \left(\frac{\partial T}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) T \right) = k \boldsymbol{\nabla}^2 T - p \boldsymbol{\nabla} \cdot \boldsymbol{v} \ ,} \qquad (28\text{-}27)$$

where the pressure is given by the ideal gas law, $p = (R/M_{\mathsf{mol}}) \rho T$. The first source term on the right hand side represents the local inflow of heat by conduction

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[2]Although this expression was derived under the assumption of adiabatic (isentropic) processes, it must be correct in general because energy is only a function of the final state and not of the particular road that leads to this state, whether adiabatic or not.

whereas the last represents the work of local compression (because it is positive when $\boldsymbol{\nabla} \cdot \boldsymbol{v} < 0$). We shall show below that if the flow velocity is much smaller than the speed of sound, the local compression source can be "moved over" to the left hand side, replacing the specific heat at constant volume by the specific heat at constant pressure,

$$\rho c_p \left(\frac{\partial T}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})T \right) = k\boldsymbol{\nabla}^2 T \; . \tag{28-28}$$

It is of the same form as for an incompressible fluid, except for the use of the specific heat at constant pressure, but it should be remembered that this is only an approximation to the correct equation (28-27) valid at small flow speeds.

The heat diffusivity is now defined as,

$$\kappa = \frac{k}{\rho c_p} \; . \tag{28-29}$$

For air at normal temperature and pressure, we have $\kappa \approx 2 \times 10^{-5}$ m^2/s which is about 140 times larger than that of water.

Derivation of the heat equation (28-28)

Using the equation of state $p = (R/M_{\mathsf{mol}})\rho T$ we get,

$$\frac{Dp}{Dt} = \frac{R}{M_{\mathsf{mol}}} \left(\frac{D\rho}{Dt}T + \rho\frac{DT}{Dt} \right) = -p\boldsymbol{\nabla} \cdot \boldsymbol{v} + (c_p - c_v)\rho\frac{DT}{Dt} \; .$$

In the last step we have used the equation of continuity (15-29) to eliminate the density derivative. The heat equation may now be written

$$\rho c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = k\boldsymbol{\nabla}^2 T \; . \tag{28-30}$$

For steady flow and in the absence of gravity, Bernoulli's theorem (16-16) relates a change in pressure to a change in flow velocity, $\Delta p \approx -\frac{1}{2}\rho\Delta(v^2)$. Since a change in the absolute temperature is related to a change in the velocity of sound (16-8), from which we get $\Delta T \sim \Delta(c_s^2)M_{\mathsf{mol}}/R\gamma$. The ratio of the two terms on the left hand side of the heat equation may thus be estimated to be,

$$\frac{\left| \dfrac{Dp}{Dt} \right|}{\left| \rho c_p \dfrac{DT}{Dt} \right|} \approx \frac{\Delta p}{\rho c_p \Delta T} \sim \frac{\Delta(v^2)}{\Delta(c_S^2)} \sim \frac{v^2}{c_S^2} \; . \tag{28-31}$$

In the last steps we have disregarded all factors of order unity. This argument suggests that the material pressure derivative can be disregarded in steady flow as long as the flow speed is much smaller than the speed of sound (which also makes the flow effectively incompressible; see section 16.4 on page 269). For general unsteady flow there will be further conditions on the rate of change of the pressure, $\partial p/\partial t$, and on the field of gravity (see [18] for a more careful discussion).

General isotropic fluid

In a general isotropic fluid, the specific internal energy will depend on both the temperature and the mass density, $u = u(T, \rho)$, and derivation of the heat equation becomes considerably more involved, requiring the full apparatus of thermodynamics. An account close in spirit to the presentation in this chapter is found in [18]. The end result is that for nearly inviscid and nearly incompressible flows the heat equation again takes the form (28-28) with heat diffusivity determined from the isobaric specific heat of the fluid (28-29). In liquids there is not a great difference between the specific heats at constant volume and at constant pressure.

