(c) Integrate over z to get

$$\int_{-d}^{0} \left\langle v_x^2 + v_z^2 \right\rangle \, dz = \int_{-d}^{0} \nabla_z \left\langle \Psi v_z \right\rangle \, dz = \left\langle \Psi v_z \right\rangle_{z=0} = \frac{1}{2} ac \coth kd \, a\omega = \frac{1}{2} a^2 g_0$$

Multiplying with $\frac{1}{2}\rho_0 A$ we get (22-45).

22.10 The kinetic energy averaged over a period is

$$\langle \mathcal{T} \rangle = \frac{1}{\tau} \int_0^\tau \frac{1}{2} m \dot{\boldsymbol{x}}^2 \, dt = -\frac{1}{\tau} \int_0^\tau \frac{1}{2} m \boldsymbol{x} \cdot \ddot{\boldsymbol{x}} \, dt \tag{22-A11}$$

$$= \frac{1}{2\tau} \int_0^\tau \boldsymbol{x} \cdot \frac{\partial \mathcal{V}}{\partial \boldsymbol{x}} \, dt = \frac{n}{2} \langle \mathcal{V} \rangle \tag{22-A12}$$

where we have integrated partially, used the periodicity of the orbit, and Newton's second law.

22.11 Consider a wave rolling in at an angle towards the beach. Since for shallow-water waves we have $c \sim \sqrt{d}$, the phase velocity of the part of a wave farther from the beach is greatest, causing the crests farther out to approach the coastline faster than the crests closer to the beach.

22.12 For $2\alpha/R = p_0$ or $R = 2\alpha/p_0 \approx 1.5$ μm.

22.13

(a) The waves cross for $c = c_g$ or $kR_c = 1$, *i.e.* for $\lambda = \lambda_c = 1.7$ cm in water. The common velocity is $c = c_g = \sqrt{2 g_0 R_c} = 23$ cm/s.

(b) The minimum of the phase velocity is obtained by

$$\frac{dc}{dk} = \frac{1}{2} c^3 g_0 \left(-\frac{1}{k^2} + R_c^2 \right) = 0 \tag{22-A13}$$

which also happens for $kR_c = 1$.

23 Whirls and vortices

23.3 Use that the field is irrotational outside the core of the Rankine vortex.

23.4 a) According to (19-52a), we have for circulating motion,

$$\frac{1}{\rho_0} \frac{dp^*}{dr} = \frac{v_\phi^2}{r} \ . \tag{23-A1}$$

Integrating this equation one gets

$$\frac{p^*}{\rho_0} = \begin{cases} -(c^2 - \frac{1}{2} r^2)\Omega^2 & 0 \le r \le c \ , \\ -\dfrac{\Omega^2 c^4}{2r^2} & c \le r < \infty \ . \end{cases} \tag{23-A2}$$

b) The surface shape is obtained by requiring the true pressure $p = p^* - \rho_0 g_0 z$ to be constant for $z = h(r)$, so that

$$h(r) = L + \frac{p^*}{\rho_0 g_0} \ , \tag{23-A3}$$

where L is the asymptotic height.

c) The depth of the depression is

$$d = L - h(0) = \frac{\Omega^2 c^2}{g_0} \ . \tag{23-A4}$$

d) $\Omega = 63\ \mathsf{s}^{-1}$ and $d = 4\ \mathsf{cm}$.

23.6 The streamlines are obtained by solving (15-15) in cylindrical coordinates,

$$r\dot{\phi} = v_\phi \ , \qquad \dot{r} = v_r \ , \qquad \dot{z} = v_z \ . \tag{23-A5}$$

The last two are elementary to integrate with the result

$$r = r_0 e^{-t/2t_a} \ , \qquad z = z_0 e^{t/t_a} \ , \tag{23-A6}$$

which after elimination of t becomes

$$z = z_0 \left(\frac{r_0}{r}\right)^2 \ . \tag{23-A7}$$

23.8

(a) Insert and verify.

(b) The angular momentum is

$$\mathcal{L}_z = \int_0^\infty r\rho_0 v_\phi(r,t)\ 2\pi r L\, dr = 16\pi\tau\nu^2\rho_0 L \tag{23-A8}$$

23.9

(a) Insert v_ϕ into (23-6) to obtain

$$\frac{d^2 f(\xi)}{d\xi^2} + \frac{df(\xi)}{d\xi} = \frac{t}{F(t)}\frac{dF(t)}{dt}\frac{1}{\xi}f(\xi) \ . \tag{23-A9}$$

(b) The t-dependent factor must be a constant, $-\alpha$, so $F(t) \sim t^{-\alpha}$.

(c) Insert and verify that the series expansion satisfies

$$\frac{d^2 f(\xi)}{d\xi^2} + \frac{df(\xi)}{d\xi} + \frac{\alpha}{\xi}f(\xi) = 0 \ . \tag{23-A10}$$

The expansion is a confluent hypergeometric function.

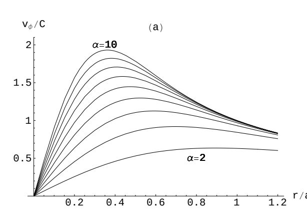
(d) For integer α the functions are Laguerre polynomials multiplied with $e^{-\xi}$. The first few such solutions are

$$f_0(\xi) = 1 - e^{-\xi} \ , \tag{23-A11a}$$

$$f_1(\xi) = \xi e^{-\xi} \ , \tag{23-A11b}$$

$$f_2(\xi) = \xi(1 - \xi/2)e^{-\xi} \ . \tag{23-A11c}$$

The Oseen-Lamb vortex corresponds to $f_0(\xi)$ and the Taylor vortex (problem 23.8) to $f_1(\xi)$.



Family of self-similar vortex shapes $f_\alpha(\xi)$ with α in steps of 0.5 from 0 (top) to 3 (bottom).