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CLASSICAL AND EXCEPTIONAL LIE ALGEBRAS
AS INVARIANCE ALGEBRAS

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ABSTRACT

A method for constructing generators of linear transformations which preserve a given set of primitive invariants is presented.

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1. INTRODUCTION AND SUMMARY

The 19th century study of invariance groups reached its peak with Cartan's classification [1] of complex Lie algebras. He gave an explicit construction of generators of all possible complex Lie algebras, but did not give the invariants associated with each algebra. Most of the entries in Cartan's list of allowable algebras were immediately identified as representations of the classical invariance algebras SO(n), SU(n) and Sp(n), but of the five exceptional algebras Cartan identified only G_2 as the algebra of generators of octonion isomorphisms. The fact that the orthogonal, unitary and symplectic groups were invariance groups of real, complex and quaternion norms suggested that the exceptional groups were associated with octonions, but it took more than another fifty years to establish the connection. The remaining four exceptional Lie algebras emerged as rather complicated constructions from octonions and Jordan algebras.

In the present paper we attempt to give a unified construction of both classical and exceptional Lie algebras as invariance algebras (without recourse to octonians and Jordan algebras) and find the most economical way of computing the values of associated invariants. From Cartan we take only the classification, as the standard for identifying invariance algebras, and occasionally as a hint of the underlying invariants. The spirit of our approach is close to the aximatic constructions of Tits[2], Springer [3] and Brown [4] (among others); instead of constructing explicit representations of group generators, we characterize the algebras by representation independent identities satisfied by the group invariants.

The unifying concept from which we strive to generate all invariance algebras is the notion of primitive invariants. They are generalizations of classical invariants such as length (corresponding to invariant tensor δ_{ab}) or volume (corresponding to ε_{abc}). By an invariance algebra we mean the algebra of a maximal set of generators of infinitesimal transformations which preserve a given set of primitive invariants. By definition, any invariant of this algebra can either be constructed from the primitives or is itself a primitive. H. Weyl calls this the first main theorem of invariant theory: "All invariants are expressible in terms of a finite number among them"[5]. Enlarging the set of primitives either restricts the number of possible realizations (δ_{ab} is preserved by SO(n) for any n, but δ_{ab}, ε_{abc} only by SO(3)) or restricts the invariance algebra to a subalgebra (δ^a_b is preserved by U(n), but δ^a_b, δ_{ab} by SO(n)). To make the notion of primitiveness an effective computational tool, we augment it by the primitiveness assumption (2.6): any invariant can be expressed in terms of tree contractions of primitives. This is possibly the most problematic step in our approach; we assume not only that every invariant can be constructed from primitives but also that it can be reduced to a particular basis set.

It is easy to state classical primitive invariants explicitly: $\delta_{11} = 1$, $\delta_{12} = 0, \ldots, \varepsilon_{132} = -1, \ldots$, but we would not make much headway if we were to define complicated primitives in this fashion. A hint of a more elegant formulation is provided by classical algebras; we note that to define SO(n) it is sufficient to specify that the primitives are δ^a_b, d_{ab} with d_{ab} symmetric, and similarly Sp(n) is defined by primitives δ^a_b, f_{ab} with f_{ab} antisymmetric. In this spirit we shall specify each primitive by its symmetries. As our intention is not to determine all possible invariance algebras but merely to establish a procedure for constructing the invariance algebra for a given set of primitives, we shall consider only three types of primitives: δ_b^a, fully symmetric $d^{ab..f}$ and fully antisymmetric $f^{ab..d}$. This will suffice to construct the lowest dimensional representations of A_n, B_n, C_n, D_n, G_2, E_6, E_7 and F_4, as well as some higher representations of classical groups.

The crucial difference between our and Cartan's approach is that we describe the invariance algebra in terms of projectors P (2.17-19), rather than in terms of generators T_i (2.8). This saves us the effort of constructing an auxilliary adjoint representation space; the projector projects out an element of the invariance algebra from an arbitrary element of U(n) without any reference to a particular T_i basis set. Furthermore, projectors are invariant tensors and by primitiveness assumption expressible in terms of primitives (2.23). The invariance conditions (2.4) then fix the constants in the projector expansion and force the primitives to satisfy certain algebraic identities.

Our attempt to carry out this programme is only partially successful, and beyond the classical invariance groups we derive G_2 (incidentally proving Hurwitz's theorem) and partially characterize F_4, E_6 and E_7, generating in the process the entire Freudenthal's magic square (Table I) and a host of algebraic identities. We summarize our results by listing the sets of primitives considered here together with the invariance algebras they generate:

δ^a_b	$\rightarrow$ U(n)	(5.5)
δ^a_b, f^{ab}	$\rightarrow$ SU(2) = Sp(2); $f^{ab} = \varepsilon^{ab}$	(6.11)
	Sp(n); n even	(6.14)
δ^a_b, f^{abc}	$\rightarrow$ SU(3); $f^{abc} = \varepsilon^{abc}$	(7.19)
δ^a_b, $f^{abc..d}$	$\rightarrow$ SU(n); f = Levi-Civita tensor	(5.6)
δ^a_b, d^{ab}	$\rightarrow$ SO(n)	(9.9)
δ^a_b, d^{abc}	$\rightarrow$ Springer's relation;	(10.10)
	E_6(27) + second row, Table I	(10.14)
δ^a_b, $d^{abc..d}$	$\rightarrow$ no realization	
δ_{ab}, f_{ab}	$\rightarrow$ SO(2); $f_{ab} = \varepsilon_{ab}$	(9.9)
δ_{ab}, f_{abc}	$\rightarrow$ Hurwitz's theorem;	
	f_{abc} = R, C, Q, $\mathcal{O}$ multiplication tensor	(19.4)
	i) Jacobi relation; SO(3)	(13.29)
	ii) alternativity relation; G_2(7)	(13.31)
δ_{ab}, $f_{abc..d}$	$\rightarrow$ SO(n); f = Levi-Civita tensor	(9.9)
δ_{ab}, f_{abcd}	$\rightarrow$ $D_2(6) = A_1(3) \oplus A_1(3)$, $G_2(7)$, $B_3(8)$.	(14.21)
δ_{ab}, d_{abc}	$\rightarrow$ i) characteristic equation;	(15.14)
	F_4(26) + first row, Table I	(15.20)
	B_1(5)	(15.38)
	A_2(8)	(15.49)
	C_3(14)	(15.62)
	ii) no characteristic equation; ?	
δ^a_b, f^{ab}, d^{abcd}	$\rightarrow$ Brown's relation:	(16.9)
	E_7(56) + third row, Table I	(16.19)
δ_{ij}, C_{ijk}, ?	$\rightarrow$ E_8(248)	
	guess for N, fourth row, Table I	(17.2)

These results are derived in Sections 5-16. To accustom the reader to the method and the notation we construct the classical invariance algebras before going on to interesting but unfamiliar exceptional algebras. The basic concepts and the diagrammatic notation for invariant tensors are introduced in Sections 2 and 3. The magic square is constructed in Sec.17, while in the remainder of the paper we compare our results with the known octonian and Jordan algebra results, and discuss some applications to Yang-Mills theories.

The present paper is a self-contained presentation of results some of which were previously stated in Ref. 6 in the context of a specific theoretical physics application. Ref. 6 contains an exhaustive list of general references. For completeness sake here we list some further references which have come to our attention since the publication of Ref. 6; Refs. 7-17 on exceptional algebras and 18, 19 on diagrammatic methods.

We conclude this introduction with few general remarks about the motivation for the above work, and its relation to the classical group theory methods.

We are motivated by quantum field theory, in particular by Quantum Chromodynamics, i.e. quark theories in which quarks differ only in one discrete label (colour), but are otherwise indistinguishable. In the interesting models the quark colours can be relabelled, but no colour is preferred (the colour symmetry is exact). The algebra of allowable colourations is defined by the colourless combinations we wish to allow. In the standard coloured quark model these are mesons formed from quarks and antiquarks (δ^a_b invariant) and baryons formed from three quarks (ε_{abc} invariant). As we show in Sec. 7, SU(3) is the unique invariance algebra for these invariants. Method of the present paper enables us to find the invariance algebra for any such model. More generally, whenever we consider an exact symmetry we do not need any explicit representation of Cartan's generators T_i, but only the colour averages (all allowable colourations summed over) given by the projectors P. This leads us to more speculative motivation for the present work; invariants studied here are crude prototypes of Feynman integrals (instead of integrating over a continuum of momenta and energy states one sums over a finite number of allowable colourations). In this model of quantum mechanics probabilities are a class of scalar invariants with direct combinatoric significance, and the colouring rules are implemented by projectors. Formulation in terms of projectors rather than the generators is reminiscent of Jordan's formulation of quantum mechanics, and study of allowable invariance algebras might lead to prototypes of quantum mechanics which cannot be formulated in the conventional Hilbert space formalism [20]. In this crude model the adjoint representation is analogous to the photon, and the arbitrariness in the definition of the adjoint representation space possibly analogous to the gauge dependence of photons (for example, the invariance condition (3.16) is analogous to a Ward identity in field theory). Our hope is that a projector formulation of invariance algebras might suggest some formulation of QED in terms of gauge invariant probabilities rather than individually gauge dependent Feynman amplitudes.

Regardless of where the above speculations might lead us, the present approach is already a useful tool for study of invariance algebras, in many ways complementary to the standard Cartan's approach. It is a method for constructing an invariance group from a given set of invariants; Cartan's construction makes no reference to primitive invariants. We have no list of all possible types of primitives; Cartan gives an exhaustive listing of all possible invariance algebras. Present approach is very convenient for computing complicated invariants [6]; these are in principle computable from Cartan's explicit canonical representations, but in practice this is too difficult for any representation beyond a few low dimensional ones. Cartan-Dynkin and Freudenthal-Tits formulations are very suitable for the study of subalgebras (this aspect is emphasized by Ramond [11]); present approach says little about subalgebras. In Cartan-Dynkin scheme all representations of a Cartan algebra are treated on equal footing; the present approach is unwieldly for higher representations.

Compared with Freudenthal-Tits construction of the magic square, present approach generates various identities and the rows of the square very quickly, but the connections between the column entries remain obscure, and the conditions that exclude columns of spurious solutions are lacking. We can easily compute a large class of invariants for exceptional algebras; it is not clear how one is to use Freudenthal-Tits construction to compute <u>any</u> invariants. Finally, we have failed to find a set of invariants that generate the E_8 row of the square, but invariants of most of the entries of this row can be determined, and we hope they might lead us to a so far unknown [7] linear realization of E_8.

Our results are in many ways preliminary, unpolished, and we would appreciate critical comments and references to any relevant literature we might have missed. There are many unanswered questions, such as;

1) what should be added to primitiveness assumption in order that the exceptional algebras are uniquely defined?

2) do there exist simple Lie algebras which do not satisfy the primitiveness assumption?

3) what is the simplicity criterion in terms of projectors?

4) how does one determine the subalgebras and the branching rules?

5) what is the connection between column entries in Table I?

6) what are E_8 primitives?

7) what is the connection between types of possible primitives, and normed algebras?

8) can one obtain Cartan's classification from projectors, rather than from generators?

9) given a Cartan-Dynkin representation, can one find the corresponding primitives?

10) what scalar invariants (beyond representation dimensions) should be integers?

2. PRELIMINARIES

Let the <u>defining space</u> V be an n-dimensional complex vector space with elements $x = (x_1, x_2, \ldots x_n)$ and $\bar{V}$ its <u>dual</u> with elements $x = (x^1, x^2, \ldots x^n)$, where $x^a \equiv (x_a)^*$. Let a finite number of arrays of complex numbers of form

$$g = g^{ab..c}_{d\,..e} \qquad , a, b, \ldots, e = 1, 2, \ldots n \tag{2.1}$$

together with their duals

$$g^\dagger = g^{\dagger\, e..d}_{\quad c..ba} \equiv (g^{ab..c}_{d\,..e})^*$$

be <u>invariants</u> (or invariant tensors) of the <u>invariance group G,</u> a group of infinitesimal linear transformations over V, $\bar{V}$

$$\begin{aligned} x'_a &= x_a + i\, D_a^b \quad x_b \\ x'^a &= x^a - iD^{\dagger a}_{b} \quad x^b, \end{aligned} \qquad a, b, = 1, 2, \ldots n. \tag{2.2}$$

where the <u>derivations</u> D_a^b are infinitesimal complex [nxn] matrices. By invariance of (2.1) we mean that the polynomials of form

$$P(x, \ldots, \bar{p}, \cdot) = g^{ab..c}_{d..e}\, x_a y_b \ldots z_c\, p^d \ldots q^e \tag{2.3}$$

are invariant under (2.2)

$$P\,(x + i\, Dx, \ldots) = P(x, \ldots).$$

(Such polynomial is sometimes referred to as a <u>form</u>).

Each such relation imposes an <u>invariance condition</u> on the derivations D:

$$\begin{aligned} &D^a_f\, g^{fb..c}_{d..e} + D^b_f \; g^{af..c}_{d..e} + \ldots + D^c_f \; g^{ab..f}_{d..c} \\ &- D^{\dagger f}_{d}\, g^{ab..c}_{f..e} - \ldots - D^{\dagger f}_{e} \; g^{ab..c}_{d..f} = 0 \end{aligned} \tag{2.4}$$

We shall be interested only in unitary transformations, i.e. transformations which preserve a complex norm

$$N(x) \equiv \bar{x}\, x = \delta_b^a\, x^b x_a \qquad (2.5)$$

The invariance condition for δ_b^a relates D and $D^\dagger$

$$D_f^a\ \delta_b^f - D^{\dagger f}{}_b\ \delta_f^a = 0$$

so that the derivations we consider are always Hermitian, $D = D^\dagger$.

Constraints imposed on D by different invariance conditions (2.4) are not necessarily independent. If D preserves g^{abc}, it automatically preserves composite invariants $g^{abc}\, g^{def}$, $g^\dagger{}_{abc}\, g^{cbd}$, $\delta_b^a\ g^{cde}$.... Clearly we have to distinguish between a finite number of primitive invariants, or primitives, each of which gives an independent constraint (2.4), and the infinite number of composite invariants which impose no further constraints on D. To clarify the notion of primitiveness let us introduce some terminology which will be self-evident in the digrammatic notation: A composite invariant can be disconnected (as $\delta_b^a\, \delta_d^c$) or connected (as $g^\dagger{}_{abc}\, g^{cbd}$). A connected composite invariant can be a tree (as $g^\dagger{}_{abc}\, g^{cde}\, g^\dagger{}_{efh}$) or can contain loops (as $g^\dagger{}_{abc}\, g^{cde}\, g^\dagger{}_{efh}\, g^{bf}$). Let W be some composite invariant (indices suppressed). Take a finite number of invariant tensors together with their duals

$$\mathcal{P} = \{ \delta_b^a,\ g^{ab},\ g^{abc},\ g_{bc}^a,\ \ldots \}$$

and compose from the invariants of this set all connected or disconnected tree invariants $T^{(m)}$ with the same free indices as W.

