3 Gravity

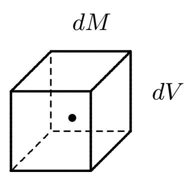
The force of gravity is all around us and determines to a large extent the way we live. It is certainly the force about which we have the best intuitive understanding. We learn the hard way to rise against it as small children, to keep it at bay as adults, only to be brought down by it in the end. A few people have experienced true absence of gravity for longer periods of time in satellites orbiting the Earth or rockets coasting towards the Moon.

Newton gave us the theory of gravity and the mathematics to deal with it. In a world where things only seem to get done by push and pull, man suddenly had to accept that the Earth could act on the distant Moon — and the Moon back on Earth. After Newton everybody had to suppress the feeling of horror for action at a distance and accept that gravity instantaneously could jump across the emptiness of space and tug at distant bodies. It took more than two centuries and the genius of Einstein to undo this learning. There *is* no action at a distance. As we understand it today, gravity is mediated by a field which emerges from massive bodies and in the manner of light takes time to travel through a distance. If the Sun were suddenly to blink out of existence, it would take eight long minutes before daylight was switched off and the Earth set free in space.

In this chapter we shall study the interplay between mass and the instantaneous Newtonian field of gravity, and derive the equations governing this field and its interactions with matter. Some basic knowledge of gravity is assumed in advance, and the presentation in this chapter aims mainly at developing elementary aspects of field theory in the comfortable environment of Newtonian gravity. More advanced concepts will be developed in chapter 6.

3.1 Mass density

In the continuum approximation the mass density is a field, $\rho(\boldsymbol{x}, t)$, assumed to exist everywhere in space and at all times. If there is no mass in a region, the mass density simply vanishes. Knowing this field, we may calculate the mass of a material particle occupying a small volume dV around the point $\boldsymbol{x}$ at time t,



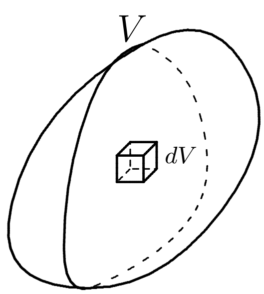
The volume dV occupied by a material particle may take any shape, here cubical. The nominal position of the particle $\boldsymbol{x}$ may be chosen anywhere within the volume.

$$\boxed{dM = \rho(\boldsymbol{x}, t)\, dV\ .} \tag{3-1}$$

We shall permit ourselves to suppress the space and time variables and just write $dM = \rho\, dV$, whenever such notation is unambiguous.

Although we usually think of the mass density field as varying smoothly throughout space, it is sometimes convenient to allow for discontinuous boundaries in material bodies (an example is shown in fig. 3.1 on page 39). Often these discontinuities are "real" in the sense that the transition between different materials may happen on the molecular scale, as for example at the interface between two solid bodies that touch each other.

In section 1.2 we discussed the continuum approximation and concluded that there are two length scales, L_{micro} and L_{macro}, that depend on the measurement precision. If $dV \gtrsim L^3_{\mathsf{micro}}$, the molecular structure may be disregarded, and if the length scale for major density changes is larger than L_{macro}, the density is effectively continuous and the shape of the minimal volume dV does not matter.



The total mass in a volume is obtained by integrating ("summing") over all material particles in the volume. The integral sign $\int$ is in fact a stylized version of the letter S (for "sum").

Total mass and center of mass

Mass density is a *local* quantity, defined in every point of space. The total mass in a volume V is a *global* quantity obtained mathematically by integrating the mass density over V,

$$M = \int_V dM = \int_V \rho(\boldsymbol{x}, t)\, dV\ . \tag{3-2}$$

Physically the integral should be understood as an approximation to a huge sum over the tiny, though not truly infinitesimal, material particles contained in the volume. If the density depends on time or if the volume changes shape and size with time, the total mass may depend on time.

In continuum physics the material contained in *any* volume V may be viewed as a "body" , and the *center of mass* of such a body is naturally defined from the average of the position $\boldsymbol{x}$ over all material particles,

$$\boldsymbol{x}_{\mathsf{M}} = \frac{1}{M}\int_V \boldsymbol{x}\, dM = \frac{1}{M}\int_V \boldsymbol{x}\, \rho(\boldsymbol{x}, t)\, dV\ . \tag{3-3}$$

In the Newtonian mechanics of particles and stiff bodies, the center of mass of a body plays an important role, because it moves like a point particle under the influence of the total force acting on the body (see appendix B). Although this

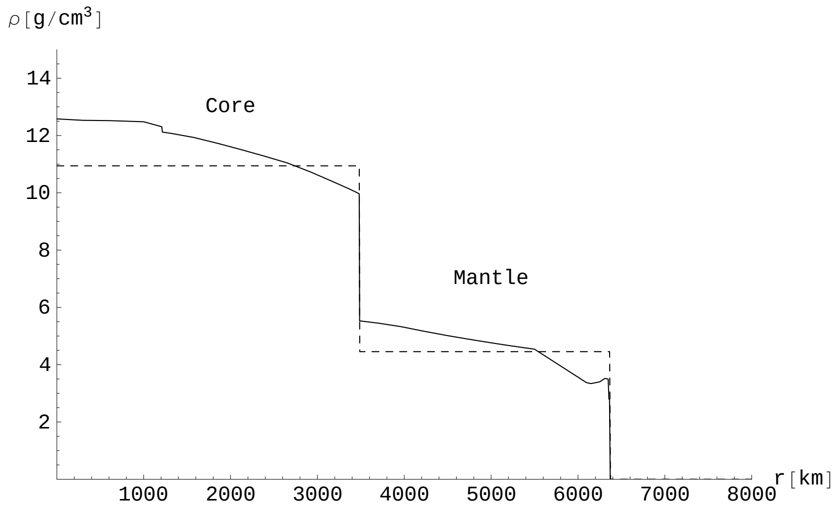


Figure 3.1: *The mass density of the Earth as a function of distance r from the center with the surface at $r = 6371$ km (from the standard Earth model [3]) . There is a sharp break in the density at the transition between the iron core and the stone mantle at $r = 3485$ km, and a smaller break at $r = 1216$ km between the outer liquid iron core and the inner solid iron core. The dashed lines indicate the average densities in mantle and core.*

is also true in continuum mechanics, it is not nearly as useful because the shape of a body may change drastically over longer time-spans. Think for example of a bucket of oil thrown into a waterfall. It may not always be physically meaningful to speak about a well-defined "body of oil" at later times.