The internal friction caused by the stresses in a viscous fluid leads to a loss of kinetic energy which reappears as source of internal energy (*i.e.* internal work) in the last term in the equation of internal energy balance (18-30). Including the dissipative power density (18-18) for general Newtonian fluids, the heat equation becomes,

$$\rho c_p \left(\frac{\partial T}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})T \right) = k\boldsymbol{\nabla}^2 T + h + 2\eta \sum_{ij} v_{ij}^2 + \zeta(\boldsymbol{\nabla} \cdot \boldsymbol{v})^2 \ . \qquad (28\text{-}32)$$

where h is the local rate of heat production, v_{ij} the symmetric strain rate (18-26), η the shear viscosity, and ζ the bulk viscosity.

In most everyday flows heat production by dissipation will be negligible in comparison with advection and conduction. In extreme situations with strong compression and shear, temperatures can become extremely high. A hypersonic object like a "shooting star" may even burn up completely when entering the atmosphere at more than 10 km/s.

28.4 Advective cooling or heating

When you take a walk on a cold day, your heat loss is amplified by wind which removes the warm air near your body and creates a large temperature gradient at the surface of the skin, resulting in a larger conductive heat transfer from your body to the air. In meteorology this is known as *wind chill*, and the local "wind chill temperature" is often announced by weather forecasters during winter. A similar phenomenon must of course occur in a hot desert wind, although the local "wind burn" is not a regular part of a summer weather forecast. In a sauna with air at 120° C, it is well-known that one should not move around too fast.

Water chill is even more important. A cold water current moving with the same speed and temperature as a cold wind results in much stronger cooling because of the 25 times higher thermal conductivity of water. This is why you are only able to survive naked for minutes in streaming water at 0 °C whereas you may survive for hours in a wind of that temperature. Similarly, hot water scalds you much faster than hot air of the same temperature.

In this section we shall discuss the limit where the temperature does not influence the motion of the fluid. In that case the flow of heat takes place on the background of a mass flow, completely controlled by the external forces that drive the fluid. This limit is often called of *forced* convection to distinguish it from the opposite limit, *free* convection, where the motion of the fluid is entirely caused by temperature differences (to be discussed at length in chapter 29).

The Prandtl number

Advective cooling (or heating) of the surface of a body involves both momentum and heat diffusion along the normal to the surface. In a time t after the start of the flow, momentum diffusion reaches to a characteristic distance $\delta_{\mathsf{mass}} \sim \sqrt{\nu t}$ from the surface whereas heat diffusion reaches to $\delta_{\mathsf{heat}} \sim \sqrt{\kappa t}$. The ratio between momentum and heat diffusivities is for this reason an important dimensionless quantity, called the *Prandtl number*,

$$\mathsf{Pr} = \frac{\nu}{\kappa} \ . \tag{28-33}$$

When the Prandtl number is large, temperature variations will take place well inside the usual boundary layer, whereas if the Prandtl number is small, the temperature distribution spreads well beyond the boundary layer.

> In contrast to other dimensionless numbers, for example the Reynolds or Péclet numbers, the Prandtl number is a property of the fluid rather than of the flow. In gases it is of order unity, for example $\mathsf{Pr} = 0.73$ for air at normal temperature and pressure. In liquids it may take a wide range of values: in water it is about 6, whereas in liquid metals it is quite small, for example 0.025 for mercury, and for isolating liquids like oil it may be quite large, of the order of 1000.

Wind chill estimate

In a steady flow with velocity scale U boundary layers will stop growing after the time, $t \sim L/U$, it takes for the fluid to move across the downstream length L of the body. The scale of the boundary layer thickness thus becomes $\delta_{\mathsf{heat}} \sim \sqrt{\kappa L/U}$, and the temperature gradient at the surface is expected to be $|\boldsymbol{\nabla} T| \approx \Theta/\delta_{\mathsf{heat}}$, where Θ is the temperature excess of the body relative to the fluid at large. From Fourier's law (28-12) we estimate that the rate of loss of heat from a body surface of area A is,

$$\dot{Q} \sim k \frac{\Theta}{\delta_{\mathsf{heat}}} A \sim k \Theta A \sqrt{\frac{U}{\kappa L}} \ . \tag{28-34}$$

We shall show below in an exact calculation that this is indeed of the right form.