Primitiveness assumption. $\mathcal{P}$ is a set of primitives if

i) any invariant tensor can be reduced to a sum over connected or disconnected trees of contractions of invariant tensors from the set $\mathcal{P}$:

$$W = \sum_m C_m T^{(m)} \tag{2.6}$$

(C_m a complex number),

ii) no element of $\mathcal{P}$ can be so reduced. For example, if $\mathcal{P} = \{\delta^a_b, g^{abc}\}$, condition i) requires that the loop contraction $g^{abc} g^\dagger{}_{cbd}$ is reducible

$$g^{abc} g^\dagger{}_{cbd} = \alpha\, \delta^a_b \; , \; \alpha > 0 \tag{2.7}$$

otherwise the set of primitives must include two rank 2 tensors, δ^a_b and $g^a_d \equiv g^{abc} g^\dagger{}_{cbd}$. Parenthetically, let us note that the set of primitive invariants is not the same thing as the set of irreducible polynomial invariants of a given Lie algebra. For example, the adjoint representation of SU(n) (A_{n-1} in Cartan's notation) has three primitive invariants $\mathcal{P} = \{ \delta_{ij}, C_{ijk}, d_{ijk}\}$, but n-1 independent homogenous polynomials.

As the notion of primitiveness is the basis of the entire construction of invariance groups attempted in this paper, it is important to emphasize that the above definition of primitiveness is maybe not satisfactory. As it stands, primitiveness assumption falls short of uniquely defining certain classes of invariance algebras, among those F_4, E_6 and E_7. I hope that the results of the present paper might suggest a sharper definition of primitiveness.

If we expand the derivation D^b_a in terms of some basis set of [nxn] Hermitian matrices, invariance conditions (2.4) become conditions on the algebra closed by the bases. We shall consider two parametrizations of D^b_a, one in terms of generators T

$$D^b_a = \varepsilon_i \, (T_i)^b{}_a \, , \quad \begin{array}{l} \varepsilon_i \text{ real, infinitesimal;} \\ T_i \text{ hermitian ,} \end{array} \tag{2.8}$$

and the other in terms of projectors P

$$D^b_a = \varepsilon^c_d \, P^{db}_{ca} \; , \qquad \varepsilon^c_d \text{ hermitian, infinitesimal;} \quad (P^{db}_{ca})^* = P^{ac}_{bd} \, . \tag{2.9}$$

Generators. In Cartan's approach one expands D in terms of N independent hermitian matrices T_i (T_i are trivially related to Cartan's non-hermitian canonical bases E_i). T_i close a Lie algebra. This can be seen by considering an N-dimensional complex vector space A with elements $X = (X_1, X_2, \ldots X_N)$ constructed by mapping

$$X_i = \bar{x} \, T_i y; \; \bar{x} \in \bar{V}, \; y \in V. \tag{2.10}$$

The transformations of the group G over the space A are generated by the adjoint (or regular) representation generators $(C_i)_{jk}$

$$X'_j = X_j + i \, \varepsilon_i (C_i)_{jk} \, X_k \tag{2.11}$$

according to the action of G on the underlying V, $\bar{V}$ spaces

$$\bar{x}' \, T_j \, y' = \bar{x} \, T_j \, y + i \, \varepsilon_i \, \bar{x} \, [T_j, T_i] \, y \tag{2.12}$$

By comparing (2.11) and (2.12) we see that if $(T_i)^a_b$ is an invariant tensor, T_i must close a Lie algebra

$$[T_i, T_j] = i \, C_{ijk} \, T_k \qquad , \quad C_{ijk} = i (C_i)_{jk} \, . \tag{2.13}$$

This is the only invariance condition of type (2.4) used in Cartan's analysis. At this point Cartan chooses T_i of canonical form whose symmetry properties make it possible to determine all solutions of (2.13). This classifies all semisimple complex Lie algebras, but it does not tell us which particular algebra preserves a given set of primitive invariants.

Before returning to this problem, let us make several observations which will be useful later. The quadratic form $\mathrm{Tr}(T_i T_j)$ is symmetric and real, so it can be diagonalized. With this redefinition the diagonal entries of $\mathrm{Tr}\,(T_i T_j)$ are non vanishing because of the independence of T_i and positive because T_i are hermitian. Hence it is always possible to choose generators in (2.8) (this amounts to a rotation and rescaling of parameters ε_i) which satisfy a <u>normalization condition</u>

$$\mathrm{Tr}\,(T_i T_j) = a \ \delta_{ij} \tag{2.14}$$

where <u>a</u> is an arbitrary positive constant. (In Cartan's approach <u>a</u> is customarily fixed by choosing root vectors of some standard length). With this normalization the structure constants C_{ijk} are real, fully antisymmetric and computable from T_i by tracing (2.13) with T_j:

$$i\, C_{ijk} = \frac{1}{a}\ \mathrm{Tr}\,[T_i T_j T_k - T_k T_j T_i] \tag{2.15}$$

<u>Projectors</u>. To clarify the relation between the generators T_i and the defining space invariants, consider an arbitrary hermitian matrix M^a_b expanded in terms of the bases T_i

$$M^a_b = \sum_{i=1}^{N} m_i (T_i)^a_b + Z^a_b \quad , \tag{2.16}$$

where Z is linearly independent of T_i.

Define a <u>projector P</u> with properties

$$P^{ce}_{df}\ P^{fb}_{ea} = P^{cb}_{da} \tag{2.17}$$

$$P^{cb}_{da}\ (T_i)^a_b = (T_i)^c_d \tag{2.18}$$

$$P^{cb}_{da}\ Z^a_b = 0\ . \tag{2.19}$$

To compute m_i in (2.16) use the projector

$$P^{cb}_{da} \; M^a_b = \sum_{i=1}^{N} m_i (T_i)^c_d \tag{2.20}$$

and trace with $(T_j)^c_d$ to obtain

$$m_i = \frac{1}{a} \; \mathrm{Tr}(T_i \, M)$$

Substituting back into (2.20) we have

$$[P^{cb}_{da} - \frac{1}{a} \; (T_i)^c_d \, (T_i)^b_a \,] \, M^a_b = 0$$

As this is true for arbitrary M^a_b, the projector which projects out the subspace of [nxn] hermitian matrices spanned by bases T_i can be constructed from the generators by

$$P^{cb}_{da} = \frac{1}{a} \; (T_i)^c_d \, (T_i)^b_a \quad . \tag{2.21}$$

That this is indeed the projector (2.9) can be seen by rewriting (2.8) using (2.14)

$$D^a_b = \varepsilon_j \; (T_j)^c_d \; \frac{1}{a} \, (T_i)^d_c \, (T_i)^a_b$$

$$= \varepsilon^c_d \;\; P^{d\,a}_{c\,b} \tag{2.22}$$

where $\varepsilon^c_d = \varepsilon_j (T_j)^c_d$ can be replaced by an arbitrary infinitesimal hermitian matrix, because by (2.20) the extra parameters do not generate any additional transformations.

Equation (2.21) establishes the connection between generator and projector descriptions of invariance groups, but it does not mean that in order to know P we must first construct T_i. P is a more fundamental object in the sense that if offers a more economical description of the invariance group; unlike T_i construction, construction of P does not require introduction of an arbitrarily labelled auxilliary space A, and an arbitrary overall normalization (2.14).

Projector P is itself an invariant tensor, and by primitiveness assumption (2.6) it must be expressible as a sum over tree contractions of primitives

$$P^{ac}_{bd} = C_1\ \delta^a_d\delta^c_b + C_2\ \delta^a_b\delta^c_d + \text{(all independent contractions of other primitives with the right index structure)} \qquad (2.23)$$

Substituting this expansion back into (2.4) we obtain conditions on C_m and the primitives which often suffice to fully determine P and the invariance algebra.

The expansion (2.23) is intimately related to the Clebsch-Gordon expansion of the Kronecker product $V \otimes \bar{V}$. The projector P projects out that part of $V \otimes \bar{V}$ which transforms as the adjoint representation (2.10)

$$x_a\, y^b = \delta^b_a\ \frac{1}{n}\, x_c y^c + \frac{1}{a}\, (T_i)^b_a\, x_c (T_i)^c_d\, y^d + \ldots.$$

As this is true for arbitrary x, $\bar{y}$ the Clebsch-Gordon series can be written as a completeness relation for a sum of projection operators, one for each irreducible representation of $V \otimes \bar{V}$ (we reserve the term "projector" for the projection operator for the adjoint representation);

$$\delta^a_b\delta^c_d = \frac{1}{n}\ \delta^a_d\delta^c_b\ + P^{ac}_{db}\ + \ldots \qquad (2.24)$$

The first term projects out the singlet, the second the adjoint representation and so forth, and the number of terms equals the number of independent tree invariants with the right index structure. Expansion (2.24) is a special case of a general $V \otimes V \ldots \otimes \bar{V} \otimes \bar{V} \otimes \ldots$ Kronecker product decomposition (indices suppressed)

$$\begin{aligned} \mathbb{1} &= \sum_\lambda P_\lambda \\ P^2_\lambda &= P_\lambda \\ P_\lambda P_\mu &= 0 \qquad \text{if } \lambda \neq \mu \end{aligned} \qquad (2.25)$$

where λ distinguishes different irreducible representations. With this in mind we can generalize the notion of invariant tensor (2.1) to tensors with many different kinds of indices, each kind being a shorthand notation for some projection operator in (2.25). For example, consider (2.1) enlarged to

$$g = {}^{a..b}_{c..d,\ i..j} \qquad \begin{matrix} a,..d &= 1,2,..n \\ i,..j &= 1,2,..N, \end{matrix} \qquad (2.26)$$

The adjoint representation indices i, ..j are a shorthand for projectors applied to an underlying defining space invariant:

$$g^{a}_{cd,\ i} (T_i)^e_f = g^{ag}_{cdh} \ P^{he}_{gf} \ .$$

In this sense we shall often use generator T_i as a shorthand for the projector P:

$$(T_i)^a_c (T_i)^e_f = a \ P^{ag}_{ch} \ P^{he}_{gf} \quad ,$$

without ever attempting an explicit construction of T_i.
Invariant polynomial (2.3) corresponding to (2.26) includes elements of the auxiliary space A of (2.10), and the invariance condition (2.4) now includes the adjoint representation derivations (2.11):

$$D_{jk} \equiv \varepsilon_i \ (C_i)_{jk}$$

Lie algebra (2.13) is simply the invariance condition for $P(x,\bar{y},Z) = (T_i)^a_b \ x_a y^b Z_i$, and similarly Jacobi relation is the invariance condition for C_{ijk}. <u>Lie algebra is an automatic consequence of the invariance of P^{ac}_{bd}.</u>

As the generalized invariants (2.26) are only a shorthand notation for defining space invariants contracted with some projection operators, the primitiveness assumption applies to generalized invariants as well: one merely contracts both sides of (2.6) with the same projection operators.

By contracting we can turn expansion (2.6) into a set of linear equations

$$W^n = C_m \ t^{mn} \qquad (2.27)$$

where $t^{mn} = T^{\dagger(m)} \cdot T^{(n)}$, $W^{m} = T^{\dagger(m)}$. W are scalar invariants formed by contracting all pairs of corresponding indices

$$T^{\dagger}.W = T^{\dagger}{}_{j..i,}{}^{d..c}_{b..a} \; W^{a..b}_{c..d,\, i..j} \qquad (2.28)$$

Tree invariants $T^{(m)}$ are generally not independent, and the summation in (2.6) can be restricted to some basis set $T^{(1)}, \ldots, T^{(\beta)}$ for which det $(t^{mn}) \neq 0$. This enables us to compute C_m in (2.6). In this way study of invariance algebras reduces to study of scalar invariants. By means of projection operators (2.23), (2.25) a generalized scalar invariant can be written as a sum over scalar invariants composed only of primitives. The primitiveness assumption guarantees that every such scalar invariant is computable. Suppose that the scalar invariant C is connected and contains ℓ loops of contractions. We can always separate out a one-loop sub-invariant W

$$C = W^{a..b}_{c..d} \; V^{d..c}_{b..a}$$

and reduce W by (2.6) to a sum over tree invariants $T^{(m)}$. By this process of rewriting each C as a sum of scalar invariants of ℓ-1 or less loops we can reduce any scalar invariant to a polynomial in $\delta^{a}_{a} = n$, the dimension of the defining space.

To illustrate the importance of scalar invariants, let us consider few simple examples. One trivial scalar invariant is the dimension of the defining space, $\delta^{a}_{a} = m$. First non-trivial scalar invariant is the dimension of the invariance algebra, $\delta_{ii} = N$. This positive integer is computed from the projector by

$$N = \delta_{ii} = \delta_{ij}\,\delta_{ji} = \delta_{ij}\,\frac{1}{a}\,(T_j)^{a}_{b}\,(T_i)^{b}_{a} = P^{ab}_{ba} \qquad (2.29)$$

The dimension of any representation in (2.25) can be similarly computed from the corresponding projection operator. Another important scalar invariant is ℓ, the index of the defining representation[22-24,12,13]

$$\ell = \frac{\mathrm{Tr}\,(T_i T_i)}{\mathrm{Tr}(C_j C_j)} \tag{2.30}$$

ℓ^{-1} turns out to be an integer for the lowest dimensional representations of all simple Lie algebras. As will be shown in the next section, ℓ can be expressed in terms of projectors as

$$\ell^{-1} = \frac{2N}{n} - \frac{2}{N} \quad P^{ac}_{bd} \; P^{bd}_{ca} \tag{2.31}$$

By hermiticity of T_i, ℓ is real and positive. For simple Lie algebras normalized by (2.14), the index is related to the Cartan-Killing metric

$$\mathrm{Tr}\,(C_i C_j) = \ell^{-1}\, a\delta_{ij} \tag{2.32}$$

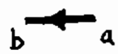
However, converse is not true; even if the explicit evaluation of Cartan-Killing metric for some invariance algebra yields (2.32), this is no guarantee that the algebra is simple. I do not know how to formulate the simplicity criterion in terms of projectors P.