Spherical systems

Planets like Earth or stars like the Sun are in the first approximation spherically symmetric. Spherical symmetry implies that the mass density ρ in any point $\boldsymbol{x}$ only depends on the distance $r = |\boldsymbol{x} - \boldsymbol{c}|$ to the center $\boldsymbol{c}$, which for symmetry reasons must also be the center of mass. Usually the origin of the coordinate system is chosen to be at the center such that $\boldsymbol{c} = \boldsymbol{0}$. For a simple spherical planet with radius a and constant density ρ_0 we have

$$\rho(r) = \begin{cases} \rho_0 & r < a \\ 0 & r > a \end{cases} . \tag{3-4}$$

Such a distribution might be used for analytic calculations for a small rock planet like the Moon, but definitely not for the Earth.

In fig. 3.1 the mass density of the Earth is plotted (fully drawn) as a function of the central distance. It cannot be measured directly, but is inferred from a combination of surface observations and modelling. For analytic calculations, its mass distribution may be approximated with two layers of constant density, $\rho_1 = 10.9$ g/cm^3 in the core and $\rho_2 = 4.4$ g/cm^3 in the mantle (see problem 3.7).

3.2 Gravitational acceleration

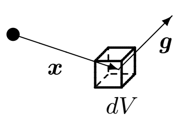
Galileo Galilei (1564–1642). *Italian natural philosopher, astronomer, mathematician, and craftsman. Carried out gravity experiments with falling objects and inclined planes. Built better telescopes than any before him. Saw as the first the mountains of the Moon, the large moons of Jupiter, and that the Milky Way is made from stars. Considered the father of the modern scientific method.*

Galileo found empirically that all bodies fall in the same way, independently of their mass. In Newton's language this shows that the force of gravity on a body is proportional to its mass. For a material particle of mass $dM = \rho dV$ the force of gravity may be written

$$\boxed{d\boldsymbol{\mathcal{F}} = \boldsymbol{g}(\boldsymbol{x}, t)\, dM = \rho \boldsymbol{g}\, dV} \ , \tag{3-5}$$

where $\boldsymbol{g}(\boldsymbol{x}, t)$ is called the *gravitational acceleration* field, or just *gravity*. The last form shows that gravity is a *body force* (also called a *volume force*) which acts everywhere in a body with a *density of force* $\boldsymbol{f} = d\boldsymbol{\mathcal{F}}/dV = \rho \boldsymbol{g}$. In fig. 3.3 on page 45 the magnitude of Earth's gravity is plotted as a function of the distance from the center of the Earth.

> In Newtonian physics the gravitational field imparts a common acceleration to all bodies. Given the same initial conditions all material bodies will follow the same orbits in a gravitational field. As a consequence there is no way we can distinguish between gravitational forces and the inertial (so-called fictitious) forces experienced in accelerated motion. The identical behavior of all bodies in a field of gravity also allows one to look upon the gravitational field as a property of space and time, rather than simply a vehicle for gravitational interaction. The indistinguishability of gravity and acceleration was raised to a fundamental law, the *Principle of Equivalence*, by Einstein in his *General Theory of Relativity* from 1915, in which gravity expresses the geometric curvature of space and time.



The weight of the matter in a volume element dV is $d\boldsymbol{\mathcal{F}} = \rho \boldsymbol{g}\, dV$.

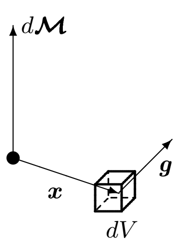
Total gravitational force and total moment of gravity

The field of gravity specifies the gravitational acceleration in every point of space and every instant of time. The total gravitational force on a body of volume V, the *weight* of the body, is

$$\boldsymbol{\mathcal{F}} = \int_V \rho \boldsymbol{g}\, dV \ . \tag{3-6}$$

The total force determines how the body as a whole moves. It is independent of the choice of origin of the coordinate system, but depends like any other vector on its orientation. If ρ, $\boldsymbol{g}$, or V change in the course of time, the total force may also change.

The total *moment of force* of gravity relative to the coordinate origin is



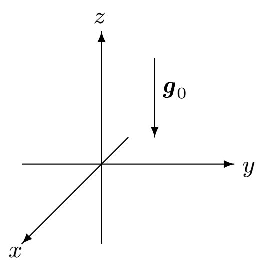
The moment of force for a volume element dV is $d\boldsymbol{\mathcal{M}} = \boldsymbol{x} \times \rho \boldsymbol{g}\, dV$.

$$\boldsymbol{\mathcal{M}} = \int_V \boldsymbol{x} \times \rho \boldsymbol{g}\, dV \ . \tag{3-7}$$

The total moment determines how a body as a whole rotates around the origin. The moment depends not only on the orientation of the coordinate system, but also on its origin. It is an improper axial vector. In a translated coordinate system with origin in $\boldsymbol{x} = \boldsymbol{c}$, the coordinates are $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{c}$, and the moment becomes

$$\boldsymbol{\mathcal{M}}' = \int_{V'} \boldsymbol{x}' \times \rho' \boldsymbol{g}' \, dV' = \int_V (\boldsymbol{x} - \boldsymbol{c}) \times \rho \boldsymbol{g} \, dV = \boldsymbol{\mathcal{M}} - \boldsymbol{c} \times \boldsymbol{\mathcal{F}} \ , \tag{3-8}$$

where the primed quantities all refer to the translated system. This shows that *the total moment is independent of the choice of origin of the coordinate system if (and only if) the total force vanishes.*



A flat-earth coordinate system.

