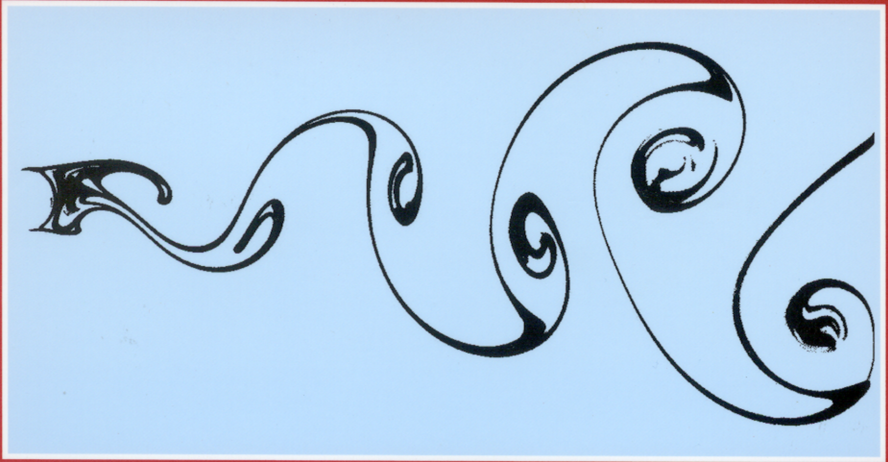
CAMBRIDGE TEXTS IN APPLIED MATHEMATICS

Introduction to Hydrodynamic Stability



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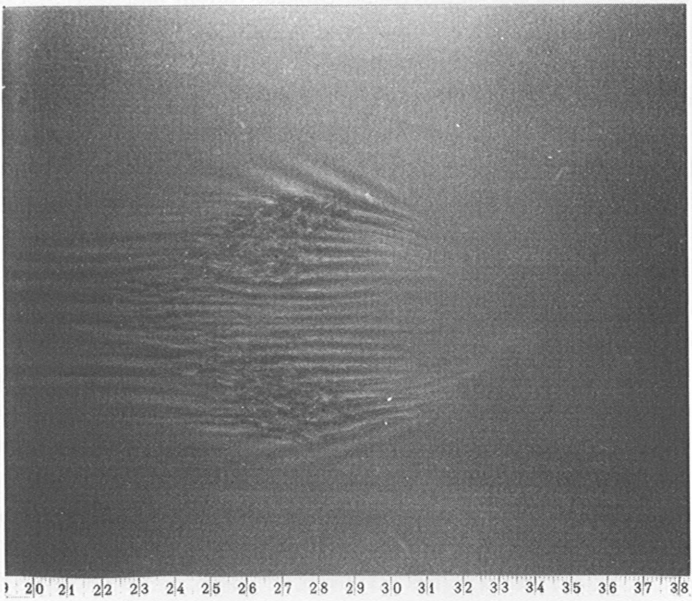

Figure 1.4 A turbulent spot triggered by jets in the wall of plane Poiseuille flow at $R = 1000$, where $R = Vd/\nu$, V is the maximum velocity of the flow, and the walls are separated by a distance $2d$. (From Carlson *et al.*, 1982, Fig. 4.)

by use of the compact disk of Homsy *et al.* (CD2000); this CD is currently more readily available than the film loops or their video versions, although briefer. Under the heading *Video Library* and subheadings ‘Reynolds Transition Apparatus’ and ‘The Reynolds Transition Experiment’, some short videos of recent experiments on Reynolds’s original apparatus are shown; further experiments can be found under the subheadings ‘Pipe Flow’, ‘Tube Flow’ and ‘Turbulent Pipe Flow’. Under the heading *Boundary Layers* and subheadings ‘Instability, Transition and Turbulence’ and ‘Instability and Transition in Pipe and Duct Flow’ more short videos are available.

