5 Buoyancy

'Buoy' mostly pronounced 'booe', probably of Germanic origin. A tethered floating object used to mark a location in the sea.

Fishes, whales, submarines, balloons and airships all owe their ability to float to *buoyancy*, the lifting power of water and air. The understanding of the physics of buoyancy goes back as far as antiquity and has probably sprung from the interest in ships and shipbuilding in classic Greece. The basic principle is due to Archimedes. His famous Law states that the buoyancy force on a body is equal and oppositely directed to the weight of the fluid that the body replaces. Actually the Law was not just one law, but a set of four propositions dealing with different configurations of body and liquid [7]. Before his time one had thought that the shape of a body determined whether it would sink or float.

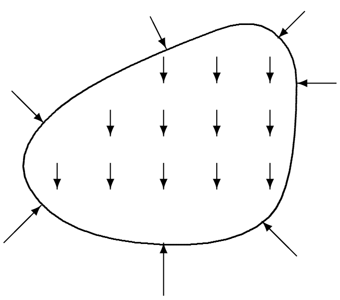
Archimedes of Syracuse (287–212 BC). *Greek mathematician. Discovered the formulas for area and volume of cylinders and spheres. Considered the father of fluid mechanics.*

The shape of a floating body and its mass distribution does determine whether it will float stably or capsize. Stability of floating bodies is of importance to shipbuilding, and to anyone who has ever tried to stand up in a small rowboat. Newtonian mechanics not only allows us to derive Archimedes' Principle for equilibrium of floating bodies, but also to characterize the deviations from equilibrium and calculate the restoring forces. Even if a body floating in or on water is in hydrostatic equilibrium, it will not be in complete mechanical balance in every orientation, because the center of mass of the body and the center of mass of the displaced water do not in general coincide. The latter is also called the center of buoyancy. For a volume of the fluid itself, the center of mass always coincides with the center of buoyancy.

The mismatch between the centers of mass and buoyancy for a floating body creates a moment of force, which tends to rotate the body towards a stable equilibrium. For submerged bodies, submarines, fishes and balloons, the stable equilibrium will always be with the center of gravity situated directly below the center of buoyancy. For bodies floating stably on the surface, ducks, ships, and dumplings, the center of gravity is mostly found directly above the center of buoyancy.

5.1 Archimedes' principle

Mechanical equilibrium takes a slightly different form than global hydrostatic equilibrium (4-14) when a body of another material is immersed in a fluid. If its material is incompressible, the body retains its shape and displaces an amount of fluid with exactly the same volume. If the body is compressible, as a rubber ball, the volume of displaced fluid will be smaller. The body may even take in fluid, like the piece of bread you dunk into your coffee, but then the physics becomes more complicated, and we shall disregard this possibility in the following. A body which is partially immersed may formally be viewed as a body that is fully immersed in a fluid for which the mass density and the equation of state vary from place to place. This also covers the case where part of the body is in vacuum which may be thought of as a fluid with the extreme properties, $\rho = p = 0$.



Gravity pulls at a body all over its volume, while pressure only presses at the surface.

Weight and buoyancy

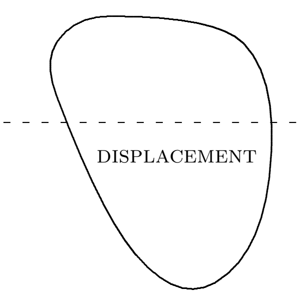
Let the actual, perhaps compressed, volume of the immersed body be V with surface S. In the field of gravity an unrestrained body is subject to two forces: its weight

$$\boldsymbol{\mathcal{F}}_G = \int_V \rho_{\text{body}}\, \boldsymbol{g}\, dV \ , \tag{5-1}$$

and the buoyancy due to pressure acting at its surface,

$$\boldsymbol{\mathcal{F}}_B = -\oint_S p\, d\boldsymbol{S} \ . \tag{5-2}$$

In general these two forces do not have to be in balance. The resultant $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_G + \boldsymbol{\mathcal{F}}_B$ determines the direction that the body will begin to move if unrestrained. In mechanical equilibrium the two forces must exactly cancel each other so that the body can remain in place.



For a body partially submerged in water the displacement is the amount of water that has been displaced by the volume of the body below the waterline.

Assuming that the body does not itself significantly contribute to the field of gravity (see however page 120), the local balance of forces in the fluid (4-18) will be the same as before. In particular the pressure in the fluid cannot depend on the volume V containing material that is different from the fluid itself. The pressure on the surface of the immersed body must for this reason be identical to the pressure on a body of fluid of the same shape. But then the global equilibrium condition (4-14) tells us that the buoyancy force will exactly balance the weight of the displaced fluid, so that

$$\boldsymbol{\mathcal{F}}_B = -\oint_S p\, d\boldsymbol{S} = -\int_V \rho_{\text{fluid}}\, \boldsymbol{g}\, dV \ . \tag{5-3}$$

This theorem is indeed Archimedes' principle: *the force of buoyancy equals (minus) the weight of the displaced fluid.*

The total force on the body may then be written

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_G + \boldsymbol{\mathcal{F}}_B = \int_V (\rho_{\text{body}} - \rho_{\text{fluid}})\boldsymbol{g}\, dV \ , \tag{5-4}$$

explicitly confirming that when the body is made from the same fluid as its surroundings, so that $\rho_{\text{body}} = \rho_{\text{fluid}}$, the resultant force vanishes automatically. In general, however, the distributions of mass in the body and in the displaced fluid will be different.