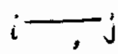
3. DIAGRAMMATIC NOTATION

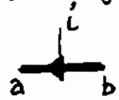
Relations like the invariance condition (2.4) can be quite cumbersome, especially if in addition one has to symmetrize various subsets of indices. The standard way of avoiding a proliferation of indices and incorporating symmetries is by defining the invariants (2.3) as abstract products. For example, instead of tensorial primitive invariants $\delta^a_b\, x_a y^b$, $z^a \equiv d^{abc}\, x_b y_c$, one introduces products $(\bar{y},x)$, $\bar{z} \equiv x\bullet y$ and demands invariance of a symmetric trilinear form $\langle x,y,z\rangle \equiv (x\bullet y, z)$:

$$\langle Dx,y,z\rangle + \langle x,Dy,z\rangle + \langle x,y,Dz\rangle = 0 \tag{3.1}$$

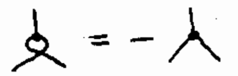
We shall give some examples of this notation later. In the present context an abstract product notation would entail too many kinds of dots and brackets, and to treat all invariance algebras in a unified manner it will be most convenient to stick to the tensor notation. It should be emphasized that all the relations studied here are <u>coordinate independent</u>, and tensorial indices have only formal meaning, indicating what representations appear in an invariant, and what summations are to be performed (see the nice discussion of tensor invariants given by Penrose[25]). We explcit this coordinate independence of tensorial equations by introducing a label-free diagrammatic notation. In this notation the tensor invariants of the preceding Section are written as

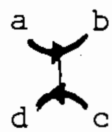
δ^a_b  (3.2)

δ_{ij}  (3.3)

$(T_i)^b_a$  (3.4)

$-i\, C_{ijk}$  (3.5)

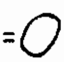
(here indices are read counterclockwise, and the skew symmetry is built in by )

$P^{ac}_{bd} = \frac{1}{a}$  (3.6)

Clebsch-Gordon series:

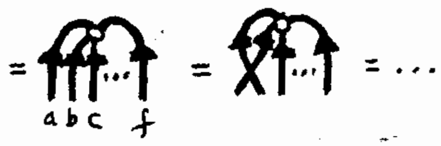
 $= \frac{1}{n}$  $+ \frac{1}{a}$  $+ \ldots$ (3.7)

Dimension of the defining space: $n =$  (3.8)

Dimension of the invariance algebra: $N =$  $= \frac{1}{a}$  (3.9)

As we shall consider only two types of primitives beyond δ_b^a, it will be convenient to introduce special diagrammatic notation for them. Fully symmetric primitives d will be denoted by

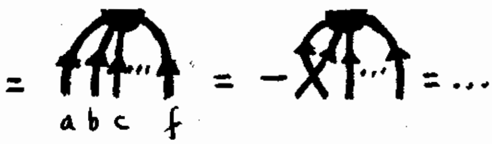
$$d: \quad d^{abc..f} = d^{bac..f} = d^{bca..f} = \ldots$$



(3.10)

and the fully antisymmetric primitives f by

$$f: \quad f^{abc..d} = -f^{bac..a} = f^{bca\,..d} = \ldots$$



(3.11)

Without fear of confusion we can suppress the dagger notation for their duals:

$$d^{\dagger} = d_{ab..f}, \; f^{\dagger} = f_{d..cba}$$

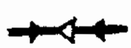
We have used a semicircle in (3.11) because even rank f's are not cyclic

$$f^{abcd} = -f^{bcda} \tag{3.12}$$

The odd rank f's are cyclically symmetric and for them we do not have to distinguish between the first and the last index:

$f^{abc..d} =$  (odd ranks only) (3.13)

We shall also find it convenient to introduce a special notation for f of rank two

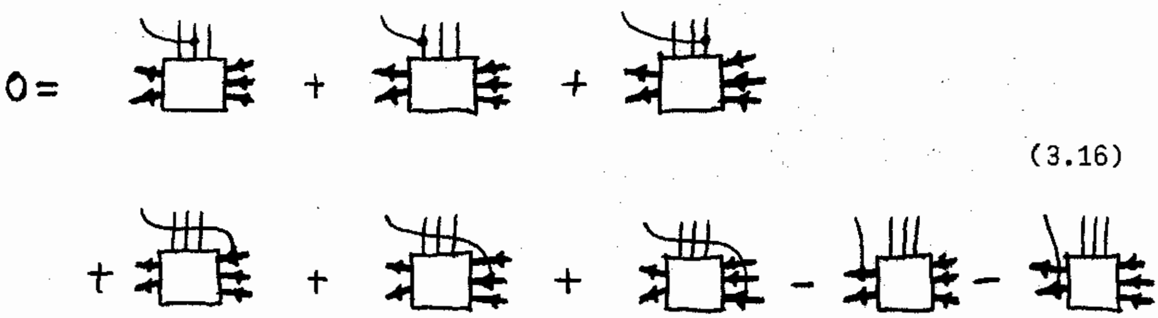
$f^{ab} =$  (3.14)

Even though in the above definitions we have labelled the lines, indices can always be omitted. An internal line signifies a summation over the corresponding indices, and for the external lines the equivalent points on the paper represent the same index in all terms of a diagrammatic equation. Consider an arbitrary tensor invariant of form (2.26)

$g^{abc}_{de,\ ijk}$ =  (3.15)

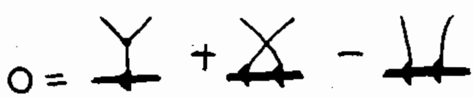
(legs are always indexed counterclockwise)

The invariance condition (2.4) is expressed diagrammatically as

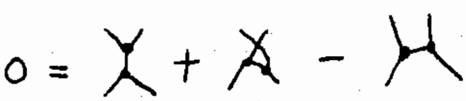


(3.16)

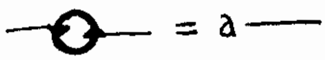
This is the most general form of the invariance condition; the tensor denoted by the box can be a primitive, or a complicated composite tensor. For example, Lie algebra (2.13) follows from the invariance of 

 (3.17)

and the Jacobi identity from the invariance of 

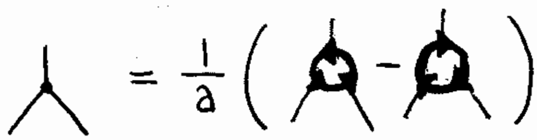
 (3.18)

The normalization (2.14) is diagrammatically



(3.19)

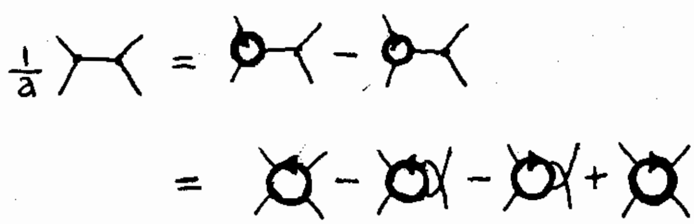
Note that (3.17) fixes the sign convention for C_{ijk}, and that if the arrows were reversed, the right-hand side would change sign. The diagrammatic version of the definition (2.15) of C_{ijk} is



(3.20)

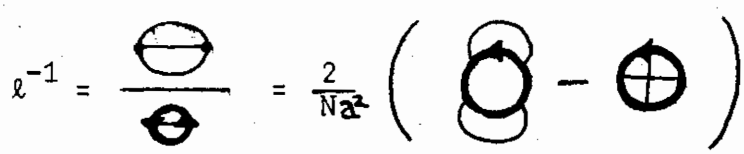
It is always sufficient to indicate the direction of a line by a single arrow, and the remaining inessential arrows can be omitted. Note also that the dual tensor $g^\dagger$ in (2.1) is obtained by flipping over the diagram for g and reversing the arrows on all directed lines.

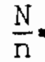
To illustrate how one computes with diagrammatic equations, we shall derive (2.31). Use (3.20) and (3.17) to obtain



(3.21)

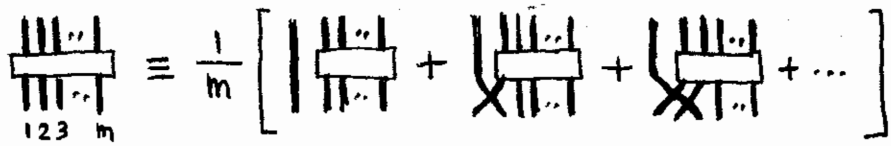
and substitute this into (2.30)



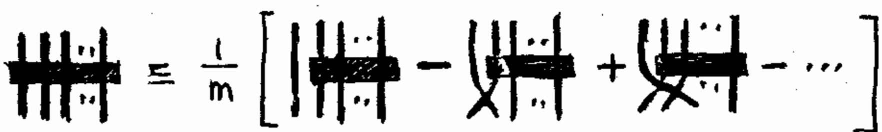
Noting that  $= a\,\frac{N}{n}$ , we have (2.31);

$$\ell^{-1} = \frac{2N}{n} - \frac{2}{N\,a^2}$$  (3.22)

Very convenient tools for the study of Kronecker products and invariants with symmetries are symmetrization and antisymmetrization operators defined recursively by

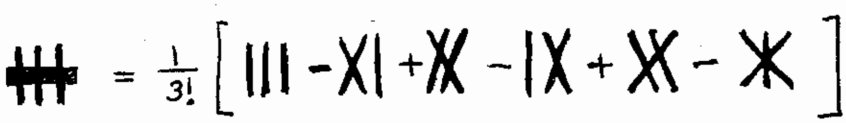


(3.23)



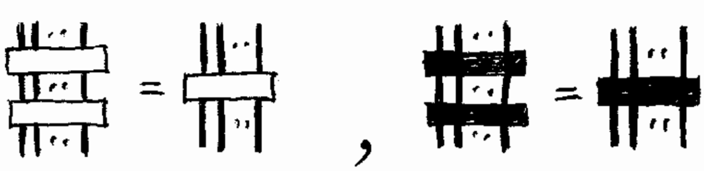
(3.24)

For example,



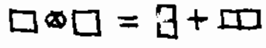
(3.25)

The factor in front counts the number of distinct permutations and insures idempotency



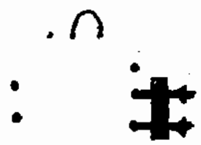
(3.26)

These operators generate the symmetric group, and give a diagrammatic version of symmetrizations and antisymmetrizations which build up

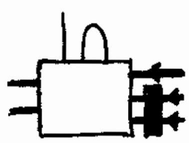
Young operators[26,27]. As a trivial example, consider the Young tableaux decomposition of a Kronecker product  ,
given by

 (3.27)

Another convenient diagrammatic tool are <u>operators</u> which we introduce by a simple example. Suppose we want to antisymmetrize (3.15) in <u>a</u> and <u>b</u> indices, and contract j and i indices. This is accomplished by the operator



which we apply to (3.15) by superimposing it over corresponding legs:



We conclude this discussion of diagrammatic notation by an illuminating observation due to Penrose[19,26]. SO(3) invariance algebra has primitives δ_{ij} and ε_{ijk}, and in the diagrammatic notation all scalar invariants are closed cubic graphs. For planar graphs the value of the corresponding scalar invariant turns out to be the number of ways of colouring the lines of the graph with the three colours meeting at each vertex. In general an invariance algebra can be interpreted as a "colouring rule" which assigns a combinatorial weight to a given diagram. This is precisely the role played by internal symmetry groups in physics - there the associated Lie algebra factors count the number of degenerate states contributing to a given process.

In this context it is quite conventional to describe the invariance algebra in terms of projectors rather than generators. In generator formalism a Yang-Mills theory with massive quarks of n colours and N massless gluons is defined by the classical Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_i^{\mu\nu} F_{i\mu\nu} + \bar{q}(i\not{D} - m)q$$

$$F_i^{\mu\nu} = \partial^\mu A_i^\nu - \partial^\nu A_i^\mu + C_{ijk} A_j^\mu A_k^\nu \qquad (3.28)$$

$$D_a^{b\mu} = \delta_a^b \partial^\mu - i A_i^\mu (T_i)_a^b$$

In the projector formulation one replaces $A_i^\mu (T_i)_a^b$ by $A^\nu \equiv A_a^{b\nu}(x)$. This is an element of the invariance algebra obtained from an arbitrary [nxn] hermitian operator matrix $\hat{A}^\nu$ by $A^\nu = P\hat{A}^\nu$. Now the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4} \mathrm{Tr}\left(F^{\mu\nu} F_{\mu\nu}\right) + \bar{q}(i\not{D} - m)q \qquad (3.29)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig\left[A^\mu, A^\nu\right]$$

$$D^\mu q = (\partial^\mu + igA^\mu)q$$

and the projectors P appear as the group theory factors in gluon propagators

$$\langle A_b^{a\mu}(x), A_d^{c\nu}(y)\rangle = P_{b\,f}^{a\,e} P_{d\,g}^{c\,h} \langle \hat{A}_e^{f\mu}(x), \hat{A}_h^{g\nu}(y)\rangle$$

$$= P_{b\,d}^{a\,c} D^{\mu\nu}(x-y) \qquad (3.30)$$

For example, 't Hooft [35] uses projector (5.5) for U(n) algebra, Callan, Coote and Gross [36] use (5.6) for SU(n) (the above even in the diagrammatic notation) and Cheng, Eichten and Li [37] use (9.13) for SO(n).

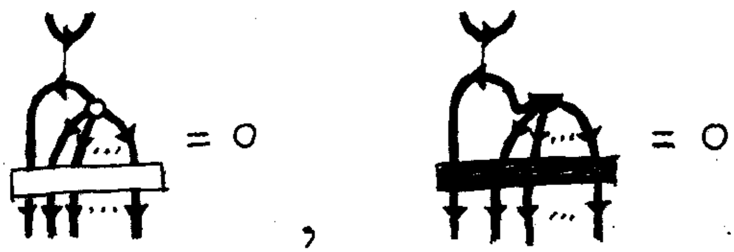
4. CONSTRUCTION OF INVARIANCE ALGEBRAS

I shall construct the projectors for various invariance algebras by the following sequence of steps:

(i) state the <u>primitives</u> and the relevant consequences of the primitiveness assumption.

(ii) list all <u>basis tensors</u> that can appear in the projector expansion and explore possible relations between them.

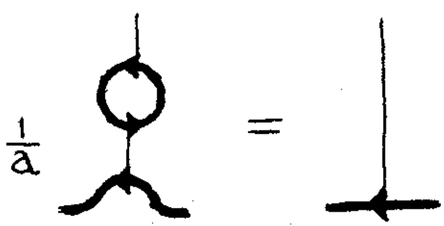
(iii) for each of the above possibilities, substitute the projector expansion into the <u>invariance condition</u>



(4.1)

and find all solutions.

(iv) compute the overall <u>normalization</u> of the projector from the normalization condition



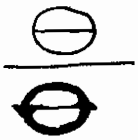
(4.2)

(v) to identify the algebra, compute its <u>dimension</u>

$N = \frac{1}{a}$  (4.3)

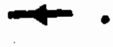
and

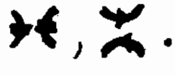
(iv) the <u>index</u> of the defining representation

$\ell^{-1} =$  $= \frac{2N}{n} - \frac{2}{Na^2}$  (4.4)

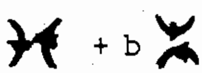
Representations will be identified by Cartan's classification as $B_1(3)$, $A_2(8)$, $F_4(26)$, $F_4(52)$, ..., where the number in the brackets is <u>n</u>, the dimension of the defining vector space (for consistency the number <u>n</u> in SU(n), SO(n) and Sp(n) refers to the dimension of the defining vector space, at variance with the conventional notation $Sp(\frac{n}{2})$). Different representations of same dimension are distinguished by their index, except for the trivial case of cogredient-contragredient pairs.

5. $\delta\vec{B}$ INVARIANCE → U(n), SU(n)

(i) <u>primitives</u>: only  .