The main consequence of the above estimate is that the heat loss, $\dot{Q} \sim \Theta\sqrt{U}$, grows with the squareroot of the velocity, thus confirming the observation that a higher velocity has the same effect as a larger temperature excess. Let the actual temperature excess be $\Theta = T_1 - T_0$ where T_1 is the exposed surface temperature
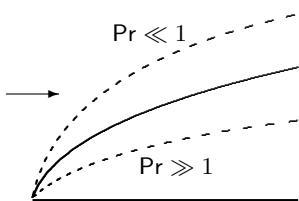
Advective cooling of a plate with wind coming in from the left. The boundary for the mass flow (fully drawn) and the heat fronts (dashed) for large and small Prandtl numbers.

and T_0 is the wind temperature, and let the actual wind velocity be U. Then the heat loss will be the same for a temperature excess $\Theta^* = T_1 - T_0^*$ and a wind speed U^*, satisfying $\Theta^*\sqrt{U^*} = \Theta\sqrt{U}$. Solving for the (fictive) wind temperature T_0^* we find,

$$T_0^* = T_1 - (T_1 - T_0)\sqrt{\frac{U}{U^*}} \ . \qquad (28\text{-}35)$$

Apart from corrections for high wind speeds, this is essentially identical to the first *wind chill formula* by Siple and Passel (1945), who — after measuring cooling rates of water in plastic containers — chose the nominal values $T_1 \approx 33°\mathsf{C}$ and $U^* \approx 5$ m/s. The Siple-Passel formula was used by the *US National Weather Service* from 1973 but was in 2001 replaced by a somewhat more conservative expression based on modern theory and experiments.

Paul Allen Siple (1908-68). American Antarctic explorer. Accompanied (as an Eagle Scout) the first Byrd expedition to Antarctica in 1928-30. Participated in Byrd's second expedition 1933-35 as a chief biologist. Coined the term "wind chill" in 1939.

> **Example 28.4.1:** At $0°\mathsf{C}$ a wind speed of $U = 10$ m/s corresponds to a fictive wind temperature of $T_0^* \approx -14°\mathsf{C}$ according to (28-35). The modern formula yields instead $T_0^* \approx -7°\mathsf{C}$.

Heat flow in the Blasius boundary layer

The simplest model of advective cooling is furnished by the steady-flow Blasius boundary layer discussed in section 25.4 on page 493, with the added condition that the plate is held at a constant temperature $T = T_0 + \Theta$ which is different from the ambient temperature T_0. Since by assumption the mass flow is not influenced by the heat flow we may take over the exact Blasius solution and insert it into the steady flow heat equation,

$$(\boldsymbol{v} \cdot \boldsymbol{\nabla})T = \kappa\boldsymbol{\nabla}^2 T \ . \qquad (28\text{-}36)$$

In the boundary layer approximation the double derivative ∇_x^2 can be disregarded in the Laplacian, so that the equation takes the same form as the Prandtl equation (25-21)

$$v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2} \ . \qquad (28\text{-}37)$$

Assuming that the temperature only depends on the scaling variable $s = y\sqrt{U/2\nu x}$, we may write

$$T(s) = T_0 + \Theta\,F(s) \ , \qquad (28\text{-}38)$$

where $F(s)$ is a dimensionless function. In the same way as for Blasius' equation (25-26) it follows that,

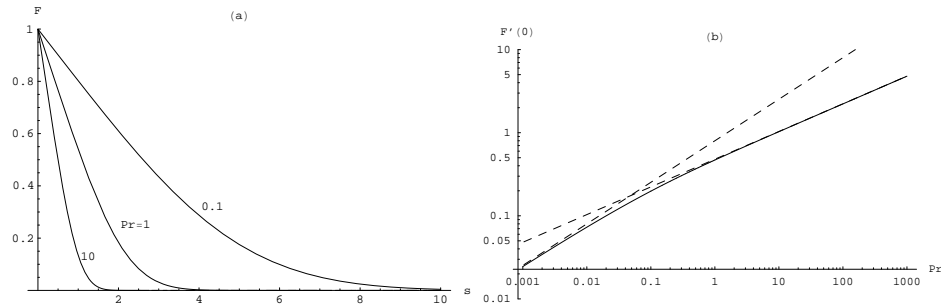$$F''(s) + \mathsf{Pr}\,g(s)F'(s) = 0 \ , \qquad (28\text{-}39)$$