Notice that Archimedes' principle is valid even if the gravitational field varies appreciably across the body. Archimedes principle fails, if the body is so large that its own gravitational field cannot be neglected, such as would be the case if an Earth-sized body fell into Jupiter's atmosphere. The extra compression of the fluid and the associated change in buoyancy caused by the body's own gravitational field is calculated section **??**.

Constant field of gravity

If the gravitational field is constant, $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0$, the weight of the body is,

$$\boldsymbol{\mathcal{F}}_G = M_{\text{body}}\, \boldsymbol{g}_0 \ , \tag{5-5}$$

and the buoyancy force becomes

$$\boldsymbol{\mathcal{F}}_B = -M_{\text{fluid}}\, \boldsymbol{g}_0 \ . \tag{5-6}$$

Since the total force is the sum of these contributions, one might say that buoyancy acts as if the displacement were filled with fluid of negative mass $-M_{\text{fluid}}$. Alternatively, one may view the buoyancy force as a kind of antigravity.

The total force on an unrestrained object is now,

$$\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_G + \boldsymbol{\mathcal{F}}_B = (M_{\text{body}} - M_{\text{fluid}})\boldsymbol{g}_0 \ . \tag{5-7}$$

If the body mass is smaller than the mass of the displaced fluid, the total force is directed upwards, and the unrestrained body will begin to move upwards. Alternatively, if the body is chained restrained from moving, the restraints must deliver a force $-\boldsymbol{\mathcal{F}}$ to keep the object in place.

For a body to hover motionless in a fluid, its mass must equal the mass of the displaced fluid,

$$M_{\text{body}} = M_{\text{fluid}} \ . \tag{5-8}$$

Fish achieve this balance by adjusting the amount of water they displace through contraction and expansion of an internal air-filled bladder. Submarines on the contrary change their mass by pumping water in and out of ballast tanks. Curiously, no animals seem to have developed balloons for floating in the atmosphere, although both the physics and chemistry of ballooning appears to be within reach of biological evolution.

5.2 The art of ballooning

Joseph Michel Montgolfier (1740-1810). *Experimented (together with his younger brother Jacques Étienne (1745-1799)) with hot-air balloons and on November 21, 1783, the first human flew in such a balloon for a distance of 9 kilometers at a height of 100 meter above Paris. Only one of the brothers ever flew, and then only once!*

Apart from large kites used in ancient China, balloons were the earliest flying machines. The first balloons made by the Montgolfier brothers in 1783 contained hot air which is lighter than cold. Hot-air balloons were a century later replaced by balloons containing light gases, hydrogen or helium, with greater lifting power. This also eliminated the need for a constant heat supply and made possible the huge (and dangerous) hydrogen airships of the 1930's. In the last half of the twentieth century hot-air balloons again came into vogue, especially for sports, because of the availability of modern strong lightweight materials (nylon) and fuel (propane).

Gas balloons

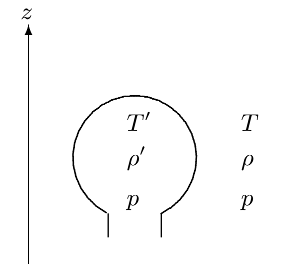
A large hydrogen or helium balloon typically begins its ascent being only partially filled, assuming an inverted tear-drop shape. During the ascent the gas expands because of the fall in ambient air pressure, and eventually the balloon becomes nearly spherical and stops expanding (or bursts) because the "skin" of the balloon cannot stretch further. Since the density of the displaced air falls with height, the balloon will reach a maximum height, a ceiling where it could hover permanently if it did not lose gas. In the end no balloon stays aloft forever.

Let the total mass of the balloon be M, including the mass of the gas, the balloon skin, the gondola, people, and what not. The condition for upwards flight is then that $M < V\rho$ where V is the total volume of air that the balloon displaces and ρ the air density. If this inequality is fulfilled on the ground, the balloon will start to rise. During the rise the volume may expand towards a maximal value while the air density falls, and the balloon will keep rising until the air density has fallen to $\rho = M/V$. In the isentropic atmospheric model the air density is given by (4-60) and solving for the maximal height z we obtain the balloon's ceiling,

$$z = h_2 \left(1 - \left(\frac{M}{\rho_0 V}\right)^{\gamma-1}\right) , \tag{5-9}$$

where $\gamma \approx 7/5$ is the adiabatic index of air, $\rho_0 \approx 1.2\ \mathsf{kg/m^3}$ its density at sea level, and $h_2 \approx 30\ \mathsf{km}$ the isentropic scale height (4-58).