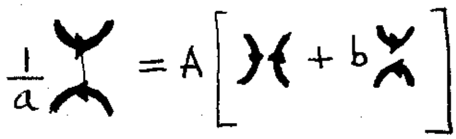
(ii) <u>projector bases</u>:  . (5.1)

Suppose that they are not independent:

0 =  (5.2)

Contracting with  and  we get $0 = n + b$ and $0 = 1 + n\,b$, with an unacceptable solution $n = -1$, and a trivial solution $n = 1$.

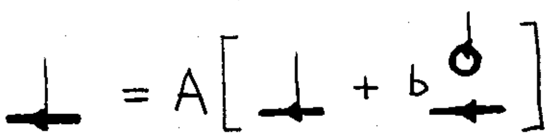
Hence the tensors (5.1) are always independent. The projector is of form



(5.3)

(iii) the invariance condition (2.5) is already built into the formalism, i.e. T_i are hermitian, and consequently A and b are real.

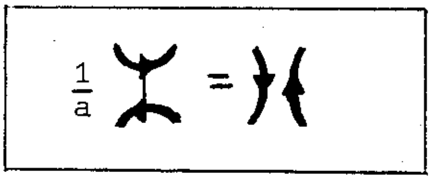
(iv) the normalization condition (4.2) yields



(5.4)

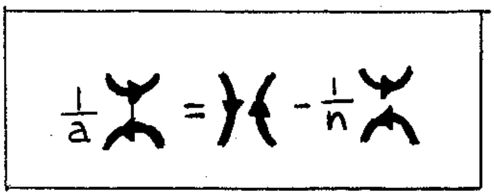
with two solutions:

Case 1. A = 1, b = 0. The projector is given by



U(n) algebra (5.5)

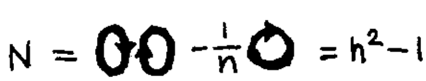
Case 2. A = 1, $\mathrm{Tr}(T_i) = 0$. This additional tracelessness condition substituted back into (5.3) yields $b = \frac{1}{n}$, and the projector is



SU(n)
$= A_{n-1}(n)$ (5.6)

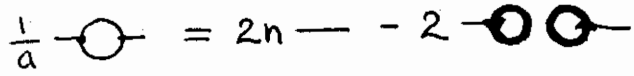
(v) algebra dimension

Case 1.  (5.7)

Case 2.  (5.8)

(vi) index

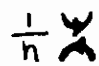
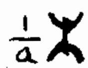
Case 1. U(n) is not semisimple:

 (5.9)

but all scalar invariants are still computable. In our definition (4.4) index is a ratio of scalar invariants and therefore computable:

$$\ell^{-1} = \frac{2(n^2-1)}{n} \tag{5.10}$$

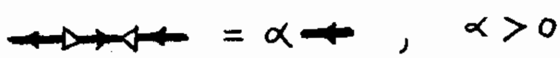
Case 2. $$\ell^{-1} = 2n \tag{5.11}$$

Comment Relation (5.6) is a reduction algorithm in the sense that it rewrites a scalar invariant with k loops as a sum of two scalar invariants with (k-1) loops. It is also a Clebsch-Gordon series in the sense that $\bar{V} \otimes V =$  is written as a sum of a singlet  and the adjoint representation 

6. δ^a_b, f^{ab} INVARIANCE → SU(2), Sp(n)

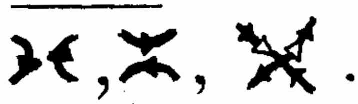
(i) primitives  (6.1)

Primitiveness assumption requires

 (6.2)

(usually $\alpha = 1$)

f^{ab} has an inverse $\frac{1}{\alpha} f_{bc}$ if det $(f_{ab}) \neq 0$, so it can be realized only in even-dimensional spaces; n is even.

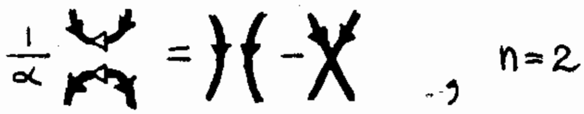
(ii) projector bases  (6.3)

Two possibilities arise:

Case 1. The above bases are not independent. Taking into account f^{ab} antisymmetry, the most general relation is of form

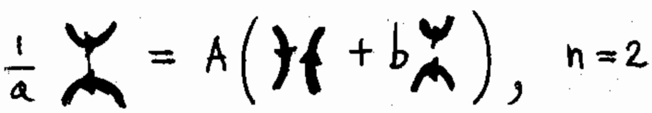
(6.4)

Contracting with  and  we get $o = n - b$ and $o = -1+b\frac{n-1}{2}$,
with $n = 2$ as the only solution:



(6.5)

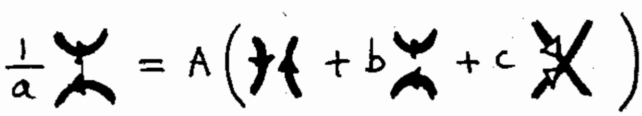
so f^{ab} is proportional to Levi-Civita tensor in two dimensions. The projector is of form



(6.6)

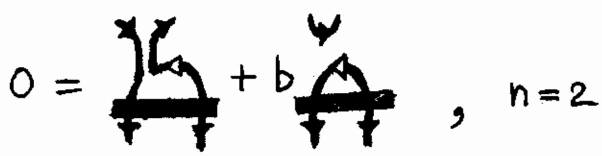
Case 2. Bases (6.3) are assumed <u>independent</u>:

(6.7)

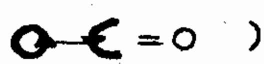


(iii) <u>invariance condition</u>:

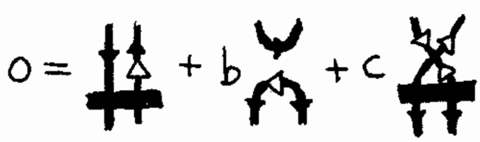
Case 1.



(6.8)

Contracting with δ_{ab}, or from (6.5) we find $b = -\frac{1}{2}$.
Note that this makes T_i traceless (i.e. from (6.6) )

Case 2.

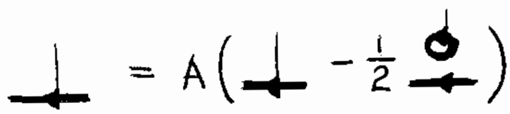


(6.9)

By assumption the above tensors are independent, hence $c = 1$, $b = 0$. Note that T_i is again traceless.

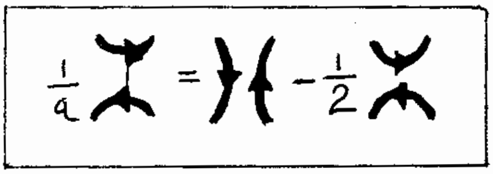
(iv) normalization:

Case 1.



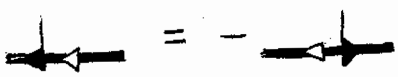
(6.10)

As T_i is traceless, $A = 1$, and the projector is



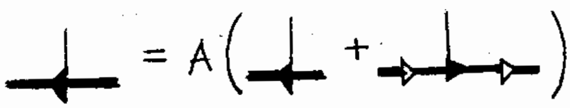
$SU(2) = A_1(2)$ (6.11)

Case 2. The invariance of f^{ab}, (3. 16) gives



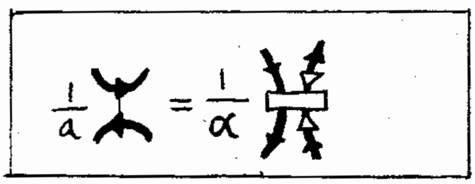
(6.12)

which reduces the second term in the normalization condition



(6.13)

so that $A = \frac{1}{2}$, and the projector is



$Sp(n) = C_{\frac{n}{2}}(n)$ (6.14)

(v) algebra dimensions

Case 1. $N = 3$ (6.15)

Case 2. $N = \frac{n(n+1)}{2}$ (6.16)

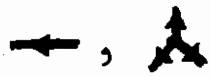
(vi) index

Case 1. $\ell^{-1} = 4$ (6.17)

Case 2. $\ell^{-1} = n+2$ (6.18)

<u>Comment</u> SU(2) = Sp(2), because by (6.5) the two projection operators are equivalent in $n = 2$ dimensions.

7. δ^a_b, f^{abc} INVARIANCE → SU(3)

(i) <u>primitives</u>  (7.1)

By primitiveness assumption

 $= \alpha$  , $\alpha > 0$ (7.2)

(usually $\alpha = 1$). f^{abc} obey a pure antisymmetry relation

 $= 0$, (7.3)

from which follow relations

 $= 0$ (7.4)

 $= 0$ (7.5)

(ii) <u>projector bases:</u> (7.6)

Two possibilities arise

Case 1. The above bases are <u>not independent.</u>

$\frac{1}{\alpha}$  $= A$  (7.7)

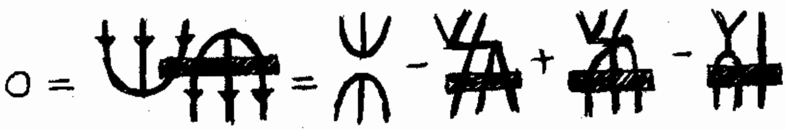
Substituting this into (7.5) we obtain

 $= 0$ (7.8)

This is equivalent to (7.7), as can be seen by expanding

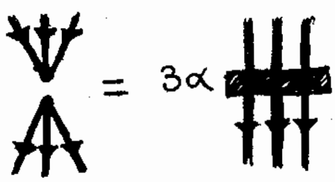
 $= 0$ (7.9)

Still another equivalent relation is obtained by expanding

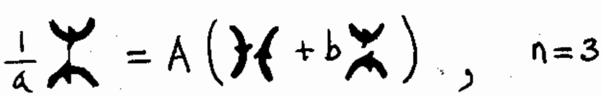
 (7.10)

Contracting (7.7) with  yields $A = 1$.

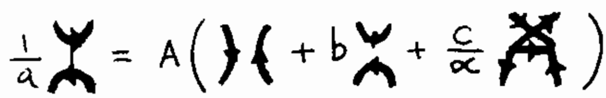
Contracting (7.7) with  yields $n = 3$, hence the relation (7.7) can be realized only in three dimensions, and f^{abc} is proportional to Levi-Civita tensor

 (7.11)

The projector is of form

 (7.12)

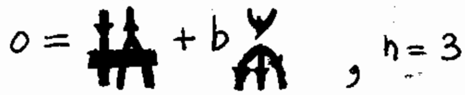
Case 2. Bases (7.5) are assumed <u>independent</u>:



(7.13)

(iii) <u>invariance condition</u>

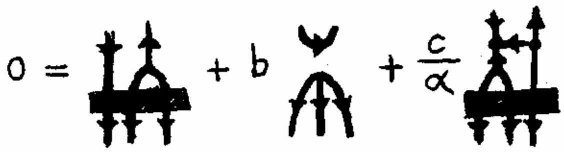
Case 1.



(7.14)

Contracting with  we obtain $b = -\frac{1}{3}$.

Case 2.

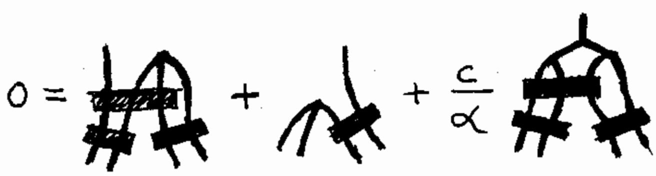


(7.15)

Antisymmetrize all out lines and use (7.5) to obtain

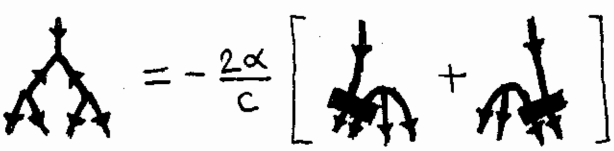
0 = (1-b)  (7.16)

By assumption the tensor is non-vanishing, because otherwise by (7.8) this case would reduce to Case 1. Hence b = 1. Now resymmetrize indices of (7.15) by applying  :



(7.17)

By expanding the third term and by (7.4) we obtain a reduction relation for a chain of three f^{abc} contractions:



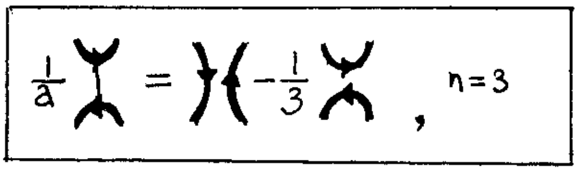
(7.18)

This reduces, by contraction with  to a relation of form (7.7), hence Case 2, again reduces to Case 1, and the primitives set (7.1) must satisfy (7.11).

Case 3. The antisymmetry relations (7.4)and (7.5) were the key to the unique n = 3 solution. For pedagogical reasons, to illustrate the type of solution I will later obtain for exceptional groups, I shall show what would have happened if I have missed the relation (7.18). From (7.16) b = 1, and contracting (7.15) with  obtain c = -n-1. T_i is now traceless.

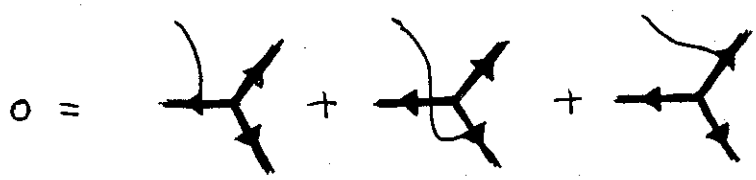
(iv) normalization:

Case 1. As in (5.4), A = 1.



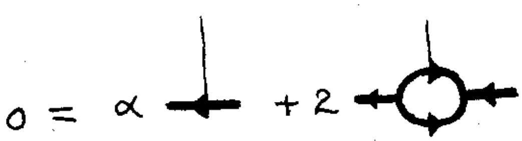
SU(3) = $A_2(3)$ (7.19)

Case 3. From the invariance (3.16) of f^{abc}



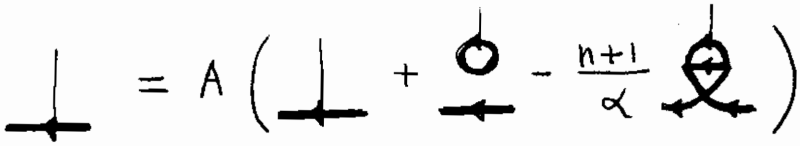
(7.20)

Contracting with f^{abc} obtain



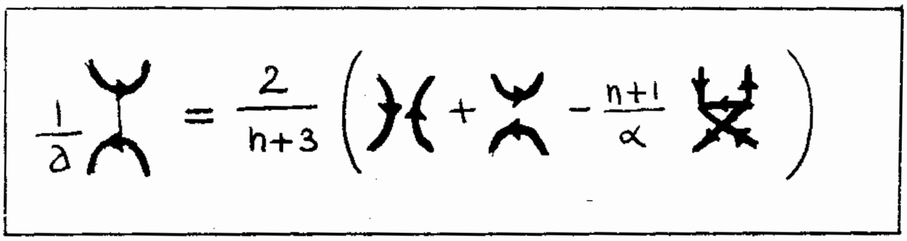
(7.21)

Hence the third term in the normalization condition is reducible



(7.22)

yielding $A = \frac{2}{n+3}$, and the projector

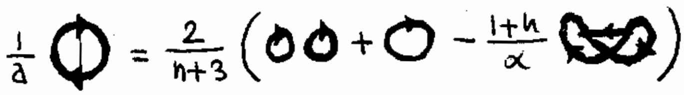


(7.23)

Note that this consistent with (7.11) and (7.19) for n = 3.