Constant gravity

In a constant gravitational field $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0$, the weight (3-6) becomes the familiar

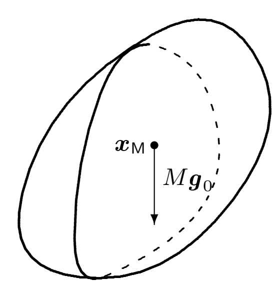
$$\boldsymbol{\mathcal{F}} = M \boldsymbol{g}_0 \ , \tag{3-9}$$

where M is the total mass (3-2). In constant gravity, it is customary to choose a "flat-earth" coordinate system with vertical z-axis, such that $\boldsymbol{g}_0 = (0, 0, -g_0)$ where g_0 is a positive constant.

> At the surface of Earth, gravity is very close to being constant with magnitude equal to the standard gravity, defined by convention to be exactly $g_0 = 9.80665 \ \mathsf{m/s^2}$ with no uncertainty. The actual gravitational acceleration at the surface of the Earth depends on many factors, for example local mass concentrations and the positions of the Moon and Sun (see fig. 7.1 on page 117). It has been determined with a relative precision of 3×10^{-9} in an experiment using atom interferometry [19]. Galileo's law was verified in the same experiment to within 7×10^{-9} by comparing the measured values of the gravitational acceleration for a macroscopic body and for a cesium atom, in effect a modern version of his famous "leaning tower in Pisa" experiment.

The moment of force may be expressed in terms of the center of mass (3-3),

$$\boldsymbol{\mathcal{M}} = \int_V \boldsymbol{x} \times \rho \boldsymbol{g}_0 \, dV = \left(\int_V \rho \, \boldsymbol{x} \, dV \right) \times \boldsymbol{g}_0 = \boldsymbol{x}_{\mathsf{M}} \times M \boldsymbol{g}_0 \ . \tag{3-10}$$



In a constant gravitational field $\boldsymbol{g}_0$, the weight of a body may be viewed as concentrated at the position of the center of mass, $\boldsymbol{x}_{\mathsf{M}}$.

This shows that in constant gravity, the total moment is the same as that of point particle with mass equal to the total mass of the body, situated at the center of mass. In a constant gravitational field, the moment of gravity calculated in a coordinate system with origin in the center of mass must vanish because $\boldsymbol{x}_{\mathsf{M}} = \boldsymbol{0}$ in these coordinates. The moment of force in constant gravity is important for understanding the stability of floating bodies (chapter 5).

Visualizing the gravitational field

A visual impression of the gravitational field may be given by a picture of the *field lines*, defined to be families of curves that at a given instant t_0 have the

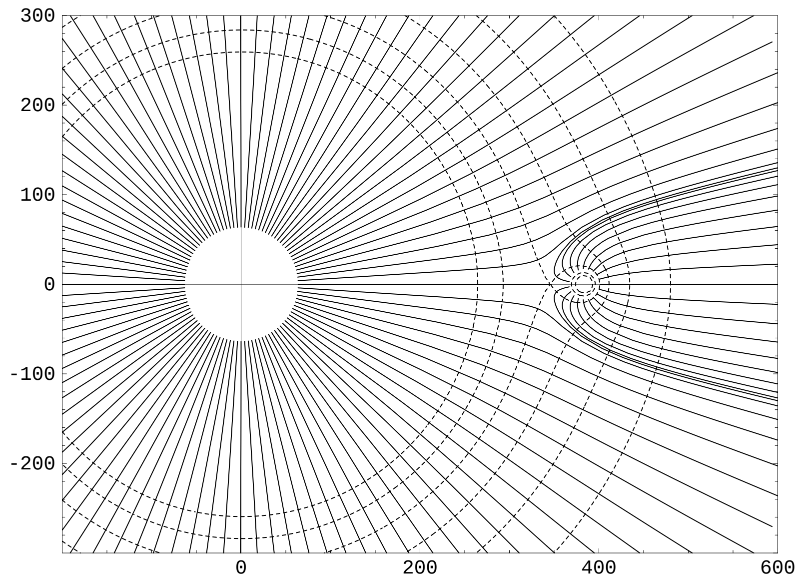
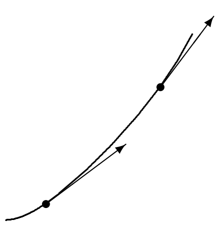


Figure 3.2: *The gravitational field (and some nearly circular equipotential surfaces) between Earth and Moon. You should imagine rotating this picture around the Earth-Moon axis. The drawing is to scale, except for two regions of 10 times the sizes of the Earth and the Moon that have been cut out for technical reasons. The field lines are everywhere plotted with a density proportional to the field strength. The numbers on the frame are coordinates centered on Earth in units of 1000 km. The Moon appears to have a streaming "mane of hair" because all the field lines ending on its surface have to come in from spatial infinity and cannot cross the lines of Earth's field.*



Field lines follow the instantaneous field everywhere. They are very different from the orbits particles would follow through the field. A thrown stone follows a parabolic orbit as it falls to the ground, whereas the field lines on the surface of the Earth are vertical.

gravitational field as tangent (see fig. 3.2). This means that the curves are solutions to the first order differential equation

$$\frac{d\boldsymbol{x}}{ds} = \boldsymbol{g}(\boldsymbol{x}, t_0) \;, \tag{3-11}$$

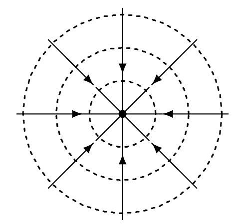
where s is a running parameter along the curve. This parameter is not the time, but has dimension of time squared because $\boldsymbol{g}$ has dimension of length per unit of time squared. The solutions are of the form $\boldsymbol{x} = \boldsymbol{x}(s, \boldsymbol{x}_0, t_0)$ with $\boldsymbol{x}_0$ being the starting point at $s = 0$. The field lines form an instantaneous picture of the field at time t_0, and cannot be directly related to particle orbits, as illustrated by the nearly circular orbit of a planet which is everywhere orthogonal to the field lines.