Example 5.2.1: A spherical balloon has a maximal diameter of 10 m yielding a volume $V \approx 524\ \mathsf{m}^3$. For the balloon to lift off at all, its mass must be smaller than $\rho_0 V = 628$ kg. Taking $M = 400$ kg the ceiling becomes $z \approx 5$ km. At this height the air pressure and temperature are 0.54 Bar and 245 K. Assuming that the balloon contains hydrogen H_2 at this temperature and pressure, the total mass of the hydrogen is merely 28 kg. The surface area of the balloon is 314 m^2, so if the skin has thickness 2 mm and density 300 $\mathsf{kg/m^3}$, its mass becomes 188 kg, which leaves about 184 kg for the proper payload. Filled with helium (He), the gas mass would double and the payload be correspondingly smaller.



A hot-air balloon has higher temperature $T' > T$ and lower density $\rho' < \rho$ but the same pressure as the surrounding atmosphere because it is open below.

Hot-air balloons

A hot-air balloon is open at the bottom so that the inside pressure is always the same as the atmospheric pressure outside. The air in the balloon is warmer ($T' > T$) than the outside temperature and the density is lower ($\rho' < \rho$). If M_0 denotes the total payload, the total mass including the hot air is $M = M_0 + \rho' V$. The total mass thus changes with height (and temperature) rather than being constant as for a gas balloon. From the ideal gas law (4-24) and the equality of the inside and outside pressures it follows that $\rho' T' = \rho T$, so that the inside density is $\rho' = \rho T/T'$. The condition for flight is as before $M < \rho V$, and the maximum payload for a given height z becomes,

$$M_0 = \left(1 - \frac{T}{T'}\right) \rho V = \left(1 - \frac{T_0}{T'}\left(1 - \frac{z}{h_2}\right)\right)\left(1 - \frac{z}{h_2}\right)^{\frac{1}{\gamma-1}} \rho_0 V \ . \qquad \text{(5-10)}$$

On the right hand side we have inserted the expressions (4-57) and (4-60) for the isentropic atmospheric temperature T and density ρ.

Example 5.2.2: A hot-air balloon with diameter 15 m is desired to reach a ceiling of 1000 m with air temperature 70 $^\circ\mathsf{C}$ = 343 K. When the ground temperature is T_0 = 20 $^\circ\mathsf{C}$ = 293 K and the density ρ_0 = 1.2 $\mathsf{kg/m^3}$, it follows that this balloon would be capable of lifting $M_0 \approx 341$ kg to the ceiling.

5.3 Stability of floating bodies

Although a body may be in buoyant equilibrium, so that the total force composed of gravity and buoyancy vanishes, $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}_G + \boldsymbol{\mathcal{F}}_B = \mathbf{0}$, it may not be in complete mechanical equilibrium. The total moment of all the forces acting on the body must also vanish; for else an unrestrained body will start to rotate.

Moments of weight and buoyancy

The total moment is like the total force a sum of two contributions,

$$\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_G + \boldsymbol{\mathcal{M}}_B \ , \qquad \text{(5-11)}$$

with one contribution from gravity, *i.e.* weight,

$$\boldsymbol{\mathcal{M}}_G = \int_V \boldsymbol{x} \times \rho_{\mathsf{body}} \boldsymbol{g} \, dV \ , \qquad \text{(5-12)}$$

and the other from pressure, *i.e.* buoyancy,

$$\boldsymbol{\mathcal{M}}_B = \oint_S \boldsymbol{x} \times (-p \, d\boldsymbol{S}) \ . \qquad \text{(5-13)}$$

If the total force vanishes, $\boldsymbol{\mathcal{F}} = \mathbf{0}$, the total moment will be independent of the origin of the coordinate system.

In hydrostatic equilibrium the total moment of any volume of fluid must vanish. Under the same assumptions as we used for deriving Archimedes principle, this implies that *the moment of buoyancy must equal the (minus) moment of gravity of the displaced fluid*,

$$\boldsymbol{\mathcal{M}}_B = -\int_V \boldsymbol{x} \times \rho_{\text{fluid}} \boldsymbol{g} \, dV \ . \tag{5-14}$$

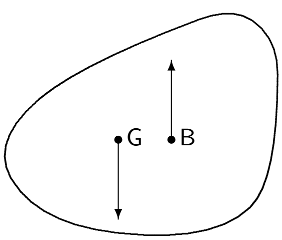
This result is a natural corollary to Archimedes' principle, and of immense help in calculating the buoyancy moment. A formal proof of the theorem is found in problem 5.6.

Constant gravity and buoyant equilibrium

In the remainder of this chapter we assume that gravity is constant, $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{g}_0$, and that the body is in buoyant equilibrium so that it displaces exactly its own mass of fluid, $M_{\text{fluid}} = M_{\textsf{body}} = M$. The densities of body and displaced fluid will, however, in general be different, $\rho_{\text{body}} \neq \rho_{\text{fluid}}$.

The moment of gravity (5-12) may as before (page 43) be expressed in terms of the center of the body mass distribution (here called the center of gravity),

$$\boldsymbol{\mathcal{M}}_G = \boldsymbol{x}_G \times M\boldsymbol{g}_0 \ , \qquad \boldsymbol{x}_G = \frac{1}{M}\int \boldsymbol{x}\rho_{\text{body}} \, dV \ . \tag{5-15}$$



Body in buoyant equilibrium but with non-vanishing total moment which here sticks out of the paper. The moment will for a submerged body tend to rotate it in the anticlockwise direction and thus bring the center of gravity below the center of buoyancy.