(v) <u>algebra dimension</u>

Case 1. $N = 8$ (7.24)

Case 3. $N =$ 

$$= 4(n-2) + \frac{24}{n+3} \tag{7.25}$$

(vi) <u>index</u>

Case 1. $\ell^{-1} = 6$ (7.26)

Case 3. $\ell^{-1} = 2\,\frac{3n^2+22n+15}{(n+3)^2}$ (7.27)

<u>Comment</u> The algebra dimension N must be an integer. This turns the "partial ignorance" result (7.25) into a Diophantine equation with only four solutions (trivial n = 1 is not realizable, because f^{abc} can be constructed only in $n \geq 3$ dimensions)

n	3	5	9	21
N	8	15	30	77
ℓ^{-1}	6			

(7.28)

"Partial ignorance" is already sufficient to sharply limit the number of n-dimensional spaces which could possibly accommodate primitives (7.1). The $n=3$ solution in the above list is trivial to construct, because any fully antisymmetric tensor of rank n in n dimensions is proportional to Levi-Civita tensor. The defining relation for Levi-Civita tensor, (7.8) and (8.6) for general n, states that no object antisymmetric in (n+1) indices can be constructed in n dimensions. However, note that I would have not known how to discard the rest of (7.28) if I had missed the relation (7.18).

8. δ^a_b, $f_{abc\ldots d}$ INVARIANCE → SU(n)

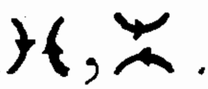
(i) primitives:  ,  ; $r>3$. (8.1)

A fully antisymmetric object can be realized only in $n \geq r$ dimensions. By primitiveness assumption

 $= \alpha$  (8.2)

 $= \frac{2\alpha}{n-1}$  etc. (8.3)

i.e., various contractions of $f_{ab\ldots d}$ must be expressable in terms of δ^a_b, otherwise there would exist additional primitives.

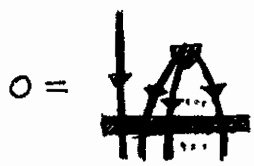
(ii) projector bases:  . (8.4)

According to (5.2) they cannot be related, so the projector is of form (5.3).

(iii) <u>invariance condition</u>

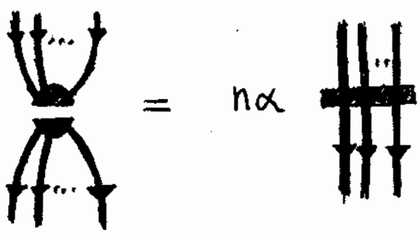
(8.5)

contracting from the top get $0 = 1+bn$. Antisymmetrizing all out legs get



(8.6)

and contracting with δ^a_b from the side get $0 = n-r$. As in the preceding Sections, (8.6) defines Levi-Civita tensor in n dimensions, and can be rewritten as



(8.7)

(Conventional Levi-Civita normalization is $n\alpha = n!$)

The above solution $b = -\frac{1}{n}$ makes T_i traceless, and it is the same as the Case 2 considered in Sec.5. To summarize: the invariance condition forces $f_{abc\ldots d}$ to be proportional to Levi-Civita tensor (essentially because in n dimensions Levi-Civita is the only fully antisymmetric tensor of rank n), and the primitives δ^a_b, $f_{ab\ldots d}$(rank n) have SU(n) as their unique invariance algebra.

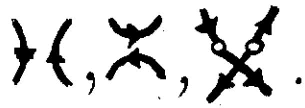
9. δ_a^b, d^{ab} INVARIANCE → SO(n)

(i) primitives  (9.1)

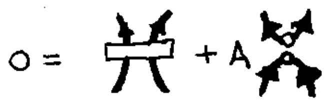
By primitiveness assumption

 (9.2)

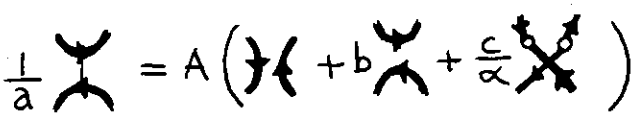
(usually $\alpha = 1$)

(ii) projector bases  (9.3)

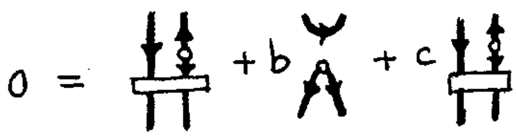
Suppose that the above bases were not independent

 (9.4)

Contracting with  and  we obtain $0 = 1 + nA$ and $0 = \frac{n+1}{2} + A$, with an unacceptable solution $n = -2$ and a trivial solution $n = 1$. Hence the bases (9.3) are always independent. The projector is of form

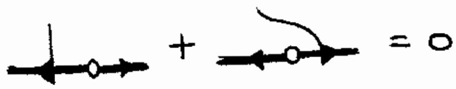
 (9.5)

(iii) invariance condition:

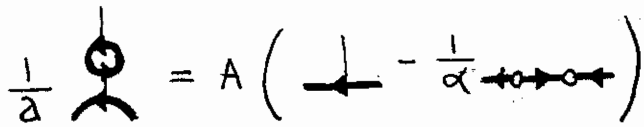
 (9.6)

The only solution is $b = 0$, $c = -1$.

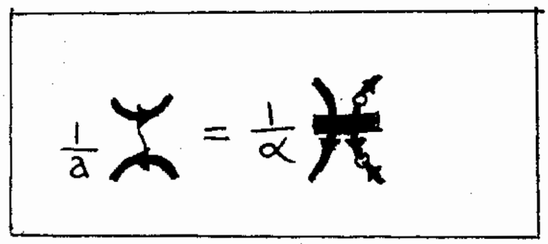
(iv) <u>normalization:</u> The invariance of d^{ab}, (4.1) gives

 (9.7)

which reduces the second term in the normalization condition

 (9.8)

so that $A = \frac{1}{2}$, and the projector is



$$= B_{\frac{n-1}{2}}(n), \text{ n odd}$$

$$SO(n) \qquad (9.9)$$

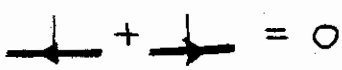
$$= D_{\frac{n}{2}}(n), \text{ n even}$$

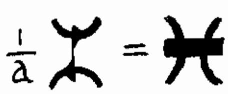
(v) <u>algebra dimension</u> $N = \frac{n(n-1)}{2}$ (9.10)

(vi) <u>index</u> $\ell^{-1} = n - 2$ (9.11)

hence semisimple only for $n > 2$.

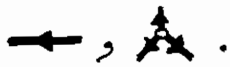
<u>Comment</u> By rotating and rescaling the defining vector space V it is always possible to bring d^{ab} to form δ^{ab}. In this case $\chi^a = \chi_a$, the <u>representation is real,</u> and there is no distinction between δ^a_b , δ^{ab} and δ_{ab}. In the future I shall always replace δ^a_b, d^{ab} primitives by δ_{ab}, omit line arrows, and replace (9.7) and (9.9) by

 (9.12)

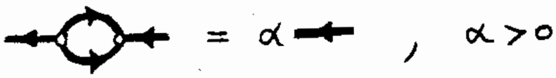
 (9.13)

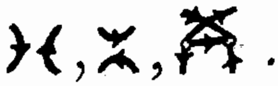
By (9.12), for real representations projector bases are antisymmetric.

10. δ_a^b, d^{abc} INVARIANCE $\rightarrow$ $E_6(27)$, ...

(i) primitives:  (10.1)

By primitiveness assumption

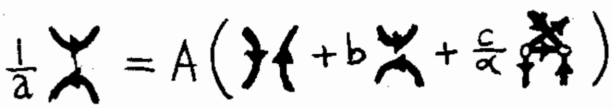
 (10.2)

(ii) projector bases:  (10.3)

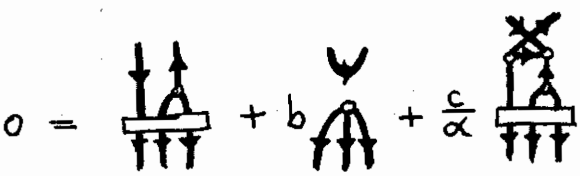
Suppose that the above bases were not independent

 (10.4)

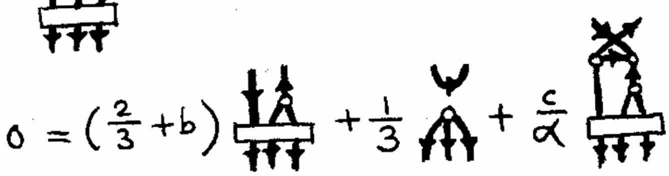
Contracting with  and  we obtain as the only solution the trivial n = 1. Hence the bases (10.3) must be assumed independent, and the projector is of form

 (10.5)

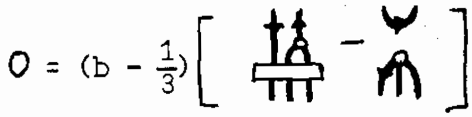
(iii) invariance condition

 (10.6)

Resymmetrize with  to obtain

 (10.7)

Subtracting from (10.6) we get

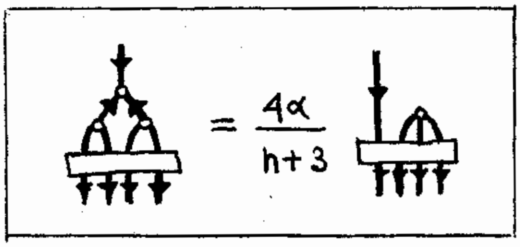


(10.8)

Suppose

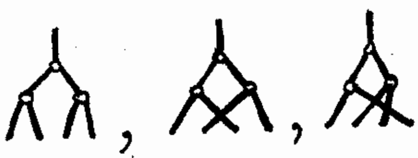
(10.9)

Contracting with  we get a trivial solution n = 1. Hence $b = \frac{1}{3}$, and contracting (10.6) with  we get $c = -\frac{n+3}{3}$. This result is written more compactly by symmetrizing all out legs on (10.6)



Springer's relation (10.10)

By Springer's relation one of 3 possible chains of 3 d_{abc} contractions



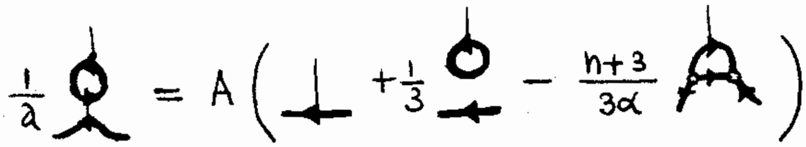
(10.11)

can be eliminated. Note that a single 3-d_{abc} chain cannot be reducible. If it were, by symmetry the reduction relation should be of form

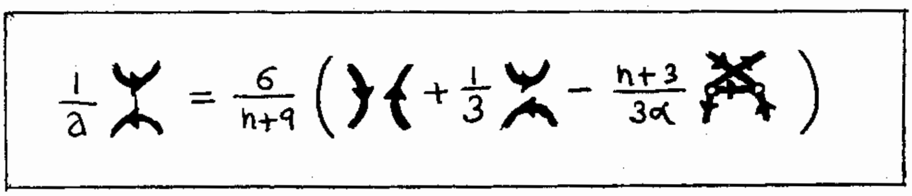
(10.12)

but upon contraction with  this reduces to (10.4), and hence can be realized only for the trivial $n = 1$ case.

(iv) normalization

 (10.13)

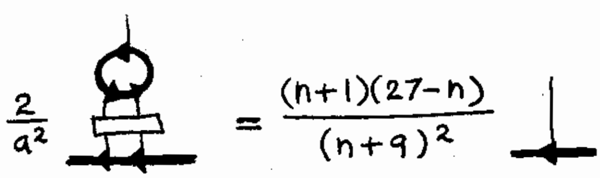
As in (7.21), the third term is reducible, $A = \frac{6}{n+9}$ and the projector is given by

 (10.14)

(v) algebra dimension $$N = \frac{4n(n-1)}{n+9} \tag{10.15}$$

(iv) index $$\ell^{-1} = 6\,\frac{n-3}{n+9} \tag{10.16}$$

Comments The solutions to the Diophantine equation (10.15) are listed in the Table I. To restrict them to $n \leq 27$, we use (10.14) to compute

 (10.17)

Define a fully symmetric tensor d_{ijk}.

$d_{ijk} \equiv$  (10.18)

By hermiticity of T_i, d_{ijk} is real, and

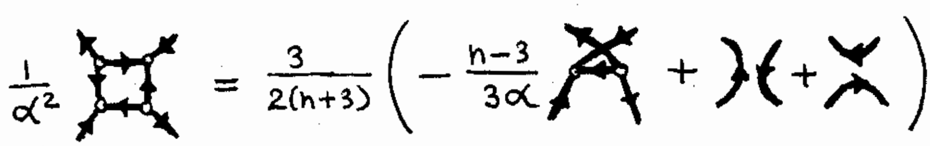
 $= \sum_{i,j,k}^{N} (d_{ijk})^2 \geq 0$ (10.19)

But from (10.17) this equals

$$= \frac{a^3}{2} \frac{(n+1)(27-n)}{(n+9)^2} N \qquad (10.20)$$

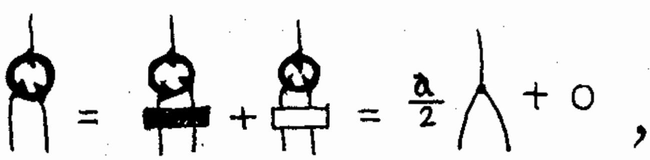
hence n is restricted to $n \leq 27$. This restricts the solutions of the Diophantine equation (10.15) to six, four of which are identifiable. I presume I can construct the first three (A_2, $A_2 + A_2$ and A_5), in the manner I shall construct a related set of algebras in Sec.15. I do not know how to eliminate n = 11 and n = 21 solutions, and for the remaining $E_6(27)$ I lack a full reduction algorithm which would enable me to compute any scalar invariant built from d_{abc} constructions. However, the scalar invariants I cannot compute are of very high order, as their shortest loop must be of length eight or longer.

