10346: Continuum Physics, spring 2002

Homework assignment 3:

Instabilities and shapes of falling fluid columns.

A narrow column of fluid with surface tension α, density ρ and viscosity ν moving rapidly downward (without rotation) in a narrow column of circular cross section with radius $h(z)$, is well-described by a rotationally symmetric velocity field of the form

$$w(r,z,t) = w_0(z,t) + w_2(z,t)r^2 + ... \tag{1}$$
$$u(r,z,t) = u_1(z,t)r + u_3(z,t)r^3 + ... \tag{2}$$
$$p(r,z,t) = p_0(z,t) + p_2(z,t)r^2 + ... \tag{3}$$

The components of the velocity field are $w = v_z$ and $u = v_r$ in cylindrical coordinates and p is the pressure.

1. Use the continuity equation to express $u_1, u_3, ...$ in terms of $w_0, w_2....$

2. Write down the boundary conditions at the free surface $r = h(z,t)$ expressed in terms of the velocity field (without expanding in r) and the derivatives of h. There are two dynamical boundary conditions and one kinematic boundary condition.

3. Insert the expansion (1)-(3) into the full Navier-Stokes equations including gravity and retain only the lowest order term in r. Then use the dynamic boundary conditions derived above and expand to lowest order in h and its derivatives to eliminate w_2 and p_0. Show that one gets

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = g - \frac{\alpha}{\rho}\frac{\partial \kappa}{\partial z} + 3\nu\frac{\partial^2 v}{\partial z^2} + \frac{6\nu}{h}\left(\frac{\partial v}{\partial z}\right)\left(\frac{\partial h}{\partial z}\right) \tag{4}$$
$$\frac{\partial h^2}{\partial t} + \frac{\partial v h^2}{\partial z} = 0 \tag{5}$$

where $v(z,t) = w_0(z,t)$. The curvature κ of the surface $r = h(z)$ is

$$\kappa = \frac{1}{h(1+(\frac{\partial h}{\partial z})^2)^{1/2}} - \frac{\partial^2 h}{\partial z^2}\frac{1}{(1+(\frac{\partial h}{\partial z})^2)^{3/2}} \approx \frac{1}{h} - \frac{\partial^2 h}{\partial z^2} \tag{6}$$

Describe the physical content of equation (5) and show that it follows from the kinematic boundary condition.

4. Show that there is a stationary solution with $v = v_0 =$ const. and $h = h_0 =$ const., when there is no gravity ($g = 0$). Demonstrate the *Rayleigh-Plateau instability*, i.e. that this stationary solution is linearly unstable.

a. Assume that $v(z,t) = v_0(1 + a(z,t))$ and $h(z,t) = h_0(1 + b(z,t))$ (with a and b small) and linearize (4)-(5) (in a and b).

b. Solve the resulting linear system by Fourier transformation, i. e. by letting

$$(a, b) = (a_0, b_0)e^{i(kx-\omega t)} \tag{7}$$

giving the dispersion relation for the complex function $\omega = \omega(k)$. What is the condition on ω for stability/instability? Are there any stable wave numbers (k)? What is the maximally unstable wavenumber when $\nu = 0$.

c. Compare the result with the exact result (from the full Navier-Stokes equation) for an inviscid fluid with $v_0 = 0$:

$$\omega^2 = -\frac{\alpha x I_1(x)}{\rho h_0^3 I_0(x)}(1 - x^2) \tag{8}$$

where $x = kh_0$ and I_0 and I_1 are modified Bessel functions. What difference does it make that the fluid moves with velocity $v_0 \neq 0$?

d. Can you check your prediction experimentally?

5. Use (4)-(5) to find a stationary solution for a thin fluid thread falling in a gravitational field. Neglect for simplicity the last term in (6), i.e., take $\kappa = 1/h$, which is a good approximation for a thin thread.

a. Show that it has the form $v(z) \sim z^\gamma$ for $z \to \infty$ and determine γ. You must

1. Determine the terms that give the asymptotics.
2. Check that the other terms are subdominant.

Does γ depend on α, ρ, ν or g? What is the corresponding form of $h(z)$.

b. Find the solution numerically. Show that $v'/\sqrt{v}$ must approach a constant when $v \to 0$ and determine the constant. What happens when you start your integration with small v' and v?

6. *Only for fun: this question does not count in the evaluation.* It turns out that the stationary solution is weakly unstable in the sens that a small perturbation near the top (say $z = 0$) will grow as

$$\delta h(z) \sim \delta h(0)e^{z^{1/8}} \tag{9}$$

can you give an argument for this? Check for yourself that you can get very long threads of, say sirup.

Due *Tuesday, May 24* — Have fun!