20.7 The pressure inside the bubble at depth $z < 0$ equals the hydrostatic pressure (disregarding the tiny effect of surface tension)

$$p = p_0 - \rho_0 g_0 z \; . \tag{20-A11}$$

where p_0 is atmospheric pressure, ρ_0 the density of the liquid, and α the liquid/gas surface tension. The gas pressure is related to the density ρ by the ideal gas law $p \sim \rho$, and the density of the gas in a bubble of fixed mass is $\rho \sim R^{-3}$. Thus the pressure in the bubble may be written

$$p = p_0 \left(\frac{R_0}{R}\right)^3 \; , \tag{20-A12}$$

where R_0 is the radius of the bubble at the surface. Combining the two equations we have obtained a relation between the depth below the surface and the radius of the bubble

$$z = h_0 \left(1 - \left(\frac{R_0}{R}\right)^3\right) \; , \qquad h_0 = \frac{p_0}{\rho_0 g_0} \; . \tag{20-A13}$$

The equation of motion for a bubble of mass m is

$$m\ddot{z} = -6\pi\eta R\dot{z} - \left(\frac{4}{3}\pi R^3 \rho_0 - m\right) g_0 z \tag{20-A14}$$

where the first term in the right hand side is the Stokes friction, and the second is the force of buoyancy. Putting $m = 0$ we get

$$\frac{\dot{z}}{z} = -\frac{2}{9}\frac{\rho_0 g_0}{\eta} R^2 \tag{20-A15}$$

which together with the relation between depth and radius leads to an ordinary differential equation for z, which may easily be solved numerically.

21 Computational fluid dynamics

21.1 The last term is easily integrated, because

$$\delta \int \boldsymbol{v} \cdot \boldsymbol{\nabla} q \, dV = \int dV [\delta \boldsymbol{v} \cdot \boldsymbol{\nabla} q + \delta \boldsymbol{v} \cdot \boldsymbol{\nabla} \delta q] = \int dV \delta \boldsymbol{v} \cdot \boldsymbol{\nabla} q$$

where we have used Gauss theorem in the last step, and dropped the surface terms.

The middle term is also easily integrated, because

$$\delta \int dV \frac{1}{2} \sum_{ij} (\nabla_i v_j)^2 = \int dV \sum_{ij} \nabla_i v_j \nabla_i \delta v_j = \int dV [-\boldsymbol{\nabla}^2 \boldsymbol{v}] \cdot \delta \boldsymbol{v}$$

where we again have used Gauss theorem and discarded boundary terms.

The problem arises from the inertia term $\delta\boldsymbol{v} \cdot (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = \sum_{ij} \delta v_i v_j \nabla_j v_i$. Assume that the integral is an expression of the form $\sum_{ijkl} a_{ijkl} v_i v_j \nabla_k v_l$ with suitable coefficients

a_{ijkl} satisfying $a_{ijkl} = a_{jikl}$. Varying the velocity and again dropping boundary terms we get (suppressing the integral as well as the sums over repeated indices)

$$\begin{aligned}\delta(a_{ijkl}v_iv_j\nabla_kv_l) &= 2a_{ijkl}\delta v_iv_j\nabla_kv_l + a_{ijkl}v_iv_j\nabla_k\delta v_l\\ &= 2a_{ijkl}\delta v_iv_j\nabla_kv_l - 2a_{ijkl}\delta v_lv_j\nabla_kv_i\\ &= 2(a_{ijkl} - a_{ljki})\delta v_iv_j\nabla_kv_l\end{aligned}$$

In order for this to reproduce the desired result $\delta v_iv_j\nabla_jv_i$ we must have

$$a_{ijkl} - a_{ljki} = \frac{1}{2}\delta_{il}\delta_{kj} \tag{21-A1}$$

but that is impossible because the left hand side is antisymmetric under interchange of i and l.

