the corners. Since their side length must vary linearly with z from $t = \frac{1}{2}\sqrt{2}$ for $z = 0$ to $t = 0$ for $z = -\frac{1}{6}\sqrt{3}$, it must be $t = \frac{1}{2}\sqrt{2} + z\sqrt{6}$. The area is thus $A(z) = \frac{1}{4}\sqrt{3}(s^2 - 3t^2)$ for $-\frac{1}{6}\sqrt{3} < z < 0$. One may verify that the two area functions reproduce the volume correctly. The center of buoyancy becomes $z_B = -\frac{13}{96}\sqrt{3}$ and the metacenter

$$z_M = z_B + \frac{I}{V} = \frac{1}{48}\sqrt{3} \tag{5-A15}$$

Thus the metacenter is above the center of gravity ($z_G = 0$) and the cube floats stably in this configuration.

6 Planets and stars

6.1 Put $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{a}p(\boldsymbol{x})$ where $\boldsymbol{a}$ is an arbitrary constant vector.

6.6 Use (6-15) and (3-28).

6.7 Integrate (6-15) using the field (3-A6), and we find

$$p(r) = \begin{cases} p_c - \frac{2}{3}\pi G\rho_1^2 r^2 & \text{for } r \le a_1 \;, \\ \\ p_c - \frac{2}{3}\pi G\rho_1^2 a_1^2 - \frac{2}{3}\pi G\rho_2^2(r^2 - a_1^2) & \\ \\ \qquad -G\rho_0(\rho_1 - \rho_2)a_1^3\left(\dfrac{1}{a_1} - \dfrac{1}{r}\right) & \text{for } a_1 \le r \le a \;. \end{cases} \tag{6-A1}$$

Apart from an additive constant and overall normalization, this is exactly the same as the minus the gravitational potential (see problem 6.6 and fig. 3.4). Setting $r = a$, the central pressure becomes

$$p_c = \frac{2}{3}\pi G\rho_1^2 a_1^2 + \frac{2}{3}\pi G\rho_2^2(a^2 - a_1^2) + \frac{4}{3}\pi G\rho_2(\rho_1 - \rho_2)a_1^3\left(\frac{1}{a_1} - \frac{1}{a}\right) \;. \tag{6-A2}$$

Again the total mass may be used to establish a relation between the densities and the radii, $M_0 = \frac{4}{3}\pi(a_1^3\rho_1 + (a^3 - a_1^3)\rho_2)$.

6.11

a) From (**??**) $\rho \sim r^{\alpha/(\gamma-1)}$ and from (6-21) one gets $\alpha - 2 = \dfrac{\alpha}{\gamma - 1}$, which implies $\alpha = 2\dfrac{\gamma-1}{\gamma-2}$. Thus $\alpha < 0$ for $1 < \gamma < 2$.

b) $M \sim \int r^2\rho\, dr \sim r^{3+\alpha/(\gamma-1)} \sim r^{(3\gamma-4)/(\gamma-2)}$. The exponent is positive for $1 < \gamma < \dfrac{3}{4}$.

c) $E \sim \int M^2/r^2\, dr \sim r^{(5\gamma-6)/(\gamma-2)}$. The exponent is positive for $1 < \gamma < \dfrac{6}{5}$.

6.12 Use the approximation (6-22).

6.13

a) Insert into (6-28).

b) $p \sim (1+s^2/3)^{-3}$, $\rho \sim (1+s^2/3)^{-5/2}$.

c) $M \sim \int \rho r^2\, dr \sim \int (1+s^2/3)^{-5/2} s^2\, ds \approx \int s^{-3} ds$ converges for $s \to \infty$.

6.14 The gravitational energy is $-E/M \approx 25,000$ J/g. Heating and melting iron takes only about $8,000$ J/g. The Earth definitely melted while its bulk accumulated.

7 Hydrostatic shapes

7.1 The centrifugal acceleration is $a\Omega^2 = g_0$ so that $\Omega = \sqrt{g_0/a}$. The potential is $\Phi = -\frac{1}{2}a^2\Omega^2$ at the cylinder. Use (3-37) to get $v_{\rm esc} = \sqrt{ag_0} \approx 316$ m/s for $a = 10$ km.

7.3 a) The extra pressure $\Delta p(\boldsymbol{x})$ created by the field of the spheres is determined by

$$w(p_0 + \Delta p) + \Phi_1 + \Phi_2 = w(p_0) \ , \tag{7-A1}$$

with the potentials of the two spheres

$$\Phi_1 = -\frac{g_1 a^2}{\sqrt{x^2+y^2+z^2}} \ , \quad \Phi_2 = -\frac{g_1 a^2}{\sqrt{x^2+y^2+(D-z)^2}} \ , \tag{7-A2}$$

where $g_1 = \frac{4\pi}{3}\rho_1 G a$ is the surface gravity of the spheres. Expanding to first order in Δp, we obtain

$$\Delta p = -\rho_0(\Phi_1 + \Phi_2) \tag{7-A3}$$

where $\rho_0 = \rho(p_0)$ is the density in the absence of the spheres. At the surface of sphere 1, $r = a$, we obtain the extra pressure

$$\Delta p_1 = \rho_0 a g_1 \left(1 + \frac{a}{D} + \left(\frac{a}{D}\right)^2 \cos\theta\right) \ , \tag{7-A4}$$

where we have expanded to leading non-trivial order in a/D, and where θ is the polar angle of the point at the surface of sphere 1. Integrating over the surface of sphere 1, we obtain the extra force in the z-direction

$$\mathcal{F}_1 = \int_{r=a} (-\Delta p_1) dS_z = -2\pi a^2 \int_0^\pi \Delta p_1(\theta) \cos\theta \sin\theta\, d\theta = -\frac{4\pi}{3}\rho_0 g_1 \frac{a^5}{D^2} \ , \tag{7-A5}$$

which is a repulsion.

b) The gravitational attraction from sphere 2 on sphere 1 is

$$\mathcal{G}_z = \frac{GM^2}{D^2} = \frac{4\pi}{3}\rho_1 g_1 \frac{a^5}{D^2} \tag{7-A6}$$

and the ratio becomes $\mathcal{F}_1/\mathcal{G}_1 = -\rho_0/\rho_1$. c) If $\rho_1 = \rho_0$, then the total force becomes $\mathcal{F}_1 + \mathcal{G}_1 = 0$, as one would expect.

7.4 Let h be the change in sea level due to p. Use constancy of (7-2) in the water close to the surface to get $p_0/\rho_0 = p/\rho_0 + g_0 h$, where p_0 is the pressure far away from the high pressure region. Then $h = -(p-p_0)/\rho_0 g_0 \approx -20$ cm.