Field lines have the very important property that they can never cross. For if two field lines crossed in a point $\boldsymbol{x}$, then by (3-11) there would have to be two different values of the gravitational field in the same point, and that is impossible (except when the field vanishes, as it does in one point of fig. 3.2). As will be shown in the following section, all gravitational field lines have to come in from infinity and end on masses, and we shall also see that field lines do not form closed loops.

3.3 Sources of gravity

The gravitational field tells us how gravity acts on material bodies. But what generates the gravitational field? What is its source? The answer is — as most people are aware — that the field is generated by mass. Quantitatively this is expressed by *Newton's law of gravity* which says that the field from a point particle of mass M at the origin of the coordinate system is

$$\boldsymbol{g}(\boldsymbol{x}) = -GM\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} , \tag{3-12}$$



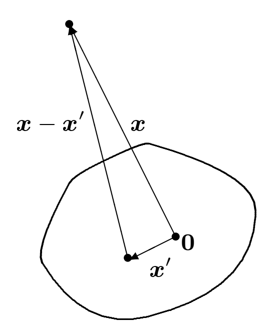
Field lines around a point particle all come in from infinity and converge upon the particle.

where G is the universal gravitational constant. The negative sign asserts that gravitation is always attractive, or equivalently that field lines always run towards masses. The last factor shows that the strength of gravity decreases with the inverse square of the distance.

> The gravitational constant is hard to determine to high precision. The recommended [1] 1998 value, $G = 6.673(10) \times 10^{-11}$ $\mathsf{Nm^2/kg^2}$, has an embarrassingly large uncertainty of more than one part in 10^3. The inverse square law has been well tested at planetary distances during the last centuries, but only recently at the submillimeter scale (see for example Long et al, Nature 421, 922 (2003)). The force of gravity is terribly weak compared to other forces. In the hydrogen atom the ratio of the force of gravity to the electrostatic force (between electron and proton) is 4.4×10^{-40}. The only reason gravity can be observed at all is the nearly complete electric neutrality of macroscopic bodies. Electrostatic and "gravistatic" forces seem superficially alike in that they are both inversely proportional to the square of the distance (which gives them infinite range; they are in fact the only fundamental forces in nature with infinite range). But where electric charge can be both positive and negative, mass is always positive, implying that there are no "neutral" bodies unaffected by gravity, nor bodies that are repelled by the gravity of other bodies (antigravity). Gravity and electromagnetism are in fact very different at a deeper level, only completely revealed in General Relativity.

Another basic property of gravity is that it is *additive*, so that matter — also from the point of view of its gravity — may be viewed as a collection of material particles of mass $dM = \rho\, dV$, each contributing its own little point-like field to the total. Using (3-12) but shifting the position of the source particle from the origin to an arbitrary point $\boldsymbol{x}'$, the collective gravitational field due to all material particles in a volume V becomes an integral,

$$\boxed{\boldsymbol{g}(\boldsymbol{x}) = -G\int_V \frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,\rho(\boldsymbol{x}')\,dV'} \;. \tag{3-13}$$



The vectors involved in calculating the field in the position $\boldsymbol{x}$

Notice that the integrand has a singularity at $\boldsymbol{x}' = \boldsymbol{x}$ (for $\rho(\boldsymbol{x}) \neq 0$). This singularity is, however, integrable and creates no problem (except problem 3.8).

Forces between extended bodies

The expression for the field (3-13) brings us full circle. We are now able to calculate the gravitational field from a mass distribution, as well as the force that this field exerts on another mass distribution, even on itself. Substituting the field (3-13) into the force (3-6), and renaming the integration variables, the total force by which a mass distribution ρ_2 in the volume V_2 acts on a mass distribution ρ_1 in V_1 becomes,

$$\boldsymbol{\mathcal{F}}_{12} = -G \int_{V_1} \int_{V_2} \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|^3} \, \rho_1(\boldsymbol{x}_1) \rho_2(\boldsymbol{x}_2) \, dV_1 \, dV_2 \; . \tag{3-14}$$

Newton's third law is fulfilled, $\boldsymbol{\mathcal{F}}_{12} = -\boldsymbol{\mathcal{F}}_{21}$, because the integrand is antisymmetric under exchange of $1 \leftrightarrow 2$ due to the first factor. Consequently, the force from a mass distribution acting on itself vanishes, as it ought to. For if the self-force did not vanish a body could, so to speak, lift itself by its bootstraps.

Asymptotic behavior

The gravitational field from the matter contained in a volume of finite extent has a particularly simple form at large distances. Since $\boldsymbol{x}'$ then only ranges over a finite region in (3-13), it follows for $|\boldsymbol{x}| \to \infty$ that

$$\frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \to \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} \; .$$

Taking this expression outside the integral we obtain

$$\boldsymbol{g}(\boldsymbol{x}) \to -GM \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} \; , \tag{3-15}$$

where M is the total mass. At sufficiently large distances the field of an arbitrary spatially bounded mass distribution always approaches that of a point particle containing the total mass of body.