Similarly the moment of the mass distribution of the displaced fluid (5-14) is,

$$\boldsymbol{\mathcal{M}}_B = -\boldsymbol{x}_B \times M\boldsymbol{g}_0 \ , \qquad \boldsymbol{x}_B = \frac{1}{M}\int \boldsymbol{x}\rho_{\text{fluid}} \, dV \ . \tag{5-16}$$

Even if each of these moments depends on the choice of origin of the coordinate system, the total moment,

$$\boldsymbol{\mathcal{M}} = (\boldsymbol{x}_G - \boldsymbol{x}_B) \times M\boldsymbol{g}_0 \ , \tag{5-17}$$

will be independent, as witnessed by the appearance of the difference of the two center positions.

As long as the total moment is non-vanishing, the body is not in mechanical equilibrium, but will start to rotate towards an orientation with vanishing moment. Except for the trivial case where the centers of gravity and buoyancy coincide, the above equation tells us that the total moment can only vanish if the centers lie on the same vertical line,

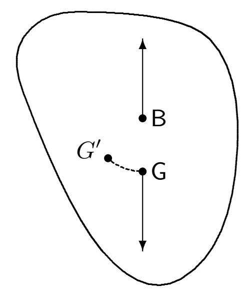
$$\boldsymbol{x}_G - \boldsymbol{x}_B \propto \boldsymbol{g}_0 \ . \tag{5-18}$$

Evidently, there are two possible orientations satisfying this condition: one where the center of gravity lies above the center of buoyancy, and another where the center of gravity is lowest. In general only one of these will be stable.

Submerged body

For a fully submerged rigid body, for example a submarine, both centers are always in the same place relative to the body. If the center of gravity does not lie directly below the center of buoyancy, but displaced a bit horizontally, the direction of the moment will always tend to turn the body so that the center of gravity is lowered with respect to the center of buoyancy. The only stable orientation of the body is where the center of gravity lies vertically below the center of buoyancy. Any small perturbation away from this orientation will soon be corrected and the body brought back to the equilibrium orientation. A similar argument shows that the other equilibrium orientation with the center of gravity above the center of buoyancy is unstable and will flip the body over, if perturbed the tiniest amount.

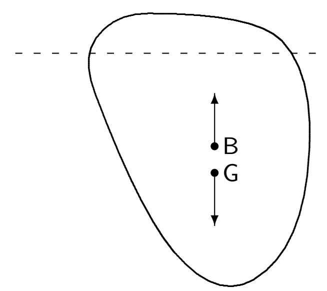
This is why the gondola hangs below an airship or balloon, and why a fish goes belly-up when it dies, because it loses control of the swim bladder which enlarges into the belly and reverses the positions of the centers of gravity and buoyancy. It also loses buoyant equilibrium and floats to the surface.



A fully submerged body in stable equilibrium must have the center of gravity directly below the center of buoyancy. If G is moved to G' a restoring moment is created which sticks out of the plane of the paper.

Floating body

At the surface of a liquid, a body such as a ship or an iceberg will according to Archimedes' principle always arrange itself so that the mass of displaced liquid exactly equals the mass of the body. Here we assume that there is vacuum or a very light fluid such as air above the liquid. The center of gravity is as always in the same place relative to the body, but the center of buoyancy depends now on the orientation of the body, because the volume of displaced fluid changes place and shape (while keeping its mass constant) when the body orientation changes.

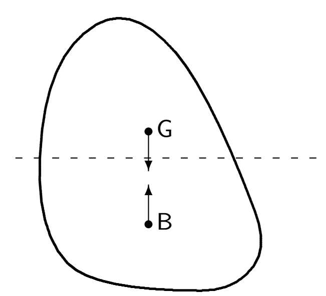


A partially submerged body may have a stable equilibrium with the center of gravity directly below the center of buoyancy.

Stability can again only occur when the two centers lie on the same vertical line, but there may be more than one point of stability. A sphere made of homogeneous wood floating on water, is stable in all orientations . None of them are in fact truly stable, because it takes no force to move from one to the other. This is however a marginal case.

A floating body may like a submerged body also possess a stable orientation with the center of gravity directly *below* the center of buoyancy. A heavy keel may, for example, used to lower the center of gravity of a sailing ship so much that this orientation becomes the only stable equilibrium. In that case it becomes virtually impossible to capsize the ship, even in a very strong wind.

The stable orientation for most floating objects, such as ships, will in general have the center of gravity situated directly *above* the center of buoyancy. This happens always when an object of constant mass density floats on top of a liquid of constant mass density, for example an iceberg on water. The part of the iceberg that lies below the waterline must have its center of buoyancy in the same place as its center of gravity. The part of the iceberg lying above the water cannot influence the center of buoyancy whereas it always will shift the center of gravity upwards.