Springer's relation (10.10) enables me to reduce loops of length four



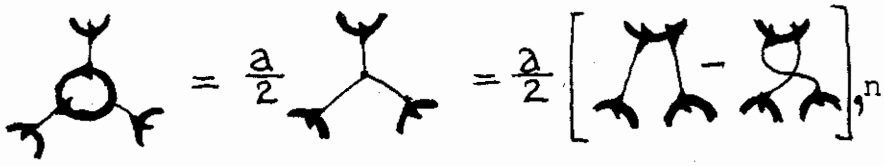
(10.21)

and the fact that for $E_6(27)$ $d_{ijk} = 0$ by (10.20) implies that



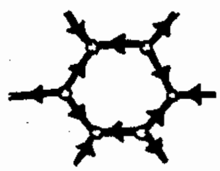
$n = 27$ (10.22)

Lie algebra (3.17) now gives a relation between projectors



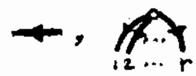
, n = 27 (10.23)

Substituting (10.14) we get on the left hand side a term of form



but the right hand side consists only of trees without loops. Hence (10.23) reduces loops of length six.

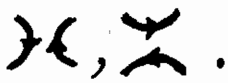
11. δ^a_b, $d_{abc\ldots d}$ INVARIANCE → NO REALIZATION

(i) primitives  $r > 3$. (11.1)

By primitiveness assumption

 $= \alpha$  (11.2)

 $= \frac{2\alpha}{n+1}$  , etc. (11.3)

(ii) projector bases:  . (11.4)

so the projector is given by (5.3).

(iii) invariance condition

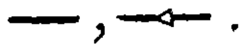
$0 =$  $+ b$  (11.5)

Contracting from the top get $b = -\frac{1}{n}$. Symmetrizing all out legs get (ignoring the trivial $n = 1$ solution)

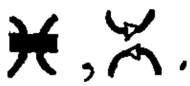
$0 =$  (11.6)

Contracting with δ^a_b get $0 = n+r$, with no acceptable solution. Hence there do not exist non-trivial Lie algebras which preserve the primitives set (11.1).

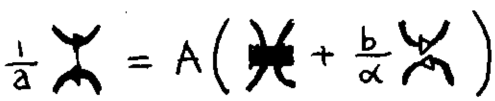
12. δ_{ab}, f_{ab} INVARIANCE → SO(2)

(i) primitives  (12.1)

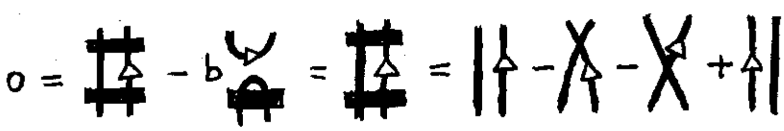
Here we have replaced δ^a_b, d_{ab} primitives by δ_{ab}, as in Sec.9, and the invariance algebra has to be a subalgebra of SO (n). Furthermore, f_{ab} is normalized as in Sec.6, and the invariance algebra must also be a subalgebra of Sp(n), n even.

(ii) projector bases:  (12.2)

(they have to be antisymmetric by (9.12)). The projector is of form

 (12.3)

(iii) invariance condition

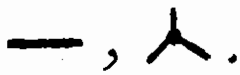
 (12.4)

Antisymmetrizing in three lets get

 (12.5)

This is just (8.6) for $r = 2$ case, and as before, one obtains (6.5) (that the lines are now not directed does not affect the derivation). Hence the second term in (12.3) is reducible, and the algebra is SO(2) of Sec. 9, with dimension $N = 1$, and index $\ell^{-1} = 0$.

13. δ_{ab}, f_{abc} INVARIANCE → SO(3), $G_2(7)$

(i) <u>primitives</u>  (13.1)

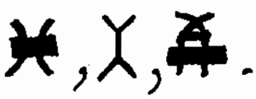
By primitiveness assumption

 $= \alpha$ $, \alpha > 0$, (13.2)

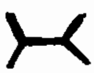
 $= \beta$  . (13.3)

(usually $\alpha=1$). As there is no distinction between out-legs and in-legs, f_{abc} satisfies, beyond relations (7.3) to (7.5), an additional symmetry relation

 $= 0$ (13.4)

(ii) <u>projector bases:</u>  (13.5)

(they have to be antisymmetric by (9.12)). There are three possibilities:

Case 1.  $= A$  (13.6)

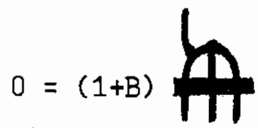
As in (7.7), this means that f_{abc} is proportional to Levi-Civita tensor in three dimensions, $A \doteq 1$, $n = 3$. The projector is of form

$\frac{1}{a}$  $= A$  , $n = 3$ (13.7)

Case 2.

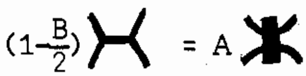
 (13.8)

Antisymmetrizing in three legs get

 (13.9)

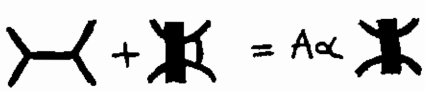
This leads to two possibilities

Case 2.1 f_{abc} obey Jacobi relation(3.18). Substituting Jacobi relation back into (13.8) we have

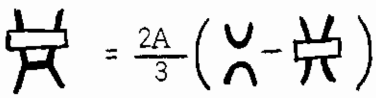
 (13.10)

Hence, if we assume any relation of form (13.8) beyond Jacobi relation ($B = 2$, $A = 0$), this case reduces to Case 1. It is quite clear that the primitives set (13.1), where f_{abc} obey Jacobi identity (i.e., f_{abc} are structure constants) has only one realization, $SO(3) = B_1(3) = A_1(3)$, but I have not been able to prove this from the primitiveness assumption.

Case 2.2 f_{abc} do not obey Jacobi relation: $B = -1$, and (13.7) becomes

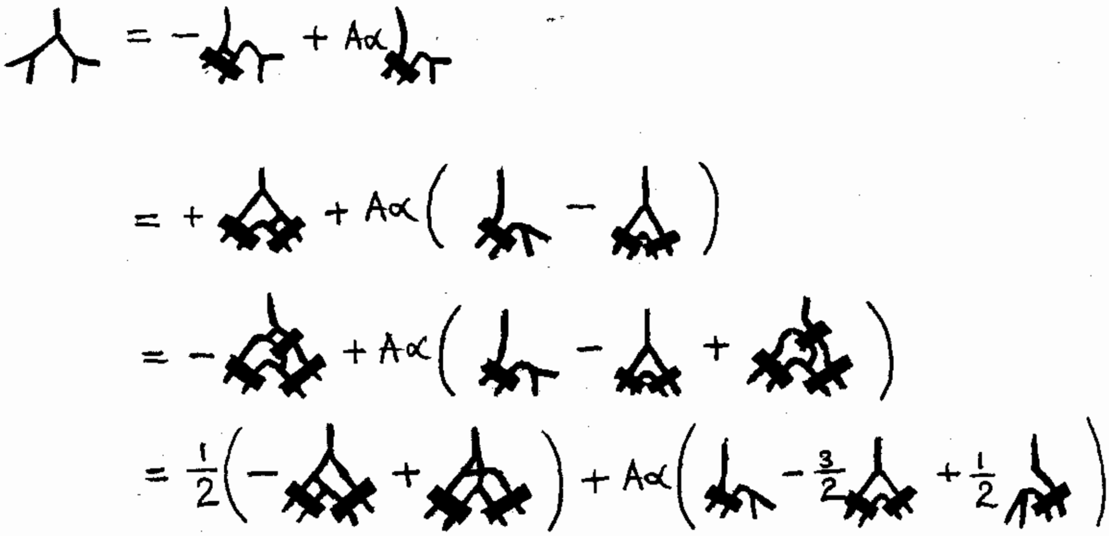
 (13.11)

By symmetrizing the two top lines, this can be rewritten as

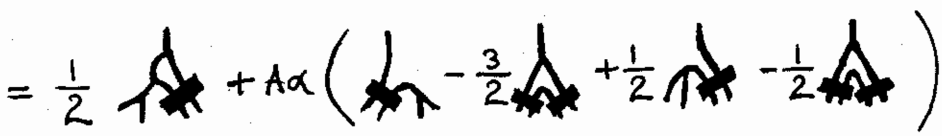


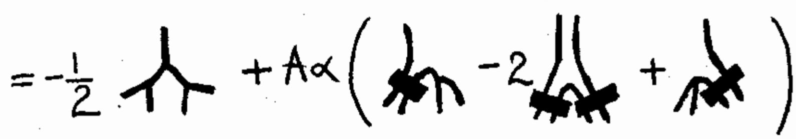
alternativity relation (13.12)

Alternativity relation is equivalent to a reduction relation for three-chains derivable from it



(second term vanishes by (7.4))

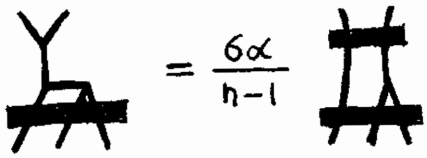




(13.13)

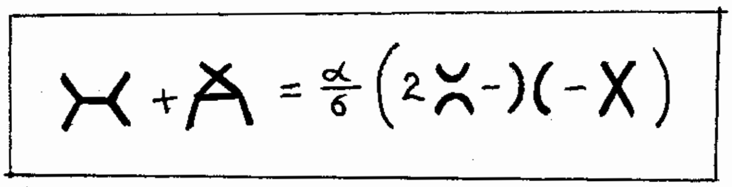
Contracting (13.12) with δ_{ab} we get $A = \frac{3}{n-1}$.

Antisymmetrizing (13.13) we obtain



(13.14)

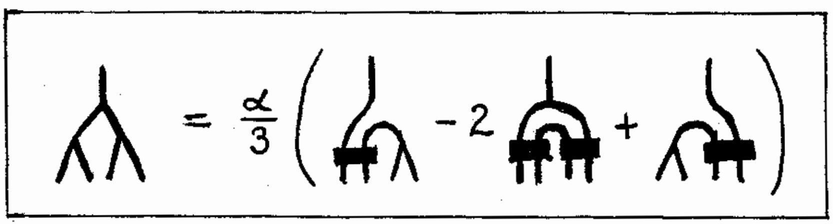
which, contracted with  yields n = 7. To summarize, if f_{abc} do not obey the Jacobi relation, a relation of form (13.8) is realized <u>only in n = 7 dimensions,</u> and f_{abc} satisfy



alternativity
n = 7

(13.15)

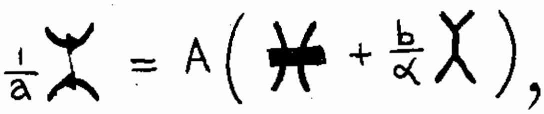
which implies the <u>reduction identity</u>



n = 7

(13.16)

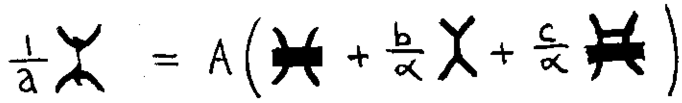
The projector is of form



n = 7

(13.17)

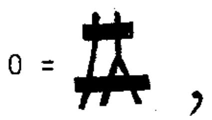
Case 3. Bases (13.5) are assumed independent:



(13.18)

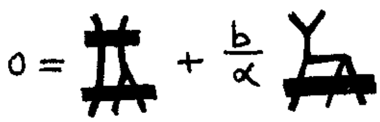
(iii) invariance condition

Case 1.

 $n = 3$ (13.19)

This is equivalent to (7.8), so f_{abc} is proportional to 3-dimensional Levi-Civita tensor.

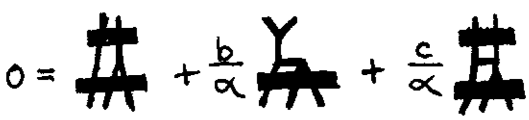
Case 2.2



(13.20)

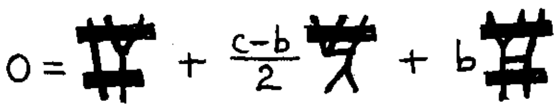
Contracting with  we get $b = -1$.

Case 3.



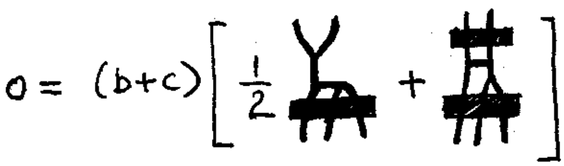
(13.21)

We assume that the Jacobi relation does not hold, otherwise this reduces to the Case 1. Now resymmetrize (13.21) with  and obtain, using (13.4)



(13.22)

Adding (13.21) yields

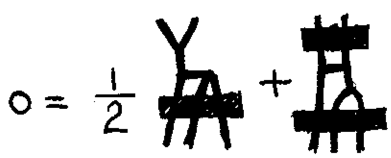


(13.23)

This leads to two possibilities.

Case 3.1 b = − c. Substituting this into (13.21) and (13.22) and subtracting yields (13.19), hence this Case reduces to Case 1.

Case 3.2

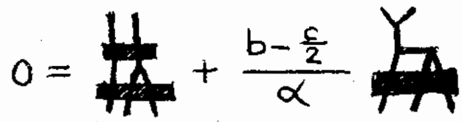


(13.24)

Compute β from (13.3) by contracting with 

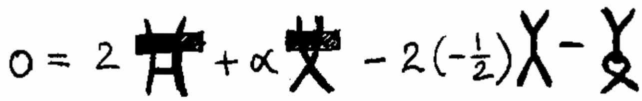
$$\beta = -\tfrac{1}{2} \tag{13.25}$$

Substituting (13.24) back into (13.21) get



(13.26)

and, as for (13.20), $b - \frac{c}{2} = -1$. Contracting with  get



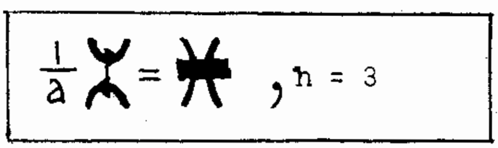
(13.27)

which is just (13.11). Hence the invariance condition reduces Case 3 to either Case 1 or Case 2.

(iv) <u>normalization</u>

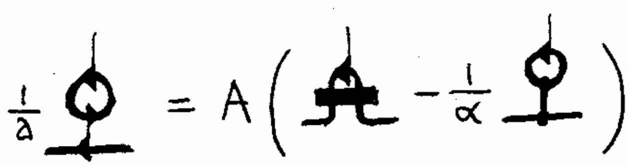
Case 1.  $A = 1$ (13.28)

The projector is



$SO(3) = B_1(3)$ (13.29)

Case 2.2.