Field of a spherical body

The mass distribution $\rho(r)$ of a spherically symmetric body is only a function of the distance $r = |\boldsymbol{x}|$ from its center, which here is taken to be at the origin of the coordinate system. Since gravity according to (3-13) is caused by the mass distribution, it should also be spherically symmetric. For a vector field this means that it is everywhere directed radially away from the center,

$$\boldsymbol{g}(\boldsymbol{x}) = g(r) \, \boldsymbol{e}_r \; , \tag{3-16}$$

where $\boldsymbol{e}_r = \boldsymbol{x}/r$ is the radial unit vector and $g(r)$ is a function of r alone. In fig. 3.3 the value of $-g(r)$ is plotted for the Earth as a function of the radial

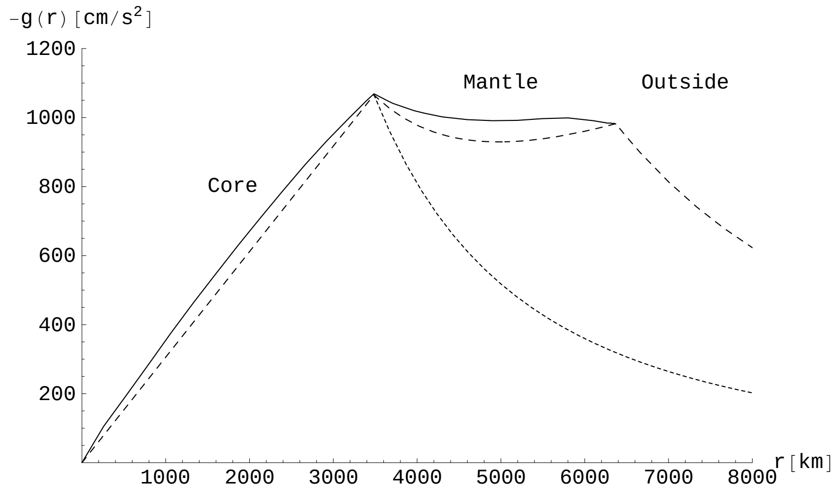


Figure 3.3: *The strength of gravity $-g(r)$ inside and outside the Earth as a function of distance from the center. The solid curve is obtained from the standard Earth data [3]. The strength of gravity grows roughly linearly from the center to the core/mantle boundary at $r = 3485$ km, and decreases slightly in the mantle due to the sharp drop in mass density at the boundary. The dotted dropping line is the core field itself. The dashed lines are obtained from the two-layer model of the Earth (problem 3.7).*

distance. One notices the surprising fact that the strength of gravity actually is larger inside Earth's mantle than on the surface.

In chapter 6 we shall see that spherical gravity takes an extremely simple form. We shall prove that the field strength $g(r)$ is everywhere — inside and outside the distribution — equal to the field of a point particle situated at the center of the distribution,

$$\boxed{g(r) = -G\frac{M(r)}{r^2}\ ,} \tag{3-17}$$

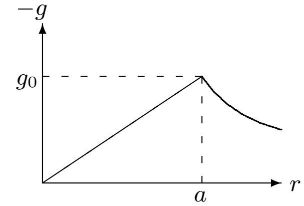
with a mass equaling the total mass inside the radius r,

$$M(r) = \int_0^r \rho(r')4\pi r'^2\, dr'\ . \tag{3-18}$$

It is also possible to prove this by direct integration of (3-13) (see problem 3.11).

For a planet with constant density we find from (3-4) that

$$g(r) = -\frac{4}{3}\pi G\rho_0 \begin{cases} r & r < a \\ \dfrac{a^3}{r^2} & r > a \end{cases} \tag{3-19}$$



The strength of gravity for a planet with constant density.

The strength of gravity grows linearly with r inside the planet because the total mass grows with the third power of r whereas the field strength decreases with the second power.

It follows from (3-18) that for every point in the vacuum *outside* a spherical mass distribution where $M(r)$ equals the total mass, the field (3-17) is *exactly* the same as that of a point particle at the center with mass equal to the total mass of the body. We have seen above that the field at great distances from an arbitrary body is always approximately that of a point particle, but now we learn that the field is of this form everywhere around a perfectly spherical body. There are *no near-field corrections to the gravitational field of a spherical body.* Without this wonderful property, Newton could never have related the strength of gravity at the surface of the Earth — iconized by the fall of an apple — to the strength of gravity in the Moon's orbit.

3.4 Gravitational potential

Although the field of gravity is a vector field with three components, there is really only one functional degree of freedom underlying the field, namely the mass distribution giving rise to it. The relationship between these two fields expressed by (3-13) is, however, *non-local*, meaning that $\boldsymbol{g}(\boldsymbol{x})$ in a point $\boldsymbol{x}$ depends on a physical quantity $\rho(\boldsymbol{x}')$ in points $\boldsymbol{x}'$ that may be arbitrarily far away.

Gravity as a gradient field

We shall now prove that the acceleration field can be derived *locally* from a single scalar field $\Phi(\boldsymbol{x})$, called the gravitational *potential*. The relation between the acceleration field and the potential is

$$\boxed{\boldsymbol{g} = -\boldsymbol{\nabla}\Phi} \tag{3-20}$$

with a conventional minus sign in front. The gradient operator (nabla) has been defined in (2-54). Because of the gradient, the potential is defined only up to an undetermined additive constant.

