D16-2 Stat Mech / Renormalization May 13, 99

P. Cvitanović, spring 1999 (p.1)

Problem set, due Tuesday May 18

1.) derive (by any method) the partition function

$$Z(K) = [2\cosh K]^N$$

for 1-d Ising model. (Hint: generate all configurations by multiplication by a [2×2] transfer matrix)

2.) Derive the $K' \to K$ recursion relations and the free energy density for "decimated" 1-d Ising model

decreasing K	or	increasing K
$K' = \frac{1}{2}\ln\cosh(2K)$		$K = \frac{1}{2}\cosh^{-1} e^{2K'}$
$f(K') = 2f(K) - \ln(2\cosh^{\frac{1}{2}}(2K))$		$f(K) = \frac{1}{2}\ln 2 + \frac{1}{2}K' + \frac{1}{2}f(K')$

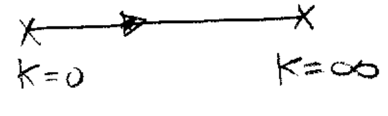
where $f(K) = \lim_{N\to\infty} \ln Z(K)/N$

For small initial K' you can neglect spin interactions, so take as initial condition

$$Z[N, 0.001] \simeq 2^N \quad \text{(just sum over all configurations)}$$

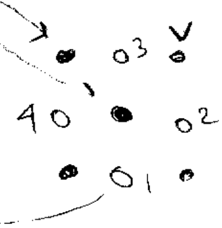
and iterate $K' \to K \to K_{-1} \to \cdots$. Compare $f(K)$ so obtained with the exact $f(K)$ from the exact $Z[K]$.

The flow is



(perhaps too hard - give it a try!)

3) Assume that decimation of every second site in the 2-d Ising model can be captured by action of form



$$Z(K) = F(K)^{N/2} \sum_{\circ}{}' \, e^{\left(K_1 \underbrace{\sum{}' \sigma_i\sigma_j}_{\text{nearest diagonal neighbors}} + K_2 \underbrace{\sum{}' \sigma_i\sigma_j}_{\text{next near.t horizontal/vertical neighbors}} + K_3 \underbrace{\sum{}' \sigma_i\sigma_j\sigma_k\sigma_l}_{\text{squares}}\right)}$$

white sites

$$= e^{K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)}$$

set up equations for
(a) all σ equal
(b) one σ different
(c) 2 neighbor σ the same
(d) 2 vis-a-vis σ the same

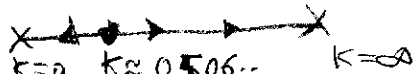
derive:

$$K_1 = \frac{1}{4}\ln\cosh(4K)$$
$$K_2 = \frac{1}{8}\ln\cosh(4K)$$
$$K_3 = \frac{1}{8}\ln\cosh(4K) - \frac{1}{2}\ln\cosh(2K)$$
$$F(K) = 2\cosh^{1/2}(2K)\cosh^{1/8}(4K)$$

Now you took Ising $\mathcal{H} = K\sum\sigma_1\sigma_2$ and you got a more complicated Hamiltonian out. Try
(a) setting $K_2, K_3 = 0$. Should get flow like Ising
(b) say K_2 is much like K_1 (neighbors), so take

$$K' = K_1 + K_2 = \frac{3}{8}\ln\cosh(4K), \text{ and derive.}$$
$$f(K') = 2f(K) - \ln\left[2\cosh^{1/2}(2K)\cosh^{1/8}(4K)\right]$$

Show new fixed point



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