2.19 Differentiate $(\boldsymbol{x}-\boldsymbol{y})^2 = (\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{y}))^2$ after $\boldsymbol{x}$ to obtain $\boldsymbol{x}-\boldsymbol{y} = (\boldsymbol{f}(\boldsymbol{x})-\boldsymbol{f}(\boldsymbol{y}))\cdot\mathbf{a}(\boldsymbol{x})$ with $\mathbf{a}(\boldsymbol{x}) = \partial\boldsymbol{f}(\boldsymbol{x})/\partial\boldsymbol{x}$. Differentiate again after $\boldsymbol{y}$ to obtain $-\mathbf{1} = -\mathbf{a}(\boldsymbol{y})^\top\cdot\mathbf{a}(\boldsymbol{x})$. This means that $\mathbf{a}(\boldsymbol{x})^{-1} = \mathbf{a}(\boldsymbol{y})^\top$. The left hand side depends only on $\boldsymbol{x}$ and the right hand side only on $\boldsymbol{y}$ which implies that both sides are independent of $\boldsymbol{x}$ and $\boldsymbol{y}$, *i.e.* the matrix $\mathbf{a}$ is a constant. Integrating $\partial\boldsymbol{f}(\boldsymbol{x})/\partial\boldsymbol{x} = \mathbf{a}$ one gets $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{a}\cdot\boldsymbol{x}+\boldsymbol{b}$.

2.20 Let $\mathbf{a}_z(\phi)$ be the matrix of the simple rotation (2-40) through an angle ϕ around the z-axis. Then the three Euler angles ϕ, θ and ψ determine any rotation matrix as a product $\mathbf{a}_z(\psi)\cdot\mathbf{a}_y(\theta)\cdot\mathbf{a}_z(\phi)$.

3 Gravity

3.2 The centripetal acceleration in a circular orbit must equal the force of gravity, $v^2/r = GM/r^2$ leading to $v = \sqrt{GM/r} = \sqrt{-\Phi}$. At ground level the velocity becomes $v = v_{\mathsf{esc}}/\sqrt{2} = 7.9\ \mathsf{km/s}$ where $v_{\mathsf{esc}} = 11.2\ \mathsf{km/s}$ is the escape velocity.

3.3 Earth's true rotation period $T = T_0 * 364/365$ is a bit shorter than $T_0 = 24$ hours because of the orbital motion which adds one full revolution in one year. Taking $v = \Omega r$ where $\Omega = 2\pi/T$ we find from the equality of centripetal acceleration and gravity that

$$r\Omega^2 = g_0\frac{a^2}{r^2}\ . \tag{3-A1}$$

which solved for r/a yields

$$\frac{r}{a} = \left(\frac{g_0}{a\Omega^2}\right)^{1/3} \approx 6.613\ . \tag{3-A2}$$

The orbit height is $h = r - a \approx 5.613a \approx 35,800\ \mathsf{km}$.

3.4 At the height z above the ground the force on a small piece dz of the line is

$$d\mathcal{F} = \left(-g_0\frac{a^2}{(a+z)^2} + (a+z)\Omega^2\right)\rho A\,dz \tag{3-A3}$$

where Ω is the angular velocity in the stationary orbit and the second term represents the centrifugal force. Since this only vanishes for $z = h$, the total force is maximal at the satellite. Integrating the force from 0 to h, we find the maximal force

$$\mathcal{F} = \int_0^h d\mathcal{F}(z) = \rho A h\left(-g_0\frac{a}{a+h} + \Omega^2\left(a+\frac{1}{2}h\right)\right)\ . \tag{3-A4}$$

The absolute value of the tension-to-density ratio becomes,

$$\frac{\sigma}{\rho} = h\left(g_0\frac{a}{a+h} - \Omega^2\left(a+\frac{1}{2}h\right)\right) \approx 4.8\times10^7\ \mathsf{m^2/s^2} \tag{3-A5}$$

The tensile strength a Beryllium-Copper alloy of density $\rho = 8230\ \mathsf{kg/m^3}$ can go as high as $\sigma \approx 1.4$ GPa, leading to $\sigma/\rho \approx 1.7\times10^5\ \mathsf{m^2/s^2}$, a factor nearly 300 below the required value.

3.7 A small volume is invariant under a rotation $dv' = dv$ and so is the amount of mass contained in it, $dm' = dm$. By the definition (3-1) we have $dm' = \rho'(\boldsymbol{x}')dv' = dm = \rho(\boldsymbol{x})dv$ and from that $\rho'(\boldsymbol{x}') = \rho(\boldsymbol{x})$.

3.8 The force on a small volume transforms according to $d\boldsymbol{F}' = \mathbf{a} \cdot d\boldsymbol{F}$ whereas the mas element is invariant $dm' = dm$. By the definition (3-5) we have $d\boldsymbol{F}' = \boldsymbol{g}'(\boldsymbol{x}')\,dm' = \mathbf{a} \cdot d\boldsymbol{F} = \mathbf{a} \cdot \boldsymbol{g}(\boldsymbol{x})dm$ and from this $\boldsymbol{g}'(\boldsymbol{x}') = \mathbf{a} \cdot \boldsymbol{g}(\boldsymbol{x})$.

3.10 Cut out a small sphere $|\boldsymbol{x}' - \boldsymbol{x}| \leq a$ around the point $\boldsymbol{x}$. Let a be so small that $\rho(\boldsymbol{x}')$ does not vary appreciably within this sphere. Then we get the contribution to gravity from the small sphere

$$\Delta \boldsymbol{g}(\boldsymbol{x}) = -G \int_{|\boldsymbol{x}'-\boldsymbol{x}|\leq a} \frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,\rho(\boldsymbol{x}')\,dv' \approx -G\rho(\boldsymbol{x}) \int_{|\boldsymbol{x}'-\boldsymbol{x}|\leq a} \frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,dv' = \mathbf{0}$$

The last integral vanishes because of the spherical symmetry (it is a vector with no direction to point in).