(13.30)

Noting that  = 0, get A = 1, so

$\frac{1}{a}$  =  $- \frac{1}{\alpha}$ , $n=7$ $\qquad G_2(7) \qquad$ (13.31)

(v) algebra dimension

Case 1. $\quad N = 3$ (13.32)

Case 2.2 $\quad N =$  $- \frac{1}{\alpha}$  $= 14$ (13.33)

(vi) index

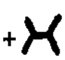
Case 1. $\quad \ell = 1$ (13.34)

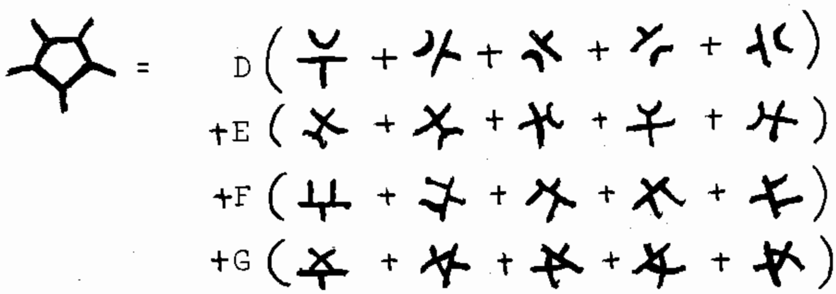
Case 2.2 $\quad \ell^{-1} = 4$ (13.35)

Comments. As will be shown in Sec. 19. the proof that the relation (13.8) has only two non-trivial realizations, three dimensional (13.6) or seven dimensional (13.15), is a simple proof of Hurwitz's theorem. I have in addition almost proven that the primitives (13.1) have only two invariance algebras, SO(3) and $G_2(7)$; I have only failed to prove from primitiveness assumption that SO(3) is the only solution compatible with Jacobi relation. Let me illustrate how far primitiveness assumption takes me - this might shed some light on the shortcomings of my approach in cases of other exceptional Lie algebras. Primitiveness assumption means that all loops have to be expressible as sums over trees, i.e.

 = (α= 1 normalization)

 = $\frac{1}{2}$  (by Jacobi relation) (13.36)

 = A{  +  } + B  + C{  +  } (13.37)

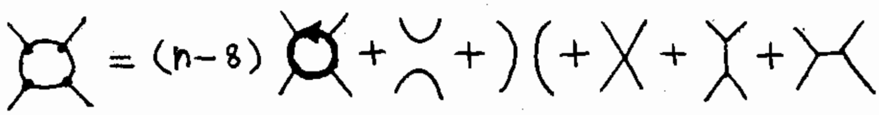


(13.38)

where I have exploited rotational and flip symmetries of the four- and five-loops. Various contractions and Jacobi relations yield

$$A = B = \frac{5}{6(n+2)}\ ,\ C = \frac{1}{6}\ ,\ D = E = \frac{5}{12(n+2)}\ ,\ F = \frac{1}{20}\ ,\ G = \frac{1}{60}\ ,$$

but I have not been able to find a further condition that would fix $n = 3$. However, we know that there is only one algebra of Cartan rank 1; $B_1 = A_1$, and no other algebra has expansions of form (13.37), (13.38). For example, using (9.13) we can compute for the adjoint representation of SO(n)



(13.39)

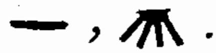
This is compatible with (13.37) only for $n = 3$.In general the adjoint representation of any algebra has higher primitive invariants, such as $d_{ijk\ell}$

$d_{ijk\ell}$ =

(13.40)

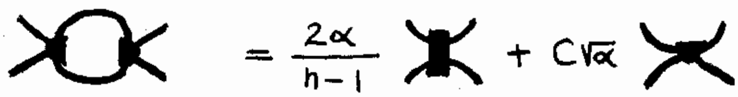
For sufficiently high rank these are reducible by the characteristic equation for [nxn] matrices (see Sec. 6 of Ref. 6).

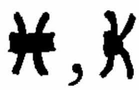
14. δ_{ab}, f_{abcd} INVARIANCE $\rightarrow D_2(6) = A_1(3) + A_1(3)+A_1(3)$, $G_2(7)$

(i) primitives  (14.1)

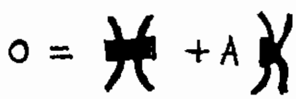
By primitiveness assumption

 (14.2)

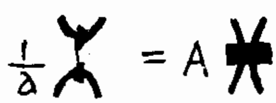
 (14.3)

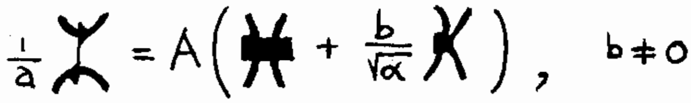
(ii) projector bases  (14.4)

They are independent, as otherwise

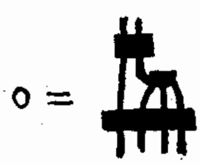
 (14.5)

antisymmetrized in all legs gives  = 0. We shall consider two possibilities for the projector:

Case 1.  (14.6)

Case 2.  (14.7)

(iii) invariance condition

Case 1.  (14.8)

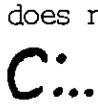
Antisymmetrizing in 4 legs we obtain (8.6), hence f_{abcd} is proportional to Levi-Civita and $n = 4$.

Case 2. $0 =$  $+ \frac{b}{\sqrt{\alpha}}$  (14.9)

Two possibilities arise:

Case 2.1 if (14.8) holds both tensors vanish, leaving b undetermined $n = 4$ in this case.

Case 2.2 Assume (14.8) does not hold.

Contracting (14.9) with  and  we obtain:

$0 = \frac{3b}{4\sqrt{\alpha}}$  $+ \frac{n-4}{8}$  (14.10)

hence $bC = \frac{n-4}{6}$. (14.11)

$0 =$  $+ \frac{2b\sqrt{\alpha}}{n-1}$  $+ bC$  (14.12)

Comparing with (14.9) we have

$$b = \frac{(n-1)\ (1+bC)}{2b} \tag{14.13}$$

hence

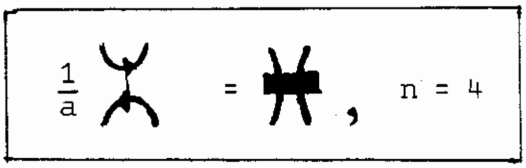
$$b_{\pm} = \pm\sqrt{\frac{(10-n)\ (n-1)}{12}} \tag{14.14}$$

$$C = \pm \frac{4-n}{\sqrt{3(10-n)\ (n-1)}} \ . \qquad (14.15)$$

$4 \leqslant n$, because no non-trivial f_{abcd} exists in less than four dimensions. As P^{ab}_{cd} is hermitian, b is real and $n \leqslant 10$. $n = 10$ is excluded, because by hypothesis $b \neq 0$.

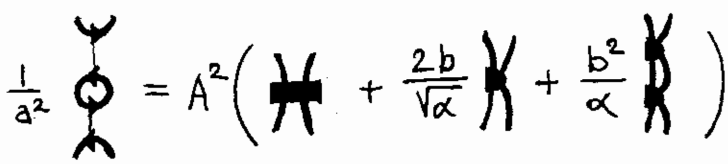
(iv) normalization

Case 1 $\quad A = 1$, and the projector is



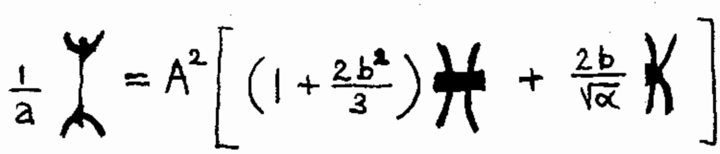
$SO(4) = D_2(4)$ (14.16)

Case 2.1 $\quad n = 4,\ b \neq 0.$

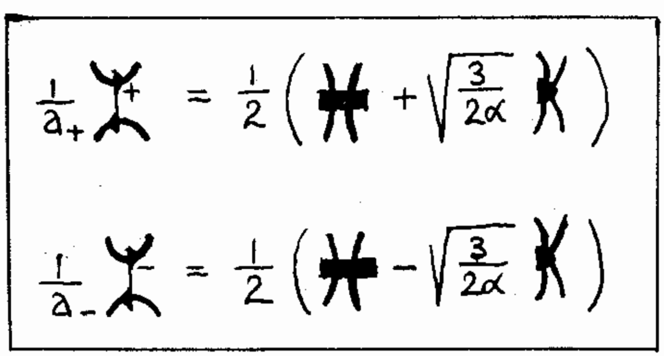


(14.17)

Reduce the third term using (8.7);



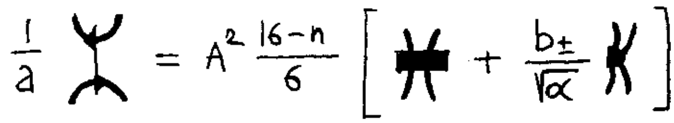
There are two solutions: $A = \frac{1}{2}$, $b = \pm\sqrt{\frac{3}{2}}$, with projectors



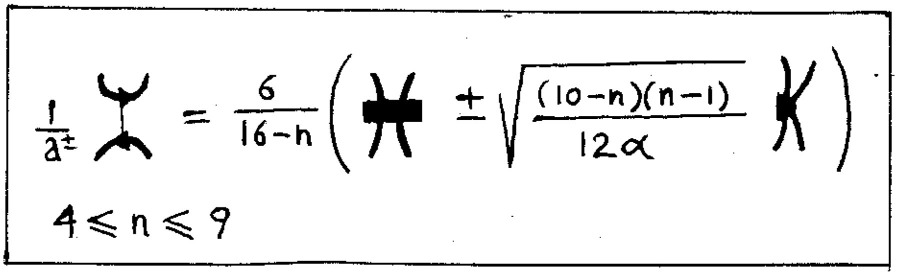
(14.18)

(14.19)

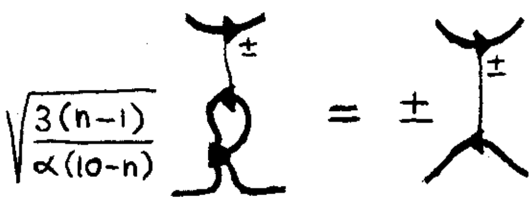
Case 2.2 Reduce the third term in (14.17) using (14.3);


(14.20)

Hence the projectors are


(14.21)

$P^{(+)}, P^{(-)}$ are "dual" and "anti-dual" projectors with respect to f_{abcd} in the sense that


(14.22)

(v) algebra dimension

Case 1	$N = 6$	(14.23)
Case 2.1	$N_+ = N_- = 3$	(14.24)
Case 2.2	$N_+ = N_- = \frac{3n(n-1)}{16-n} = -3n-45 + \frac{2^4 . 3^2 . 5}{16-n}$	(14.25)

(vi) index

Case 1	$\ell^{-1} = 2$	(14.26)
Case 2.1	$\ell_+^{-1} = \ell_-^{-1} = 2$	(14.27)

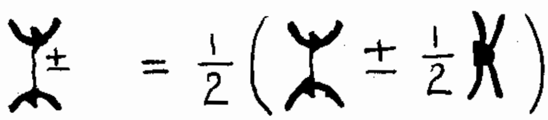
Case 2.2  $= \frac{6}{16-n}\left[\frac{N}{2} + \frac{b_\pm}{\sqrt{\alpha}}\right.$  $\left.\right]$

$$= \frac{6}{16-n}\left[\frac{N}{2} - \frac{6nb^2}{16-n}\right]$$

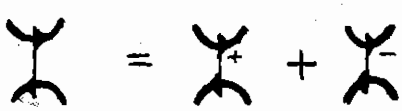
$$\ell_+^{-1} = \ell_-^{-1} = \frac{4(n+2)}{16-n} = -4 + \frac{2^3 . 3^2}{16-n} \qquad (14.28)$$

Comments

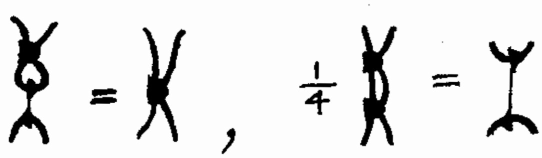
Case 2.1 The two projectors (14.19), (14.20) are an explicit decomposition of the adjoint representation of the semisimple algebra $D_2(6)$ of Case 1 into $A_1(3) + A_1(3)$. We list various properties (setting for simplicity $a = a_+ = a_- = 1$ and using the Levi-Civita (8.7) normalization $\alpha = 6$)



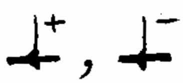
(14.29)



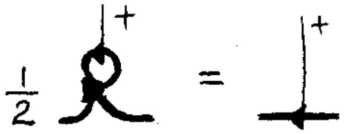
(14.30)



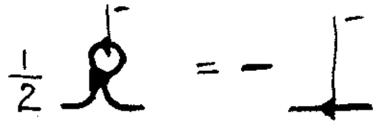
(14.31)

Using  as a shorthand for the corresponding projectors we have

(14.32)



self-dual projection (14.33)



anti-self-dual projection (14.34)

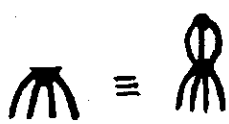
Case 2.2 All solutions of the Diophantine equations (14.25) and (14.28) are

n	4	6	7	8
N	3	9	14	21
ℓ^{-1}	2	/	4	10
	$A_1(2)$	/	$G_2(7)$	$B_3(8)$

(14.35)

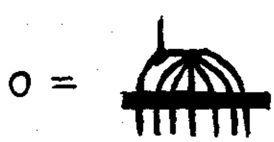
The first column(Case 2.1) corresponds to SU(2), $n = 4$ because complex 2-dimensional vectors are here represented by real 4-dimensional vectors.

$G_2(7)$ appears in this series because the primitive f_{abc} of Sec. 13 can be replaced by its dual f^*_{abcd}



, $n = 7$, (14.36)

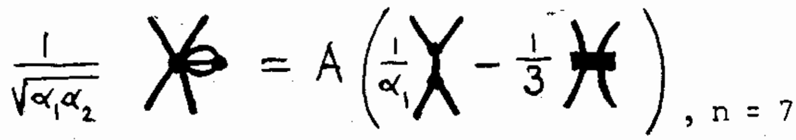
where  is defined by (13.16) and  is the seven dimensional Levi-Civita tensor $f_{abcdefg}$) preserved by $G_2(7)$ because



(14.37)

(this can be obtained by expanding trivial identities like

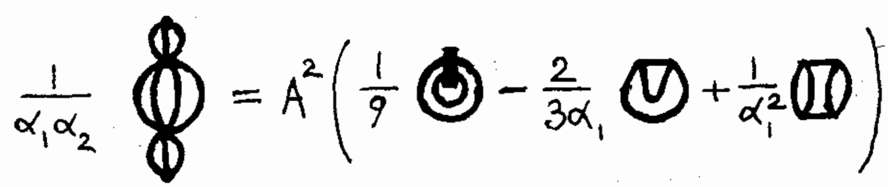
From primitiveness assumption for Sec. 13 primitives



(14.38)

where

Squaring both sides we have

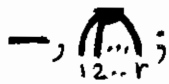


(14.39)

Using (8.7) we obtain $A = \pm\sqrt{\frac{3}{70}}$, $\alpha = \frac{\alpha_1\alpha_2}{35}$. It can be easily checked that (14.38) construction of f_{abcd} is consistent with all above relations for Case 2.1 and that substituted in (14.21) it reduces the projector to (13.31).

I do not know how to eliminate n = 6 solution of (14.35), and have not checked whether n = 8 really corresponds to $B_3(8)$.