In order to demonstrate (3-20) we first calculate the gradient of the central distance (see also problem 2.9)

$$\boldsymbol{\nabla}|\boldsymbol{x}| = \boldsymbol{\nabla}\sqrt{\boldsymbol{x}^2} = \frac{1}{2|\boldsymbol{x}|}\boldsymbol{\nabla}\boldsymbol{x}^2 = \frac{1}{2|\boldsymbol{x}|}\boldsymbol{\nabla}(x_1^2 + x_2^2 + x_3^2) = \frac{\boldsymbol{x}}{|\boldsymbol{x}|} \ , \tag{3-21}$$

and from this

$$\boldsymbol{\nabla}\frac{1}{|\boldsymbol{x}|} = -\frac{1}{|\boldsymbol{x}|^2}\boldsymbol{\nabla}|\boldsymbol{x}| = -\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} \ . \tag{3-22}$$

Comparing with (3-12) we conclude that the potential of a point particle of mass M is

$$\Phi(\boldsymbol{x}) = -\frac{GM}{|\boldsymbol{x}|} \ , \tag{3-23}$$

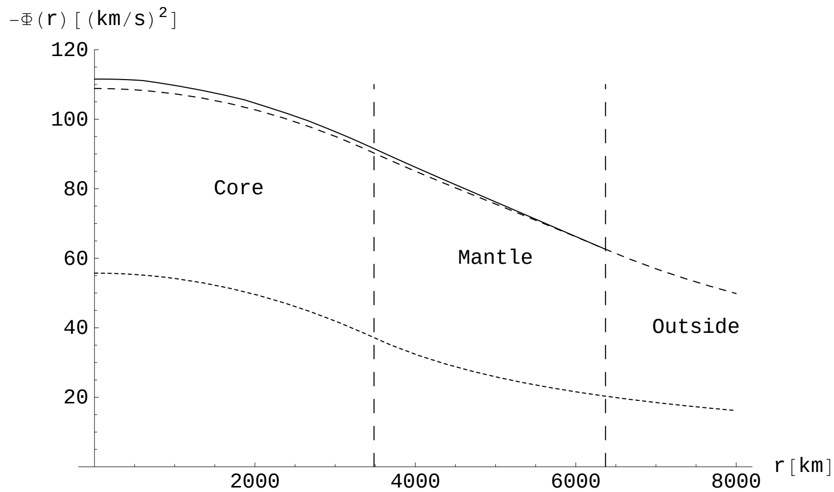


Figure 3.4: *The gravitational potential $-\Phi(r)$ of the Earth as a function of distance from the center. The fully drawn curve is obtained by numerically integrating the standard Earth data [3]. The dotted curve is the potential of the core alone, and the dashed curve is obtained from the two-layer model (problem 3.7). The vertical dashed lines indicate the positions of the sharp transitions in the mass density (see fig. 3.1), which have completely disappeared from view in this plot.*

apart from the arbitrary additive constant which we here choose so that the potential vanishes at spatial infinity. In the same way we derived (3-13), we find the potential from a mass distribution in V

$$\boxed{\Phi(\boldsymbol{x}) = -G \int_V \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} dV'} \quad , \tag{3-24}$$

Since the mass density is always positive, the gravitational potential is always negative, a direct consequence of the attractive nature of gravity and the normalization to zero potential at infinity.

Even if the mass distribution may jump discontinuously, the gravitational field will always be continuous (as is evident from fig. 3.3), because of the integration over the mass distribution in (3-13) or (3-18). The potential is still smoother, because its definition (3-20) requires it to have a continuous derivative. This is also borne out by fig. 3.4 which shows the potential of the Earth as a function of central distance. Almost all traces of the original discontinuities have vanished from sight.

The gravitational potential may be visualized by means of surfaces of constant potential, also called *equipotential surfaces*. The field lines are always orthogonal to the equipotential surfaces, and if they are plotted with constant potential difference, the strength of the field will be inversely proportional to the distances between them. A few equipotential surfaces have been shown in the Earth-Moon plot in fig. 3.2 on page 42.

Asymptotic behavior

For a mass distribution of finite extent, the denominator will for $|\boldsymbol{x}| \to \infty$ become independent of $\boldsymbol{x}'$, so that

$$\Phi(\boldsymbol{x}) \to -G\frac{M}{|\boldsymbol{x}|} \tag{3-25}$$

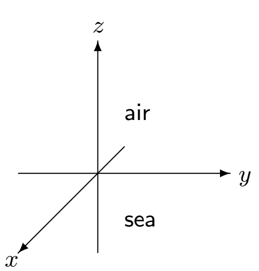
where M is the total mass, in complete accordance with (3-15). At large distances the potential of a body thus approaches that of a point mass carrying the total mass of the body (see however problems 3.14 and 3.15).

The flat Earth limit

For a constant gravitational field $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0$ we may take

$$\Phi(\boldsymbol{x}) = -\boldsymbol{x} \cdot \boldsymbol{g}_0 \ . \tag{3-26}$$

This seems to be at variance with the expression (3-24) and does certainly not vanish at infinity. Constant gravitational fields should, however, always be understood as an approximation valid within limited regions of space and time, and then the difficulty disappears.



The flat-earth coordinate system with the sea level at $z = 0$.

For length scales much smaller than the radius of the Earth, the surface of the sea may be considered to be flat and the gravitational field constant. In a flat-earth coordinate system with the z-axis vertical and the sea level at $z = 0$, we may take $\boldsymbol{g}_0 = (0, 0, -g_0)$ and find

$$\Phi = g_0 z \ . \tag{3-27}$$

The gravitational acceleration in the z-direction becomes $g_z = -\partial\Phi/\partial z = -g_0$ and is directed downwards everywhere, as it should.

The spherical case

The potential of a spherical mass distribution can only depend on $r = |\boldsymbol{x}|$. Using that $\boldsymbol{\nabla}\Phi(r) = (d\Phi(r)/dr)\ \boldsymbol{\nabla} r$ and $\boldsymbol{\nabla} r = \boldsymbol{x}/\,|\boldsymbol{x}| = \boldsymbol{e}_r$, we find by comparison with (3-16)

$$\boxed{g(r) = -\frac{d\Phi(r)}{dr}} \ . \tag{3-28}$$

Conversely, integrating this from r to ∞ and using that the potential vanishes at infinity, we obtain

$$\Phi(r) = \int_r^\infty g(s)\, ds = -G\int_r^\infty \frac{M(s)}{s^2}\, ds \ , \tag{3-29}$$

where we have also made use of (3-17). Performing a partial integration we obtain

$$\Phi(r) = G \int_r^\infty M(s)\, d\left(\frac{1}{s}\right) = -G\frac{M(r)}{r} - 4\pi G \int_r^\infty s\rho(s)\, ds\ . \qquad (3\text{-}30)$$

The final expression is not quite as pretty as (3-17) because of the second term, which is necessary to secure the continuity of the derivative of $\Phi(r)$. But outside the mass distribution the second term vanishes, and the potential becomes as expected that of a point particle carrying the total mass of the body.