3.5

a) Minimal kinetic energy: $\frac{1}{2}v_{\rm esc}^2 \approx 63\ (\mathsf{km/s})^2 = 63 \times 10^6\ \mathsf{J/kg}$.

b) Melting, heating and evaporating ice: $\approx 3 \times 10^6\ \mathsf{J/kg}$.

3.6 Energy conservation: $\frac{1}{2}\dot{r}^2 + \Phi(r) = \Phi(a)$. Use (3-31).

a) $v_0 = -\left.\dot{r}\right|_{r=0} = a\sqrt{\frac{4}{3}\pi\rho_0 G} = \sqrt{g_0 a} = 7.9\ \mathsf{km\ s^{-1}}$.

b) $t_0 = \int_0^a \dfrac{dr}{\sqrt{2(\Phi(a)-\Phi(r))}} = \int_0^a \dfrac{dr}{\sqrt{\frac{4}{3}\pi\rho_0 G(a^2-r^2)}} = \dfrac{\pi a}{2v_0} = 1267\ \mathsf{s}$.

3.11 From (3-17) we get

$$g(r) = -\frac{4}{3}\pi G \begin{cases} r\rho_1 & r \leq a_1 \\ \dfrac{a_1^3}{r^2}\rho_1 + \left(r - \dfrac{a_1^3}{r^2}\right)\rho_2 & a_1 < r \leq a \\ \dfrac{a_1^3\rho_1 + (a^3 - a_1^3)\rho_2}{r^2} & r > a \end{cases} . \tag{3-A6}$$

and from (3-28)s

$$\Phi(r) = -\frac{2}{3}\pi G \begin{cases} (3a_1^2 - r^2)\rho_1 + 3(a^2 - a_1^2)\rho_2 & r \leq a_1 \\ 2\dfrac{a_1^3}{r}\rho_1 + \left(3a^2 - r^2 - 2\dfrac{a_1^3}{r}\right)\rho_2 & a_1 \leq r \leq a \\ 2\dfrac{a_1^3}{r}\rho_1 + 2\dfrac{a^3 - a_1^3}{r}\rho_2 & r \geq a \end{cases} . \tag{3-A7}$$

3.12 Using the two-layer model it follows from $|g(a_1)| > |g(a)|$, that $a_1\rho_1 > (a_1^3\rho_1 + (a^3 - a_1^3)\rho_2)/a^2$ which may be rewritten in the form of the inequality (3-43). For the Earth the left hand side becomes 1.42 and the right hand side 1.18, so the inequality is fulfilled.

3.13

a) $g(r) = -4\pi G \dfrac{A}{3+\alpha} r^{1+\alpha}$, $\Phi(r) = 4\pi G \dfrac{A}{2+\alpha}\left(\dfrac{r^{2+\alpha}}{3+\alpha} - a^{2+\alpha}\right)$.

b) $\alpha > -3$.

c) $-3 < \alpha < -1$.

3.14 Use eq. (3-17). Setting $u = r/a$ one gets

$$M(r) = \int_0^r \rho(s) 4\pi s^2\, ds = 4\pi\rho_0 \int_0^r e^{-s/a} s^2\, ds = 4\pi\rho_0 a^3 \left(2 - (2 + 2u + u^2)e^{-u}\right)$$

Similarly, using (3-30) one finds

$$\int_r^\infty s\rho(s)\, ds = \rho_0 \int_r^\infty s e^{-s/a}\, ds = \rho_0 a^2 (1+u) e^{-u}$$

and from this

$$\Phi = -\frac{4\pi G \rho_0 a^3}{r}\left(2\left(1 - e^{-u}\right) - u e^{-u}\right)$$

3.15 Multiplying (3-13) by $\boldsymbol{e}_r = \boldsymbol{x}/r$ and using (3-16) one gets

$$g(r) = -G \int_{|\boldsymbol{x}'| \le a} \frac{\boldsymbol{x} \cdot (\boldsymbol{x} - \boldsymbol{x}')}{r\,|\boldsymbol{x} - \boldsymbol{x}'|^3} \rho(\boldsymbol{x}')\, dv'$$

Introducing $s = |\boldsymbol{x}'|$ and the angle θ between $\boldsymbol{x}$ and $\boldsymbol{x}'$, so that $dv' = 2\pi \sin\theta d\theta s^2 ds$, this becomes

$$g(r) = -2\pi G \int_0^a \rho(s) s^2 ds \int_{-1}^1 d\cos\theta \frac{r - s\cos\theta}{(r^2 + s^2 - 2rs\cos\theta)^{\frac{3}{2}}}$$

Integrating over $u = \cos\theta$ one obtains

$$\int_{-1}^1 du \frac{r - su}{(r^2 + s^2 - 2rsu)^{\frac{3}{2}}} = -\frac{\partial}{\partial r}\int_{-1}^1 du \frac{1}{\sqrt{r^2 + s^2 - 2rsu}}$$

$$= \frac{\partial}{\partial r}\left[\frac{\sqrt{r^2 + s^2 - 2rsu}}{rs}\right]_{u=-1}^1 = \frac{\partial}{\partial r}\frac{|r-s| - (r+s)}{rs}$$

$$= -2\frac{\partial}{\partial r}\begin{cases} \dfrac{1}{r} & r > s \\ \dfrac{1}{s} & r < s \end{cases} = \frac{2}{r^2}\theta(r-s)$$

which leads to the desired result (3-17).