Figure 28.2: **a)** *The function $F[(s)$ for three different values of the Prandtl number.* **b)** *The slope $F'(0)$ as a function of* Pr. *The dashed lines indicate the asymptotic power law behaviors (see problem 28.3).*

where $g(s)$ is the Blasius solution (25-23). Solving it with the boundary conditions $F(0) = 0$ and $F(\infty) = 1$, we find

$$F(s) = 1 - \frac{H(s)}{H(\infty)} , \tag{28-40}$$

where

$$H(s) = \int_0^s e^{-\text{Pr}\, G(u)}\, du , \qquad G(s) = \int_0^s g(u)\, du . \tag{28-41}$$

The function $F(s)$ is plotted in fig. 28.2a for three different values of the Prandtl number and illustrates clearly how the heat front depends on this number. Notice that for Pr $= 1$ we have $1 - F(s) = g'(s) = f(s)$ according to (25-26).

Total rate of heat loss

The gradient of the temperature field on the plate

$$\nabla_y T|_{y=0} = \Theta F'(0)\sqrt{\frac{U}{2\nu x}} , \tag{28-42}$$

determines the heat flow $q_y = -k\nabla_y T$ according to Fourier's law. Integrating q_y along an area A of the plate of length L in the x-direction and width A/L in the z-direction, we find the total rate of heat loss,

$$\dot{Q} = \frac{A}{L}\int_0^{L_x} q_y(x)dx = -F'(0)k\Theta A\sqrt{\frac{2U}{\nu L}} . \tag{28-43}$$

The slope $-F'(0) = 1/H(\infty)$ is plotted as a function of the Prandtl number in fig. 28.2b. Notice that this is not quite of the same form as the previous estimate (28-34) because it has the ν instead of κ in the denominator of the squareroot. The explanation is that for Pr $\ll 1$ the slope varies like the squareroot of the

Prandtl number, $-F'(0) \approx 0.8\sqrt{\text{Pr}}$, bringing the estimate into agreement with the exact calculation (see problem 28.3).

Example 28.4.2: The grown-up human body has a skin surface area of about $A \approx 2$ m^2. A naked human standing with the shoulders aligned with the wind will roughly present this area as $L \approx 0.5$ m and $A/L \approx 4$ m. For air with $\text{Pr} = 0.73$ we have $-F'(0) = 0.42$, so that in a wind with $U = 1$ m/s and a temperature $\Theta = 10$ K below the skin temperature we find the heat loss rate $\dot{Q} \approx 100$ W. Since the human body produces heat at this rate, a skin temperature of $27°$C can thus be maintained essentially indefinitely in a gentle breeze with velocity 1 m/s and temperature $17°$C. In this calculation we have ignored natural convection and evaporative heat losses.

Under the same conditions in water where $-F'(0) = 0.87$ the rate of heat loss becomes enormous, $\dot{Q} \approx 23$ kW, and will almost instantly cool the skin to the temperature of the water. Everybody is familiar with the (relatively mild) skin chock that is experienced when one jumps into water as warm as $17°$C. At this temperature the loss of heat will eventually lead to severe hypothermia and death in the course of a some hours, depending on what you wear and how you behave.

Problems

28.1 Show that the average of r^n over a sphere of radius a is,

$$\langle r^n \rangle = \frac{3}{n+3} a^n \ . \tag{28-44}$$

28.2 The blood circulation in the human body actively attempts to maintain constant temperature everywhere in the body. Consider a "spherical human" of radius a and mass M with constant heat production h_0 everywhere in the body (example 28.2.5). The circulation maintains constant temperature T_c in the core of the body of radius $c < a$. Calculate the temperature drop $\Theta = T_c - T_s$ in the skin and estimate its value for an average "skin thickness' of $a - c = 5$ cm.

28.3 Show that the asymptotic behavior of $-F'(0) = 1/H(\infty)$ is

$$-F'(0) \approx \begin{cases} 0.80\mathsf{Pr}^{1/2} & \mathsf{Pr} \to 0 \\ 0.48\mathsf{Pr}^{1/3} & \mathsf{Pr} \to \infty \end{cases} \tag{28-45}$$