1.2 The Methods of Hydrodynamic Stability

It may help at the outset to recognize that hydrodynamic stability has a lot in common with stability in many other fields, such as magnetohydrodynamics, plasma physics, elasticity, rheology, combustion and general relativity. The physics may be very different but the mathematics is similar. The mathematical essence is that the physics is modelled by nonlinear partial differential equations and the stability of known steady and unsteady solutions is examined. Hydrodynamics happens to be a mature subject (the Navier–Stokes equations having been discovered in the first half of the nineteenth century), and a given motion of a fluid is often not difficult to produce and to see in a laboratory, so hydrodynamic stability has much to tell us as a prototype of nonlinear physics in a wider context.

We learn about instability of flows and transition to turbulence by various means which belong to five more-or-less distinct classes:

(1) *Natural phenomena and laboratory experiments.* Hydrodynamic instability would need no theory if it were not observable in natural phenomena, man-made processes, and laboratory experiments. So observations of nature and experiments are the primary means of study. All theoretical investigations need to be related, directly or indirectly, to understanding these observations. Conversely, theoretical concepts are necessary to describe and interpret observations.

(2) *Numerical experiments.* Computational fluid dynamics has become increasingly important in hydrodynamic stability since 1980, as numerical analysis has improved and computers have become faster and gained more memory, so that the Navier–Stokes equations may be integrated accurately for more and more flows. Indeed, computational fluid dynamics has now reached a stage where it can rival laboratory investigation of hydrodynamic stability by simulating controlled experiments.

(3) *Linear and weakly nonlinear theory.* Linearization for small perturbations of a given basic flow is the first method to be used in the theory of hydrodynamic stability, and it was the method used much more than any other until the 1960s. It remains the foundation of the theory. However, weakly nonlinear theory, which builds on the linear theory by treating the leading nonlinear effects of small perturbations, began in the nineteenth century, and has been intensively developed since 1960.

(4) *Qualitative theory of bifurcation and chaos.* The mathematical theory of differential equations shows what flows *may* evolve as the dimensionless parameters, for example the Reynolds number, increase. The succession of bifurcations from one regime of flow to another as a parameter increases cannot be predicted quantitatively without detailed numerical calculations, but the admissible and typical routes to chaos and thence turbulence may be identified by the qualitative mathematical theory. Thus the qualitative theory of dynamical systems, as well as weakly nonlinear analysis, provides a useful conceptual framework to interpret laboratory and numerical experiments.

(5) *Strongly nonlinear theory*. There are various mathematically rigorous methods, notably Serrin's theorem and Liapounov's direct method, which give detailed results for arbitrarily large perturbations of specific flows. These results are usually bounds giving sufficient conditions for stability of a flow or bounds for flow quantities.

The plan of the book is to develop the major concepts and methods of the theory in detail, and then apply them to the instability of selected flows, relating the theoretical to the experimental results. This plan is itemized in the list of contents. First, in this and the next chapter, many concepts and methods will be described, and illustrated by simple examples. Then, case by case, these methods and concepts, together with some others, will be used in the later chapters to understand the stability of several important classes of flows. The theory of hydrodynamic stability has been applied to so many different classes of flow that it is neither possible nor desirable to give a comprehensive treatment of the applications of the theory in a textbook. The choice of applications below is rather arbitrary, and perhaps unduly determined by tradition. However, the choice covers many useful and important classes of flow, and illustrates well the five classes of general method summarized above.

1.3 Further Reading and Looking

It may help to read some of the following books to find fuller accounts of many points of this text. Many of the books are rather out of date, being written before the advent of computers had made much impact on the theory of hydrodynamic stability. (Perhaps computational fluid dynamics has led to the most important advances in recent years, and perhaps the theory of dynamical systems or applications of the theory has led to a wider physical range of new problems.) However, the subject is an old one, with most of the results of enduring importance, so these books are still valuable.