21.2 Under a small variation $\delta p(\boldsymbol{x})$ we find

$$\delta\mathcal{E} = \int_V (\boldsymbol{\nabla}p\cdot\boldsymbol{\nabla}\delta p + s\delta p)\,dV = \int_V \left(-\boldsymbol{\nabla}^2p + s\right)\delta p\,dV \tag{21-A2}$$

where the surface terms in the integral have been dropped (assuming either $p = 0$ or $\boldsymbol{n}\cdot\boldsymbol{\nabla}p = 0$ on the surface). This vanishes only for arbitrary variations when the Poisson equation is fulfilled. Choosing

$$\delta p = \epsilon(\boldsymbol{\nabla}^2p - s) \tag{21-A3}$$

will make $\delta\mathcal{E}$ negative and make the field converge towards the desired solution.

22 Surface waves

22.1 The wave becomes

$$\begin{aligned}h &= \mathcal{R}e\int_{-\infty}^{\infty} a(k)\exp[i(kx - \omega(k)t + \chi(k))]\,dk\\ &= \frac{1}{\Delta k\sqrt{\pi}}\mathcal{R}e\int_{-\infty}^{\infty}\exp\left(i(k_0x - \omega_0t + \chi_0) + i(k-k_0)(x - c_gt - x_0) - \frac{(k-k_0)^2}{\Delta k^2}\right)dk\\ &= \frac{1}{\sqrt{\pi}}\mathcal{R}e\int_{-\infty}^{\infty}\exp\left(i(k_0x - \omega_0t + \chi_0) + iu\Delta k(x - c_gt - x_0) - u^2\right)du\\ &= \frac{1}{\sqrt{\pi}}\mathcal{R}e\int_{-\infty}^{\infty}\exp\left[i(k_0x - \omega_0t + \chi_0) - \left(u - \frac{i}{2}\Delta k(x - c_gt - x_0)\right)^2 - \frac{1}{4}\Delta k^2(x - c_gt - x_0)^2\right]du\\ &= \frac{1}{\sqrt{\pi}}\mathcal{R}e\int_{-\infty}^{\infty}\exp\left[i(k_0x - \omega_0t + \chi_0) - u^2 - \frac{1}{4}\Delta k^2(x - c_gt - x_0)^2\right]du\\ &= \cos(k_0x - \omega_0t + \chi_0)\exp\left[-\frac{1}{4}\Delta k^2(x - c_gt - x_0)^2\right] .\end{aligned}$$

In the second line we have substituted $k = k_0 + u\Delta k$ and in the third we have rearranged the resulting quadratic form. In the fourth we shift $u \to u + \frac{i}{2}\Delta k(x - c_gt - x_0)$ and in the fifth we use that $\int_{-\infty}^{\infty}\exp(-u^2)\,du = \sqrt{\pi}$.

The wave contains a single wave packet with a Gaussian envelope of width $\sim 1/\Delta k$ with the center moving along $x = x_0 + c_gt$. The phase shift derivative $x_0 = -d\chi/dk$ determines the position of the center at $t = 0$.

22.15

(a) For $n = 0$ it is trivial. For $n \neq 0$, the sum is geometric with progression factor $F = \exp(2\pi i n/N)$

$$\sum_{m=0}^{N-1} \exp\left[2\pi i \frac{nm}{N}\right] = \sum_{m=0}^{N-1} F^m = \frac{1 - F^N}{1 - F} = 0 \tag{22-A1}$$

because $F \neq 1$ but $F^N = 1$.

(b) Write the last expression as a double sum

$$h_n = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} h_k \exp\left[2\pi i \frac{(k-n)m}{N}\right] \tag{22-A2}$$

and do the m-sum first.

(c) Do the triple sum

$$\sum_n |h_n|^2 = \frac{1}{N} \sum_{n,m,k} \hat{h}_m \hat{h}_k^\times \exp\left[2\pi i \frac{(k-m)n}{N}\right] \tag{22-A3}$$

Do the sum over n first.

22.2 From the solution (22-35) we find

$$\frac{\partial v_z}{\partial t} + \frac{1}{\rho_0} \nabla_z p + g_0 = -a\omega^2 \left(1 + \frac{z}{d}\right) \cos(kx - \omega t) \tag{22-A4}$$

which ought to vanish. Since the finite-depth solution does satisfy the field equations, the problem must lie in the higher-order terms in kz we have dropped in the shallow-water limit.