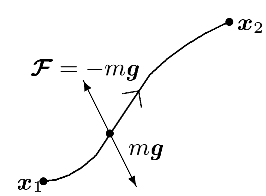
For a planet with constant mass density we obtain from the above expression and (3-19),

$$\Phi(r) = -\frac{2}{3}\pi G\rho_0 \begin{cases} 3a^2 - r^2 & r < a \\ 2\dfrac{a^3}{r} & r > a \end{cases} . \qquad (3\text{-}31)$$

One may avoid integrating and instead verify that the potential is continuous at $r = a$ and that the derivative $-d\Phi/dr$ indeed is identical to (3-19).

3.5 Potential energy

According to the laws of elementary particle mechanics, work is calculated from the product of a force and the distance it moves a particle. It is important to make completely clear who does work on whom. When a particle falls freely in a gravitational field, it is the force of gravity which performs work on the particle while the particle follows the path of its natural motion and gains kinetic energy. If we want the particle to follow any other path, we must "by hand" cancel the force of gravity with an equal and opposite force, and slowly move the particle along the desired path.

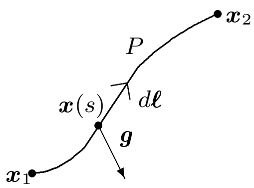


The gravitational force $m\boldsymbol{g}$ must be cancelled by an external force $\boldsymbol{\mathcal{F}} = -m\boldsymbol{g}$ in order to move the particle slowly along any desired path between the end points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

For so-called *conservative* forces, the work we perform on the particle while moving it along the path depends only on its end points and not on where the path goes between the end points. The work we must perform in moving the particle from a fixed position to any desired point $\boldsymbol{x}$ in space is only a function of the end point $\boldsymbol{x}$ of the path. This function, $V(\boldsymbol{x})$, is called the *potential energy* of the particle in the point $\boldsymbol{x}$ because it represents the work that a particle in $\boldsymbol{x}$ would potentially perform on us, should we move it back to the fixed position.

Gravitational work

Suppose we move a point particle of mass m from position $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ along a path P in a static field of gravity, $\boldsymbol{g}(\boldsymbol{x})$. To keep the particle on this path, we must provide a force $-m\boldsymbol{g}$ to counter gravity. The work performed by us on the particle while moving it along the curved path is the sum of all the tiny contributions $-m\boldsymbol{g} \cdot d\boldsymbol{\ell}$ from each little path element $d\boldsymbol{\ell}$, and the total work we



A path $\boldsymbol{x}(s)$ with $s_1 \leq s \leq s_2$ running from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$.

perform becomes a line integral,

$$W = -\int_P m\boldsymbol{g} \cdot d\boldsymbol{\ell} = -m \int_{s_1}^{s_2} \boldsymbol{g}(\boldsymbol{x}(s)) \cdot \frac{d\boldsymbol{x}(s)}{ds}\, ds\ . \tag{3-32}$$

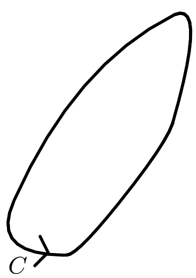
In the last expression the line integral along the path $\boldsymbol{x}(s)$ from $\boldsymbol{x}_1 = \boldsymbol{x}(s_1)$ to $\boldsymbol{x}_2 = \boldsymbol{x}(s_2)$ has been made explicit by means of a running parameter s varying in the interval $s_1 \leq s \leq s_2$ along the path.

Because the field of gravity is the gradient of the gravitational potential, the line integral may be carried out. Inserting (3-20) we find

$$W = m \int_{\boldsymbol{x}_1}^{\boldsymbol{x}_2} \nabla\Phi(\boldsymbol{x}) \cdot d\boldsymbol{\ell} = m \int_{s_1}^{s_2} \frac{d\Phi(\boldsymbol{x}(s))}{ds}\, ds \tag{3-33}$$

$$= m\,\Phi(\boldsymbol{x}_2) - m\,\Phi(\boldsymbol{x}_1)\ . \tag{3-34}$$

Since this result is independent of the actual path of the particle, it follows that the gravitational potential energy of the particle is $V(\boldsymbol{x}) = m\Phi(\boldsymbol{x})$, and that the gravitational potential $\boldsymbol{\Phi}(\boldsymbol{x})$ is the potential energy per unit of mass.



A gravitational field line cannot form a closed path C.

No closed loops of gravity

For a closed path, C, which begins in the same point as it ends, the line integral must vanish

$$\oint_C \boldsymbol{g} \cdot d\boldsymbol{\ell} = 0 \tag{3-35}$$

because the potential is the same in the end-points of the path.

This result is true for all gradient fields, such as gravity $\boldsymbol{g}$ and the electrostatic field $\boldsymbol{E}$. It implies that there can be no closed loops of field lines. For if there were, the product $\boldsymbol{g} \cdot d\boldsymbol{\ell}$ would have the same sign all around the loop, because the tangent of a field line is everywhere proportional to the field, *i.e.* $d\boldsymbol{\ell} \sim \boldsymbol{g}$, and such an integral cannot vanish.

Place	km/s
Earth surface	11.2
Mars surface	5.0
Moon surface	2.4
Sun surface	617.6
Earth orbit	42.1
Moon orbit	1.4
Neutron star	1×10^5
Black hole	3×10^5

Escape velocities from some places in the solar system, and a couple of exotic ones. Notice that escaping from the orbit of Earth means escaping from the solar system whereas escaping from the orbit of the Moon only gets you free of Earth's gravity. The neutron star is assumed to have solar mass.