Betchov & Criminale (1967) is a monograph largely confined to the linear theory of the stability of parallel flows, covering numerical aspects especially well. Chandrasekhar (1961) is an authoritative treatise, a treasure house of research results of both theory and experiment. It emphasizes the linear stability of flows other than parallel flows, with influence of exterior fields such as magnetohydrodynamic, buoyancy and Coriolis forces. Its coverage of the literature is unusual, informative and of great interest. Drazin & Reid (1981) is a monograph with a broad coverage of the subject. It has several problems for students, but few of them are easy. Huerre & Rossi (1998) is a set of 'lecture notes', though at an appreciably higher level than this book. It is an account, mostly of linear stability of mostly parallel flows, with good modern coverage of numerical and experimental as well as theoretical results. Joseph (1976) is a monograph which emphasizes nonlinear aspects, especially the energy method, but has a broad coverage of basic flows. Landau & Lifshitz (1987) is a great treatise masquerading as a textbook; it summarizes the physical essentials of hydrodynamic stability with masterly brevity. Lin (1955) is a classic monograph, largely confined to the linear stability of parallel flows of a viscous fluid, the complement of Chandrasekhar's treatise. Schmid & Henningson (2001) is an up-to-date comprehensive research monograph on instability and transition of parallel flows.

We have already referred to pictures to enrich understanding of Reynolds's experiment. Such pictures are, of course, as valuable in the understanding of many other hydrodynamic instabilities. Van Dyke (1982) is a beautiful collection of photographs of flows, including hydrodynamic instabilities. Nakayama (1988) is another fine collection of photographs of flows, including hydrodynamic instabilities. Look at the photographs relevant to hydrodynamic stability, think about them, and relate them to the theory of this book. However, hydrodynamic instability is a dynamic phenomenon, best seen in motion pictures. So, many relevant films, film loops and videos, and the compact disk of Homsy *et al.* (CD2000), are listed in the Motion Picture Index at the end of the list of references. It is appropriate to add some words of caution here. The results of visualization of *un*steady flows are liable to be misinterpreted. Be careful. In particular, make sure that you understand the difference between streamlines, streaklines and particle paths before you jump to too many conclusions.

Therefore

$$u'(t) = u(0)e^{st},$$

where the exponent $s = a, = k(R - R_c)$ for $k > 0$ (in realistic applications). Then there is linear stability, with exponential decay, if $R < R_c$ and linear instability, with exponential growth, if $R > R_c$. This solution of the linearized equation which grows exponentially with time is an example of a *normal mode*.

If $U = \pm[k(R - R_c)/l]^{1/2}$ for $l > 0, R > R_c$, then we similarly find $u'(t) = u(0)e^{st}$, but where now

$$s = a - 3lU^2 = -2k(R - R_c) < 0$$

and so gives supercritical stability, as indicated in Figure 2.12(a).

All these results may be confirmed by use of the 'exact' explicit solution of the Landau equation. □

Example 2.9: A Hopf bifurcation. Consider

$$\frac{dx}{dt} = -y + (a - x^2 - y^2)x, \qquad \frac{dy}{dt} = x + (a - x^2 - y^2)y,$$

where $a = k(R - R_c)$, $k > 0$. The only steady solution of this system is the null solution $x = y = 0$. To find its stability we linearize the system with respect to small perturbations of the null solution, finding

$$\frac{dx}{dt} = -y + ax, \qquad \frac{dy}{dt} = x + ay.$$

We solve this linearized system by again using the *method of normal modes*, that is, by supposing that $x, y \propto e^{st}$, and deducing that