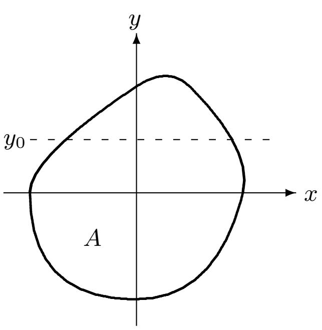
A partially submerged body may have a stable equilibrium with the center of gravity directly above *the center of buoyancy.*

How can that situation ever be stable? Will the restoring moment not be of the wrong sign? Why don't ducks and tall ships capsize spontaneously? The qualitative answer is that when the body is rotated away from such an equilibrium orientation, the volume of displaced water will change position and shift the center of buoyancy back to the other side of the center of gravity, reversing the direction of the restoring moment.

5.4 Ship stability

Sitting comfortably in a small rowboat, it is fairly obvious that the center of gravity lies above the center of buoyancy, and that the situation is stable with respect to small movements of the body. But many a fisherman has learnt that suddenly standing up may compromise the stability and send him out among the fishes. There is, as we shall see, a strict limit to how high the center of gravity may be above the center of buoyancy.

Most ships are mirror symmetric in a plane, but we shall be more general and consider a "ship" of an arbitrary shape. We shall assume that the ship initially is in mechanical equilibrium and calculate the moment that arises when it is brought slightly out of equilibrium. If the moment tends to turn the ship back into equilibrium, the initial orientation is stable. To lowest order of approximation, the stability turns out to be an essentially two-dimensional problem, depending mainly on the shape of the outline of the ship's hull in the waterline.

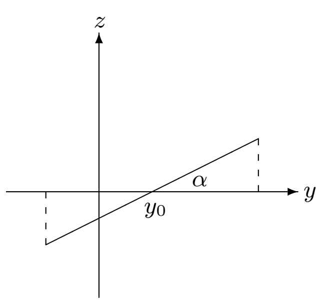


The area A of the ship in the waterline may be of quite arbitrary shape. The z-axis is vertical, sticking out of the paper. The ship is tilted around the line $y = y_0$.

Center of roll

A flat-earth coordinate system is introduced in which the outline of the ship's hull is A in the waterline, $z = 0$. The ship is now tilted slightly through a small angle α around a line $y = y_0$, parallel to x-axis, so that the previous waterline area A comes to lie in the plane $z = \alpha(y - y_0)$. The net change in the displacement due to the tilt is to lowest order in α given by the difference in volumes of the two wedge-shaped regions between new and the old waterlines,

$$\delta V = \int_A (-z)\, dA = -\alpha \int_A (y - y_0)\, dA \; . \tag{5-19}$$



Tilt around the axis $y = y_0$. The change in displacement is negative in the wedge to the right and positive in the wedge to the left.

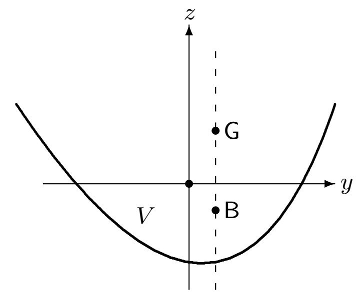
Here we have disregarded the small corrections of order α^2 due to the actual shape of the hull just above and below the waterline.

For the ship to remain in buoyant equilibrium after the tilt, the change in displacement must vanish, $\delta V = 0$, which is only possible for $y_0 = \frac{1}{A}\int_A y\, dA$. Including also tilts around the y-axis, this defines a unique point,

$$(x_0, y_0) = \frac{1}{A}\int_A (x, y)\, dA \; , \tag{5-20}$$

which we shall call the *center of roll*. A roll of the ship around any axis through this point will generate no change in displacement. We shall from now on place the origin of the coordinate system at the center of roll, so that $x_0 = y_0 = 0$.

The restoring moment



The ship in an equilibrium orientation, stable or unstable, shown in a vertical plane containing the aligned centers of gravity and buoyancy.

A roll through an angle α around the x-axis generates a restoring moment, which may be calculated from (5-17),

$$\mathcal{M}_x = -(y_G - y_B) M g_0 \ . \tag{5-21}$$

Since we have $y_G = y_B$ in the original mechanical equilibrium, the difference in coordinates after the tilt may be written,

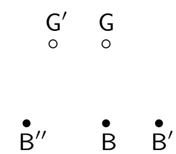
$$y_G - y_B = \delta y_G - \delta y_B \ , \tag{5-22}$$

where δy_G and δy_B are the small shifts in centers of gravity and buoyancy caused by the tilt.

The center of gravity is (hopefully!) fixed with respect to the ship and is to first order in α shifted horizontally by a simple rotation,

$$\delta y_G = -\alpha z_G \ . \tag{5-23}$$

There will also be a vertical shift, $\delta z_G = \alpha y_G$, but that is of no importance to the stability.



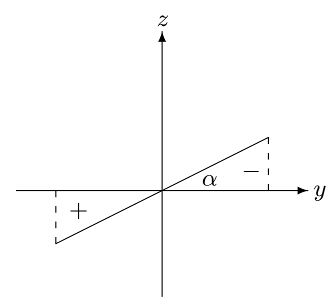
The tilt rotates the center of gravity from G to G′, and the center of buoyancy from B to B′. In addition, the change in displaced water shifts the center of buoyancy back to B″. In stable equilibrium this point must for $\alpha > 0$ lie to the left of the new center of gravity.