22.3 From the flat-bottom solution (22-32) we find the particle orbit equations

$$\frac{dx}{dt} = v_x = a\omega \frac{\cosh k(z+d)}{\sinh kd} \sin(kx - \omega t) \ ,$$
$$\frac{dz}{dt} = v_z = a\omega \frac{\sinh k(z+d)}{\sinh kd} \cos(kx - \omega t) \ .$$

Under the assumption of small amplitude $ak \ll 1$, we have the approximative solution

$$x = x_0 - a \frac{\cosh k(z_0 + d)}{\sinh kd} \sin(kx_0 - \omega t) \tag{22-A5}$$

$$z = z_0 + a \frac{\sinh k(z_0 + d)}{\sinh kd} \cos(kx_0 - \omega t) \tag{22-A6}$$

This orbit is an ellipse centered at (x_0, z_0) with major axis $a \cosh k(z_0 + d)/\sinh kd$ and minor axis $a \sinh k(z_0+d)/\sinh kd$. For $z_0 \to -d$, the minor axis vanishes and the ellipse degenerates into a horizontal line. In the deep-water limit, the ellipses degenerate into circles of radius ae^{kz_0}.

22.5 Expanding to first order in h, we have

$$\bar{v}_x = \frac{1}{d}\int_{-d}^{0} v_x\, dz + \frac{h}{d}\, v_x|_{z=0} - \frac{h}{d^2}\int_{-d}^{0} v_x\, dz \tag{22-A7}$$

and

$$\langle \bar{v}_x \rangle = \frac{1}{d}\,\langle h v_x\rangle_{z=0} - \frac{1}{d^2}\int_{-d}^{0} \langle h v_x\rangle\; dz \tag{22-A8}$$

Inserting (22-32) and integrating, the desired result is obtained. The expression vanishes for $d \to 0$.

22.6 The total amount of water in a shallow-water wave is $M = \rho_0 \lambda L d$. The ratio between the transported mass and the total mass is

$$\frac{\langle Q\rangle\,\tau}{\rho_0 L \lambda d} = \frac{a^2}{2d^2} \tag{22-A9}$$

For $a/d \approx 0.1$, it is only half a percent.

22.7 Since $v_x = \nabla_x \Psi$ we have

$$\int_{-d}^{h} v_x\, dz = \nabla_x \int_{-d}^{h} \Psi\, dz - \nabla_x h\; \Psi|_{z=h} \tag{22-A10}$$

Since the function only depends on $kx - \omega t$, the average over a period is equivalent to an average over a wavelength. But then the average of the first term vanishes because of periodicity. For small amplitudes the last term may similarly be recast as $\langle -\nabla_x h \Psi\rangle_{z=h} \approx \langle h v_x\rangle_{z=h}$.

22.8 Using (22-19) we get

$$\begin{aligned}\langle \mathcal{F}_x\rangle &= -\left\langle \int_{-d}^{h} p\, L dz\right\rangle = \left\langle \int_{-d}^{h} \left(p_0 - \rho_0\left(g_0 z + \frac{\partial \Psi}{\partial t} + \frac{1}{2}(v_x^2 + v_z^2)\right)\right) L dz\right\rangle \\ &= p_0 L d + \frac{1}{2}\rho_0 g_0 d^2 L - \frac{1}{2}\rho_0 g_0 a^2 L - \rho_0 L \left\langle \int_{-d}^{h} \frac{\partial \Psi}{\partial t}\, dz\right\rangle\end{aligned}$$

where we have used the expression for the total energy (22-46). Using the periodicity we find

$$\rho_0 L \left\langle \int_{-d}^{h} \frac{\partial \Psi}{\partial t}\, dz\right\rangle$$

22.9

(a) Use mass conservation $\nabla_x v_x + \nabla_z v_z = 0$ to get

$$\nabla_x(\Psi v_x) + \nabla_z(\Psi v_z) = v_x \nabla_x \Psi + \psi \nabla_x v_x + v_z \nabla_z \Psi + \Psi \nabla_z v_z = v_x^2 + v_z^2$$

(b) Since Ψv_x is a periodic function of $x - ct$ we have

$$\langle \nabla_x(\Psi v_x)\rangle = \frac{1}{\tau}\int_0^{\tau} \nabla_x(\Psi v_x)\, dt = \frac{1}{\lambda}\int_0^{\lambda} \nabla_x(\Psi v_x)\, dx = \left[\Psi v_x\right]_0^{\lambda} = 0$$