$$sx = ax - y, \qquad sy = x + ay,$$

and therefore that s is an eigenvalue of the matrix

$$\mathbf{J} = \begin{bmatrix} a & -1 \\ 1 & a \end{bmatrix}.$$

Therefore

$$0 = \det(\mathbf{J} - s\mathbf{I}) = (a - s)^2 + 1.$$

Therefore

$$s = a \pm i = k(R - R_c) \pm i.$$

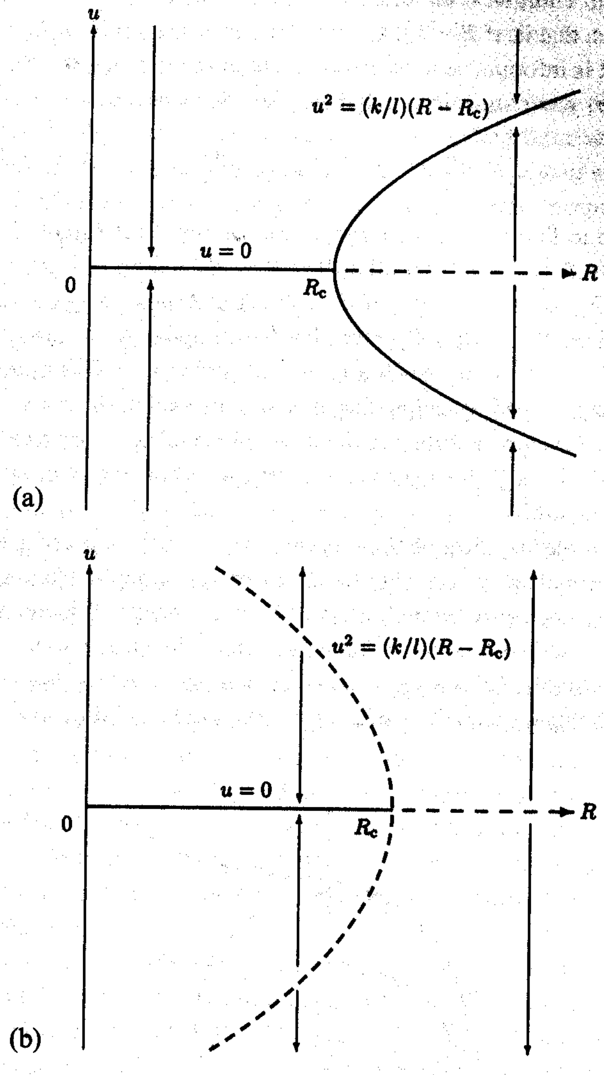

Figure 2.12 Bifurcation diagrams in the (R, u)-plane for the Landau equation: (a) supercritical stability, $l > 0$; (b) subcritical instability, $l < 0$.

Therefore

$$x(t) = \frac{1}{2}\left(Ae^{it} + A^*e^{-it}\right)e^{at}, \qquad y(t) = -\frac{1}{2}i\left(Ae^{it} - A^*e^{-it}\right)e^{at}$$

for some complex constant A, which may be determined by use of the initial conditions, where an asterisk is used as a superscript to denote complex conjugation. This gives stability, with exponential decay, if $\mathrm{Re}(s) < 0$ for both eigenvalues, that is, if $R < R_c$, and similarly instability if $R > R_c$.

In fact it is informative to transform to polar coordinates r, θ, where $r \geq 0$, $x = r\cos\theta$, $y = r\sin\theta$, in which the system decouples as

$$\frac{dr}{dt} = r(a - r^2), \qquad \frac{d\theta}{dt} = 1,$$

and thence to find the exact solution. The solution (in Example 2.8) implies that $r(t) \to 0$ as $t \to \infty$ for all $r(0)$ if $R \leq R_c$ and $r(t) \to a^{1/2} = [k(R - R_c)]^{1/2}$ as $t \to \infty$ for all $r(0)$ if $R > R_c$. Also $\theta(t) = \theta_0 + t$ for all $\theta(0) = \theta_0$. This gives, for all $R > R_c$, a nonlinear solution $x = r\cos\theta$, $y = r\sin\theta$ of period 2π as $t \to \infty$. Such a periodic solution of a differential equation which is approached by neighbouring solutions as time increases is called a *limit cycle*. Two typical orbits in the phase plane of (x, y), as t increases, are shown in Figure 2.13 for the case $R > R_c$; note how the limit cycle attracts neighbouring orbits.