The center of buoyancy is at first shifted in the same way as the center of gravity by the tilt, but because the displacement also changes there will be another contribution Δy_B,

$$\delta y_B = -\alpha z_B + \Delta y_B \ . \tag{5-24}$$

The change in displacement consists in moving the water in wedge-shaped region from $y > 0$ into the region $y < 0$. Due to the choice of coordinates the two regions have equal volumes. The horizontal change in the center of buoyancy is calculated from the change in the center of mass of the displaced water,

$$\Delta y_B = \frac{1}{V}\int_A y(-z)dA = -\frac{\alpha}{V}\int_A y^2\, dA \ .$$



The change in displacement consists in moving the water in the wedge to the right into the wedge to the left.

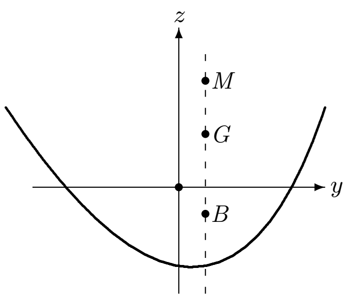
Finally, putting it all together we find the restoring moment

$$\boxed{\mathcal{M}_x = -\alpha\left(z_B + \frac{I}{V} - z_G\right) M g_0 \ ,} \tag{5-25}$$

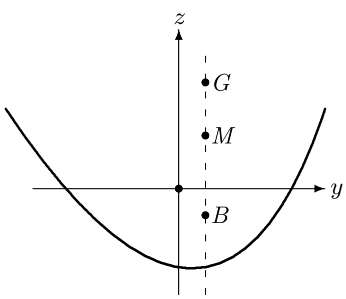
where

$$\boxed{I = \int_A y^2\, dA} \ , \tag{5-26}$$

is the second "moment of inertia" of the area A around the tilt axis. It is a purely geometric quantity which in principle may be calculated for any choice of tilt axis when the shape of the hull in the waterline is known.



Stable ship. The metacenter lies above the center of gravity.



Unstable ship. The metacenter lies below the center of gravity.

The metacenter

For the ship to be stable, the restoring moment must counteract the tilt and thus have opposite sign of the tilt angle α. This implies that the expression in parenthesis in (5-25) must be positive, or

$$\boxed{z_G < z_M \equiv z_B + \frac{I}{V}} \ . \tag{5-27}$$

The quantity z_M on the right hand side defines the z-coordinate of a fictitious point situated vertically above the original center of buoyancy, called the *metacenter*. The ship is stable when the center of gravity lies below the metacenter, $z_G < z_M$. A good captain should always know the positions of the center of gravity and the metacenter of his ship before he sails, or else he may capsize when casting off.

Example 5.4.1: It is this condition you violate when you stand up in a small rowboat. Taking the boat to be roughly rectangular with area $A = 1\ \mathsf{m} \times 2\ \mathsf{m}$, the moment of the area becomes $I = 2\ \mathsf{m} \times \frac{2}{3}(1/2\ \mathsf{m})^3 = 0.167\ \mathsf{m}^4$. If your mass is 75 kg and the boat's is 50 kg, the displacement will be $V = 0.125\ \mathsf{m}^3$, and the draught $d = 6.25\ \mathsf{cm}$. The coordinate of the center of buoyancy becomes $z_B = -3.1\ \mathsf{cm}$ and the metacenter $z_M = 1.3\ \mathsf{m}$. A person getting up from his seat may indeed raise the center of gravity so much that this inequality is violated and the boat becomes unstable.

The orientation of the coordinate system with respect to the ship's hull was not specified in the analysis and is therefore valid for a tilt in any direction. For a ship to be fully stable, the stability condition must be fulfilled for all possible tilt directions. Since the displacement V is the same for all tilt directions, the second moment of the area on the right hand side of (5-27) should in fact be chosen to be the smallest one. Often it is quite obvious which moment is the smallest. Many modern ships are extremely long with the same cross section for most of their length and a mirror symmetry through a vertical plane. For such ships the smallest moment is clearly obtained with the roll axis parallel to the longitudinal axis of the ship.

The position of the center of gravity depends on the way the ship is loaded and how the load is balanced, but once the weight of the ship and the center of gravity is known, the metacenter becomes a purely geometric quantity which may be calculated from the shape of the ship. We have previously seen the important role played by the shape of the ship in the waterline, $z = 0$. Let now $A(z)$ denote the area of the ship at depth z below the waterline. From the position of the coordinate system we have $-d < z < 0$, where d is the maximal depth of the ship's hull, also called the ship's *draught*. The displacement of the ship may be written

$$V = \int_{-d}^{0} A(z)\, dz \ , \tag{5-28}$$

and similarly the center of buoyancy becomes

$$z_B = \frac{1}{V}\int_{-d}^{0} z\, A(z) dz \ . \tag{5-29}$$

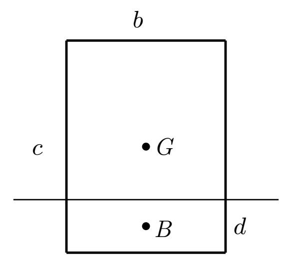
These expressions are quite useful in practical stability calculations (see for example problem 5.11).