Escape velocity

The total energy of a point particle in $\boldsymbol{x}$ with velocity $\boldsymbol{v}$ is the sum of its kinetic energy and its potential energy, $\frac{1}{2}m\boldsymbol{v}^2 + m\Phi(\boldsymbol{x})$. From elementary mechanics we know that the total energy is conserved, *i.e.* constant in time. In other words, a particle starting in the point $\boldsymbol{x}_0$ with velocity $\boldsymbol{v}_0$ must at all times obey the equation

$$\frac{1}{2}\boldsymbol{v}^2 + \Phi(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{v}_0^2 + \Phi(\boldsymbol{x}_0)\ . \tag{3-36}$$

Taking $\boldsymbol{x}_0$ at infinity, where the potential vanishes, it follows immediately that in order to escape the grip of gravity with $|\boldsymbol{v}_0| \neq 0$ from a point $\boldsymbol{x}$ with potential Φ, an object at $\boldsymbol{x}$ must be given a velocity that is larger than

$$v_{\text{esc}} = \sqrt{-2\Phi}\ . \tag{3-37}$$

Knowing the potential in a point is simply equivalent to knowing the escape velocity from that point.

A particularly interesting case happens when the potential becomes so deep that the escape velocity equals or surpasses the velocity of light c. In that case the body has turned into a black hole. Using the potential of a point mass (3-23) we find that this happens when the radius a of a spherical mass distribution satisfies

$$a < \frac{2GM}{c^2} \ . \tag{3-38}$$

Being a non-relativistic calculation this is of course highly suspect, but accidentally it is exactly the same as the correct condition obtained in general relativity [4], where the right hand side is called the Schwarzchild radius. For the Earth the Schwarzchild radius is about a centimeter, and for the Sun three kilometers.

Problems

3.1 Show that the gravitational field outside a spherical planet may be written

$$g(r) = -g_0 \frac{a^2}{r^2} \ , \tag{3-39}$$

where a is the radius of the planet and g_0 the magnitude of the surface gravity.

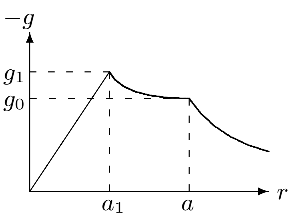
3.2 Show that a satellite moving in a circular orbit around a spherical planet has velocity $v = \sqrt{-\Phi}$, where Φ is the gravitational potential in the satellite's orbit. Calculate the velocity of a satellite moving at ground level.

3.3 Arthur C. Clarke proposed (*Wireless World*, October 1945, pages 305-308) that communication satellites should be put into a circular equatorial orbit with revolution time equal to Earth's rotation period, so that the satellites would stay fixed over a point at equator. Calculate the height of the orbit above the ground, taking also into account that the Earth circles the Sun in one year.

3.4 A space elevator (fictionalized by Arthur C. Clarke in *Fountains of Paradise* (1978)) can be created if it becomes technically feasible to lower a line down to Earth from a geostationary satellite at height h (see problem 7.5). Assume that the line is unstretchable and has constant cross section A and constant density ρ. Calculate the maximal tension $\sigma = -\mathcal{F}/A$ (force per unit of area) in the line. Determine the numerical value of the ratio of tension to density, σ/ρ, and compare with the tensile strength (breaking tension) of a known material.

3.5 A comet consisting mainly of ice falls to Earth. a) Estimate the minimum energy released in the fall per unit of mass. b) Compare with the an estimate of the energy needed to evaporate the comet.

3.6 A stone is set in free fall from rest through a mine shaft going right through the center of a planet with constant density. a) Calculate the speed with which the stone passes the center. b) Calculate the time it takes to fall to the center.



The strength of gravity for a two-layer planet with a dense core.

3.7 A planet consists of two layers with constant mass density,

$$\rho(r) = \begin{cases} \rho_1 & r \leq a_1 \\ \rho_2 & a_1 < r \leq a \\ 0 & r > a \end{cases} \ . \tag{3-40}$$

a) Calculate the strength of gravity and the potential. b) Show that the strength of gravity at the boundary between the layers is greater than at the surface when

$$\frac{\rho_1 - \rho_2}{\rho_2} > \frac{a^2}{a_1(a + a_1)} \ . \tag{3-41}$$

Verify that this is fulfilled for the Earth.

∗ **3.8** Show by direct integration in a small spherical volume around the singularity in (3-13) that it gives a finite contribution to the integral.

∗ **3.9** Show that the mass density is a scalar field.

∗ **3.10** Show that the gravitational field is a vector field.

∗ **3.11** Show that gravitational field of a spherical body (3-17) may be derived by integration of (3-13).

∗ **3.12** A spherical planet with mass distribution of the form $\rho(r) = Ar^\alpha$ for $r \leq a$. a) Calculate the gravitational field strength and the potential inside the planet for this distribution. b) For what values of α is the problem solvable with finite planet mass. c) For what value of α does gravity grow stronger towards the center.

∗ **3.13** An "exponential star" has a mass density $\rho = \rho_0 e^{-r/a}$, where ρ_0 is the central mass density and a is the "radius". Calculate the gravitational field and potential.

∗ **3.14** a) Calculate the gravitational potential and field from a mass distribution shaped like a very thin line (a model of a cosmic string) of length $2a$ with uniform mass λ per unit of length. b) Calculate the behaviour of the potential at infinity orthogonally to the line. c) Discuss what happens in the limit of $a \to \infty$.

∗ **3.15** a) Write an expression for the gravitational potential from a mass distribution shaped like a very thin circular plate of radius a with uniform mass σ per unit of area (a model of the luminous matter of a galaxy). b) Calculate the value of the potential along the central normal of the plate. c) Calculate its form far from the disk. d) What happens for $a \to \infty$?