This example is typical of *Hopf bifurcations*, in which the real part $\mathrm{Re}(s)$ of a complex conjugate pair of eigenvalues increases through zero as a parameter increases or decreases through a critical value, here as R increases through R_c, and an oscillatory solution bifurcates from the steady solution where it becomes unstable. Of course, it is no accident that a real system often has a complex conjugate pair of eigenvalues, so we meet Hopf bifurcations for partial

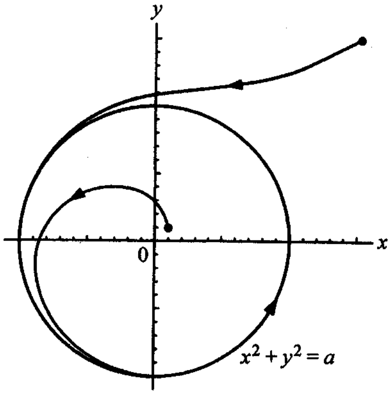

Figure 2.13 Two orbits in the (x, y)-plane for the system $dx/dt = dx/dt = -y + (a - x^2 - y^2)x$, $dy/dt = x + (a - x^2 - y^2)y$ of Example 2.9 when $R > R_c$. (After P. Drazin & T. Kambe, *Ryutai Rikigaku – Anteisei To Ranyu (Fluid Dynamics – Stability and Turbulence)*, University of Tokyo Press, 1989, Fig. 2.10. Reproduced by permission of the University of Tokyo Press.)

differential systems governing flows as well as for this simple example of an ordinary differential equation. So it is important to determine from the linear problem whether the exponent s is zero or purely imaginary at the margin of stability: in the former case a turning point, a transcritical or pitchfork bifurcation typically occurs, and in the latter case a Hopf bifurcation. In the *fluid dynamical context*, it is sometimes said that the *principle of exchange of stabilities* is valid when the time exponent of the least stable normal mode is zero at the margin of stability. □

Before moving on, note that we have used a complex representation of a real solution of a real problem in Example 2.9. This idea, based on the property that if a complex function satisfies a real homogeneous equation, then the real and imaginary parts of the function satisfy the equation separately, will be exploited often in the pages that follow. We shall write the complex solution of a real linearized equation or system of equations, meaning implicitly that its real part represents the appropriate physical quantity such as a perturbation of a velocity component or the pressure; for example, we may write $x(t) = A\mathrm{e}^{(a+\mathrm{i})t}$, where A is some complex constant, to mean its real part $\frac{1}{2}(A\mathrm{e}^{\mathrm{i}t} + A^*\mathrm{e}^{-\mathrm{i}t})\mathrm{e}^{at} = |A|\mathrm{e}^{at}\cos(t + \arg A)$. This is the traditional way to use the method of normal modes.

These examples have been chosen for their simplicity rather than to illustrate all aspects of hydrodynamic stability. One common phenomenon they do not illustrate is the instability of the supercritically stable bifurcated flow itself as the Reynolds number increases substantially above the critical value for a pitchfork or Hopf bifurcation. Then we call the first flow the *primary flow*, its instability the *primary instability*, the supercritically stable flow the *secondary flow* and its instability the *secondary instability*. These successive instabilities are discussed further in §9.1.

This section as a whole serves to introduce some important concepts (basic solution, stability, bifurcation) and methods (linearization, normal modes) of the theory of hydrodynamic stability by use of simple ordinary differential equations. Ordinary differential equations will be used later to illustrate other important concepts (such as quasi-periodic solutions and chaos) and methods (weakly nonlinear perturbation) of hydrodynamic stability. However, it should not be forgotten that the motion of a fluid involves space as well as time, and that it is modelled by *partial* differential equations. This means that the use of ordinary differential models is limited, albeit valuable pedagogically. More realistic models with the partial differential equations of hydrodynamics are treated in the next section, which covers some fundamental concepts and methods of the theory of hydrodynamic stability, especially the linear theory.