The restoring moment becomes

$$\mathcal{M}_x = -\alpha\rho_0 V g_0 (z_M - z_G) \ , \tag{5-30}$$

and is proportional to the vertical distance between the metacenter and the center of gravity. The closer the center of gravity comes to the metacenter, the smaller will the restoring moment be, and the longer will the period of rolling oscillations. The actual roll period depends also on the true moment of inertia of the ship around the roll axis (see problem 5.9).

Floating board



Board floating on water. The board has length a into the paper.

The simplest example to which we may apply the stability criterion is that of a rectangular board or box of dimensions a, b and c in the three coordinate directions. The board is assumed to be made from a uniform material with constant density ρ_1 and floats in water of constant density ρ_0. The draught is d, and in hydrostatic equilibrium we must have $dab\rho_0 = abc\rho_1$ or $\rho_1/\rho_0 = d/c$.

The position of the center of gravity is $z_G = 0$ and the center of buoyancy $z_B = -(c-d)/2$. The second moment of the waterline area becomes (for tilts around the x-axis)

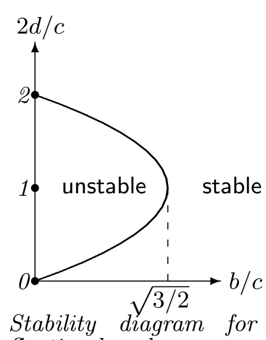
$$I = \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy\, y^2 = \frac{1}{12}ab^3 \ . \tag{5-31}$$

Since $V = abd$, the position of the metacenter is

$$z_M = \frac{1}{2}(d-c) + \frac{b^2}{12d} \ . \tag{5-32}$$

The board is stable for $z_M > 0$. Rearranging this condition, it may be written as

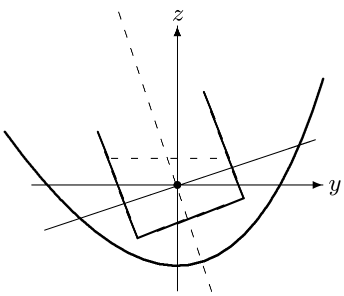
$$\left(\frac{2d}{c} - 1\right)^2 > 1 - \frac{2}{3}\left(\frac{b}{c}\right)^2 \ . \tag{5-33}$$



Stability diagram for the floating board.

When the board dimensions obey $b/c > \sqrt{3/2} = 1.2247\ldots$, the right hand side becomes negative and the inequality is always fulfilled. On the other hand, when $b/c < \sqrt{3/2}$, *i.e.* if the width of the board is less than 122% of the height, there is a range of draft values around $d = c/2$ (corresponding to density values around $\rho_1 = \rho_0/2$), for which the board is unstable and will keel over and come to rest in another orientation (see problem 5.11).

This stability analysis is valid for other configurations of mass inside the board, as long as the total mass and the position of the center of gravity are unchanged, as is for example the case for a box-shaped "ship" with a hull constructed from plates of uniform thickness.



Tilted ship with an open container filled with liquid.

Ship with liquid cargo

Many ships carry liquid cargos, oil, water, etc. When the tanks are not completely filled this kind of cargo may strongly influence the stability of the ship. In heavy weather or due to accidents, car ferries may inadvertently also get a layer of water on the car deck. The main effect of an open liquid surface inside the ship is that the center of mass is shifted in the same direction by the redistribution of real liquid as the shift in the center of buoyancy due to the change in displaced water, *i.e.* towards negative y-values. This disturbs the stability and creates a competition between the liquid carried by the ship and the water displaced by the ship.

For the case of a single open tank, let as before the displaced volume (including the weight of the liquid) be V and the second moment I. A similar calculation then shows that the change in center of gravity due to the movement of a wedge of liquid of density ρ_1 becomes

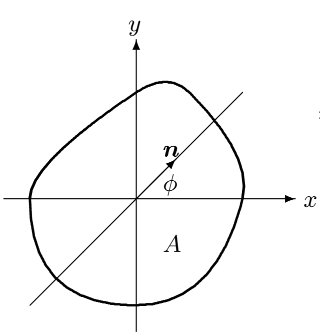
$$\Delta y_G = -\alpha \frac{\rho_1 I_1}{M} = -\alpha \frac{\rho_1}{\rho_0} \frac{I_1}{V} \tag{5-34}$$

where I_1 is the second moment of the open liquid surface. The metacentric height now becomes

$$\boxed{z_M = z_B + \frac{I}{V} - \frac{\rho_1}{\rho_0} \frac{I_1}{V}} \tag{5-35}$$

The effect of the moving liquid is to lower the metacentric height with possible destabilization as result (see problem 5.5). The unavoidable sloshing of the liquid may further compromise the stability.

The destabilizating effect of a liquid cargo may be counteracted by having a number of smaller containers or compartments, instead of a single container with an open surface. In car ferries this option is not available because it would hamper efficient loading of the cars, and such ships remain susceptible to the destabilizing effects of water on the car deck.



Tilt axis $\boldsymbol{n}$ *forming an angle* ϕ *with the* x*-axis.*

* Principal roll axis

It has already been remarked that the metacenter for absolute stability is determined by the smallest second moment of the waterline area. Instead of tilting the ship around the x-axis, it is tilted around an axis $\boldsymbol{n} = (\cos\phi, \sin\phi, 0)$ forming an angle ϕ with the x-axis. Since this configuration is obtained by a simple rotation through ϕ around the z-axis, the transverse coordinate to be used in calculating the second moment becomes $y' = y\cos\phi - x\sin\phi$ (see eq. (2-40b)), and we find

$$I = \int_A (y')^2 \, dA = I_{xx}\cos^2\phi + I_{yy}\sin^2\phi + 2I_{xy}\sin\phi\cos\phi = \boldsymbol{n}\cdot\mathbf{I}\cdot\boldsymbol{n} \tag{5-36}$$

where I_{xx}, I_{yy} and I_{xy} are the elements of the matrix

$$\mathbf{I} = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{pmatrix} = \int_A \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} dA \tag{5-37}$$

The extrema of the positive definite quadratic form $\boldsymbol{n} \cdot \mathbf{I} \cdot \boldsymbol{n}$ are found from the eigenvalue equation $\mathbf{I} \cdot \boldsymbol{n} = \lambda \boldsymbol{n}$ (see problem 5.8). The eigenvector corresponding to the smallest eigenvalue is called the *principal roll axis* of the ship and its eigenvalue determines the metacenter for absolute stability.

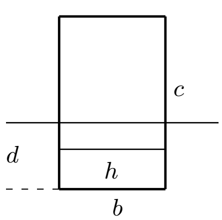
Problems

5.1 Show that when the total force on a body vanishes, the moment of force becomes independent of the origin of the coordinate system.

5.2 A stone weighs 1000 N in air and 600 N when submerged in water. Calculate the volume and average density of the stone.

5.3 A hydrometer with mass $M = 4\ g$ consists of a spherical glass container filled with air and a long thin cylindrical stem of radius $a = 2$ mm. The sphere is weighed down so that the apparatus will float stably with the stem pointing vertically upwards and crossing the fluid surface at at some point. How much deeper (h) will it float in alcohol (mass density $\rho_1 = 0.78\ \mathrm{g/cm^3}$) than in oil (mass density $\rho_0 = 0.82\ \mathrm{g/cm^3}$)?

5.4 A cylindrical wooden stick (density $\rho_1 = 0.65\ \mathrm{g/cm^3}$) floats in water (density $\rho_0 = 1\ \mathrm{g/cm^3}$). The stick is loaded down with a lead weight (density $\rho_2 = 11\ \mathrm{g/cm^3}$) at one end such that it floats in vertical position with a fraction $f = 1/10$ of its length out of the water. a) What is the ratio (M_1/M_2) between the masses of the wooden stick and the lead weight? b) How large a fraction can stick out of the water (disregarding questions of stability)?



A 'car ferry' with water on the deck

5.5 A car ferry is extremely sensitive to water on the car deck. Consider a ferry in the shape of a long box-shaped hull of mass M and essentially no thickness. The length of the ferry is a, the width b, and the height c. The density of water is ρ_0. a) Calculate the draft d_0 when there is no water in the hull, b) Calculate the draft (d) when there is water of depth h inside the hull. c) Determine the center of mass height in equilibrium. d) Evaluate the stability condition.

* **5.6** Prove without assuming constant gravity that the hydrostatic moment of buoyancy equals (minus) the moment of gravity of the displaced fluid (corollary to Archimedes' law).

* **5.7** Assuming constant gravity, show that for a body not in buoyant equilibrium (*i.e.* for which the total force does not vanish), there is always a well-defined center of gravity $\boldsymbol{x}_G$, such that the total moment of gravitational and buoyant forces is given by $\boldsymbol{\mathcal{M}} = \boldsymbol{x}_G \times \boldsymbol{\mathcal{F}}$.

* **5.8** Show that the extrema of a 2×2 quadratic form $\boldsymbol{n} \cdot \mathbf{I} \cdot \boldsymbol{n}$, where $\boldsymbol{n} = (\cos\phi, \sin\phi)$ is a unit vector, are determined by the eigenvectors of $\mathbf{I}$ satisfying $\mathbf{I} \cdot \boldsymbol{n} = \lambda \boldsymbol{n}$.

* **5.9** Show that in a stable orientation the angular frequency of small oscillations around around a major axis of a ship is

$$\omega = \sqrt{\frac{Mg_0}{J}(z_M - z_G)}$$

where J is the moment of inertia of the ship around this axis.

∗ **5.10** A ship has a waterline area which is a regular polygon with $n \geq 3$ edges. Show that the area moment tensor (5-37) has $I_{xx} = I_{yy}$ and $I_{xy} = 0$.

∗ **5.11** Consider a homogeneous cubic block with density equal to half that of the water it floats on. Determine the stability properties of the cube when it floats a) with a horizontal face below the center, b) with a horizontal edge below the center, and c) with a corner vertically below the center. Hint: problem 5.10 is handy for a) and c).