14A. δ_{ab}, $f_{abc\ldots d}$ INVARIANCE $\rightarrow$ SO(n)

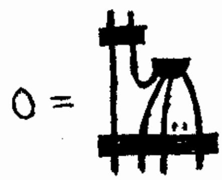
(i) primitives  ; $r > 4$ (14A.1)

As in Sec. 8, various loop contractions of $f_{ab\ldots d}$ must be reducible.

(ii) projector basis:  (14A.2)

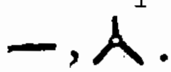
The projector is simply SO(n) projector (9.13).

(iii) invariance condition

 (14A.3)

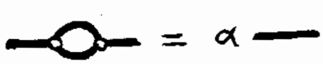
Antisymmetrizing in r legs we get (8.6), the defining equation for Levi-Civita in n = r dimensions. Hence the invariance condition forces $f_{abc..d}$ to be proportional to Levi-Civita tensor, and the invariance algebra is SO(n) algebra of Sec.9; the volume (defined by $\varepsilon_{ab\ldots d}$) is automatically preserved if the length (defined by δ_{ab}) is preserved.

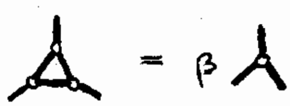
15. δ_{ab}, d_{abc} INVARIANCE $\rightarrow$ $B_1(5)$, $A_2(8)$, $C_3(14)$, $F_4(26)$

(i) primitives  (15.1)

By primitiveness assumption

 (15.2)

 , $\alpha > 0$ (15.3)

 (15.4)

(usually $\alpha = 1$)

(iii) <u>projector bases</u>  (15.5)

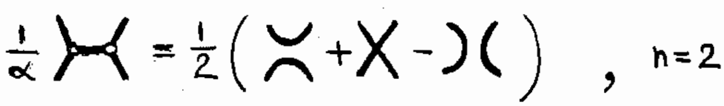
There are three possibilities:

Case 1.

(15.6)

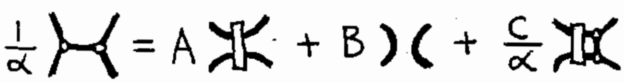
Contracting with  get A = 1. Contracting with  and  get $0 = 1 + Bn$ and $1 = \frac{n+1}{2} + B$, with solutions $n = -1,2$.

For n = 2 (15.6) becomes

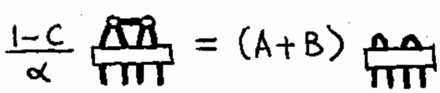
 (15.7)

The projector is of form (9.13), with n = 2.

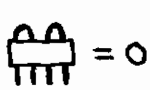
Case 2

 (15.8)

Symmetrization in all legs yields

 (15.9)

Neither of the tensors can vanish, because

 (15.10)

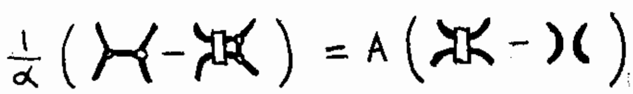
leads upon contraction with  to $n + 2 = 0$, and

 $= 0$

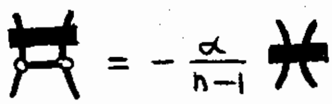
leads upon contraction with  to $2\alpha = 0$, also unacceptable.

The remaining possibilities are

Case 2.1 $\quad 1 - C = 0,\ A + B = 0$

 (15.11)

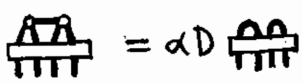
Contracting with  get $1 = A(1-n)$. Antisymmetrizing from the top obtain

 (15.12)

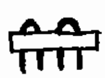
The projector is of form (9.12).

Case 2.2. $\quad 1 - C \neq 0,\ A + B \neq 0$, and (15.9) can be rewritten as

(15.13)

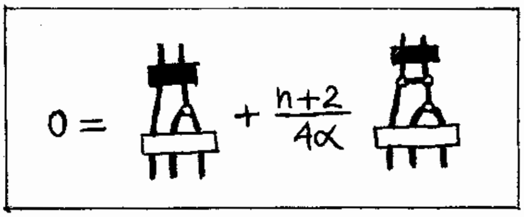


Contracting with  we obtain $2 = D(n+2)$, and in this case d_{abc} satisfy

 $= \frac{2\alpha}{n+2}$  characteristic equation (15.14)

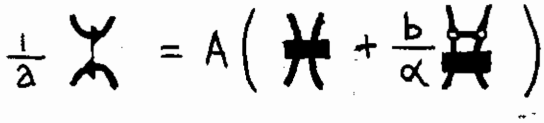
Furthermore, by contracting with 

we obtain



Jordan identity (15.15)

The projector is of form



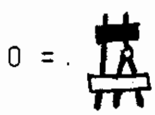
(15.16)

Case 3. Tensors 

are assumed independent. The projector is given by (15.16).

(iii) invariance condition

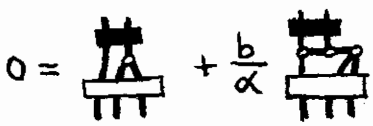
Cases 1 and 2.1



(15.17)

Contracting with  obtain 0 = n+1, no solution.

Case 2.2



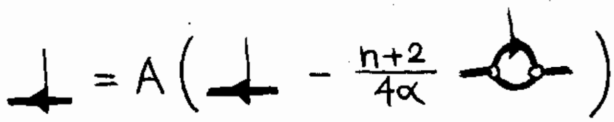
(15.18)

From (15.15) $b = \frac{n+2}{4}$

Case 3. The invariance condition is again (15.18), but lacking analogue of (15.15) I do not know how to continue. The primitiveness assumption has to be complemented by a characteristic equation of type (15.14) to fix the algebra. A nice example is given by the adjoint representation of SU(m) which has primitives (15.1) for any n (taking $n = N = m^2 - 1$ dimensional representation as <u>defining</u>), but (15.14) is satisfied only for $m = 3$. This is discussed in the Appendix B of Ref. 6.

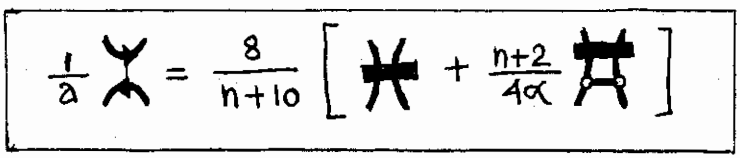
(iv) <u>normalization</u>

Case 2.2. As in (7.20), the second term in the normalization condition is reducible



(15.19)

and the projector is given by

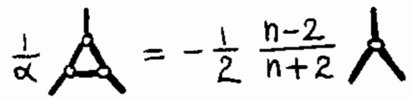


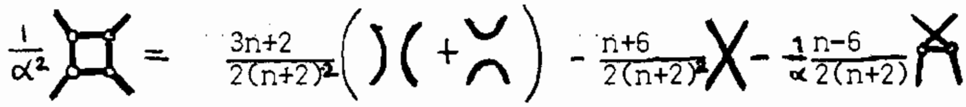
(15.20)

(v) <u>algebra dimension</u> $N = \dfrac{3n(n-2)}{n + 10}$ (15.21)

(vi) <u>index</u> $\ell^{-1} = \dfrac{5n-22}{n + 10}$ (15.22)

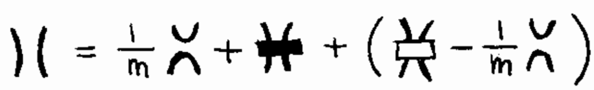
<u>Comments</u> The 14 solutions of the Diophantine equation (15.21) are given in the Table 1. The four are identifiable and belong to the Freudenthal magic square - I shall now give their explicit construction (this is not in the spirit of this paper, but I include it as it might suggest ways of eliminating the remaining 10 possible solutions). Note that (15.15), which I shall later show to be equivalent to Jordan identity (the defining identity for Jordan algebras), was a trivial consequence of the "characteristic equation" (15.14). At this point I can reduce loops of length three and four

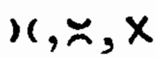
 (15.23)

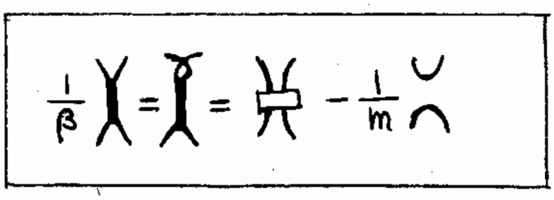
 (15.24)

but lack an algorithm for reducing loops of arbitrary length. I will now proceed to <u>construct</u> such an algorithm for the first three identifiable solutions in Table I.

<u>$B_1(5)$ algebra</u> Consider the Clebsch-Gordon series for the product of two vector representations of SO(m)

 (15.25)

The first term is the singlet (N=1), the second term is the adjoint representation (9.13) and the third term is the symmetric rank 2 tensor representation ($N = \frac{m(m+1)}{2} - 1$). (Clebsch-Gordon series amounts to replacing the three basis  by the above three orthonormal bases). Let us introduce a notation for the symmetric representation



$B_{\frac{m-1}{2}}\left(\frac{m(m-1)}{2} - 1\right)$, m odd (15.26)

$D_{\frac{m}{2}}\left(\frac{m(m-1)}{2} - 1\right)$, m even

Normalization:  (15.27)

(here $\beta = 1$). By construction this representation is traceless

 $= 0$ (15.28)

Now I take this symmetric representation to be the <u>defining representation</u> of the algebra to be constructed. The dimension of the defining representation <u>n</u> is related to the dimension of the underlying SO(m) by

$n =$  $= \frac{m(m+1)}{2} - 1$ (15.29)

Define a series of symmetric invariants d_{abc}, d_{abcd}, ... by

 (15.30)

 (15.31)

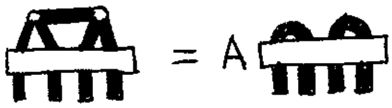
Expansion (15.26) enables us to compute <u>all</u> scalar invariants constructed from d_{abc}, d_{abcd}, ..., and perform reductions such as

 $= \left(\frac{m+2}{4} - \frac{2}{m} \right)$  (15.32)

 $= \left(\frac{m+4}{8} - \frac{3}{m} \right)$  (15.33)

 $=$  $- \frac{1}{m}$  (15.34)

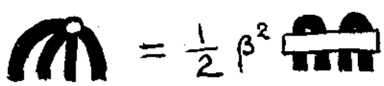
We note that (15.14) does not hold for this algebra unless d_{abcd} is reducible, and

 (15.35)

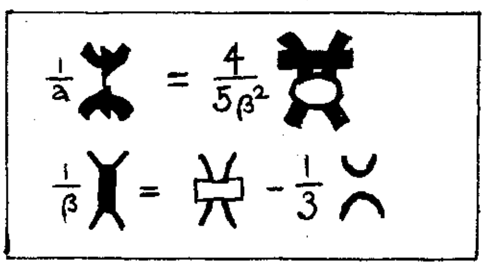
Contracting with  and  we obtain

$$m+2 - \frac{8}{m} = [m(m+1)+2]\, A$$
$$\frac{m+3}{2} - \frac{8}{m} = 2A \qquad (15.36)$$

yielding $(m-3)\ (m+6) = 0$, <u>so $m = 3$, $A = \frac{1}{6}$,</u> <u>$n = 5$ is the only SO(m) algebra</u> satisfying the characteristic equation (15.14). From (15.13) and (15.32) $\alpha = \frac{7}{12}\beta^2$. d_{abcd} is reduced by

 , $m=3$ (15.37)

Substituting (15.26) into (15.20) we obtain for the projector

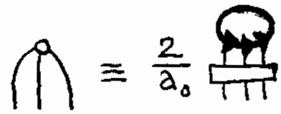


$B_1(5)$ (15.38)

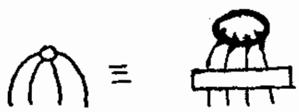
$,m=3$ (15.39)

with normalization (15.27). It is easily checked that (15.38) is a projection operator, and that N=3. Above two relations give a reduction algorithm that enables us to compute scalar invariants constructed from any number of d_{abc} (15.30).

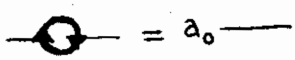
<u>$A_2(8)$ algebra</u> By analogy with (15.30) define fully symmetric invariants for the adjoint representation of SU(m) ($=A_{m-1}(n)$, $n=N=m^2-1$, the <u>defining representation</u> for the algebra presently considered):

 (15.40)

(normalization of the Appendix B of Ref.

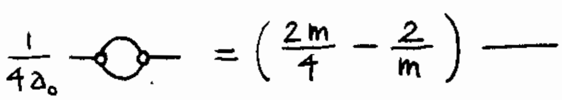
 , etc... (15.41)

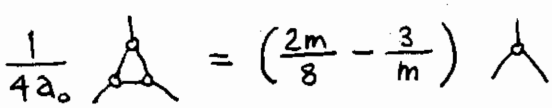
The projector (5.6), normalized by

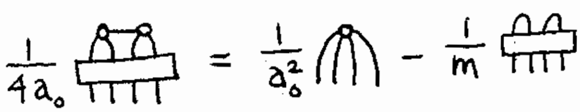


(15.42)

gives following reductions:

 (15.43)

 (15.44)

 (15.45)

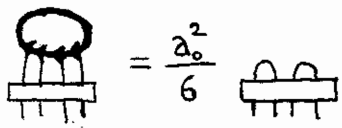
As before, we look for the solution of (15.35) and obtain

$$2m - \frac{8}{m} = [2m^2 + 2]\,\frac{A}{4} \qquad (15.46)$$

$$m - \frac{8}{m} = 2\,\frac{A}{4}$$

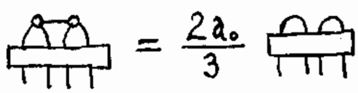
yielding $(m-3)\,(m+3) = 0$, so $\underline{m = 3,\ A = \frac{2}{3},\ n = 8 \text{ is the only solution}}$

satisfying the characteristic equation (15.14). By (15.45) $d_{ijk\ell}$ is reducible



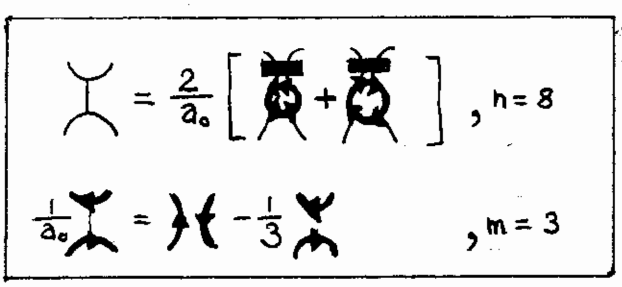
(15.47)

and (15.35) becomes



(15.48)

It was shown in Ref. 6, Fig.22c that the first relation is the characteristic equation for traceless [3x3] Hermitian matrices, and that (15.48) is the SU(3) relation of Macfarlane *et al*.[28] By (15.3) and (15.43) the normalizations are related by $\alpha = \frac{10}{3}$ a. Noting further that $a_0/a = \ell$, the index, we have from (5.11) $a = 6a_0$. Substituting (5.6) into (15.20) we obtain for the projection operator

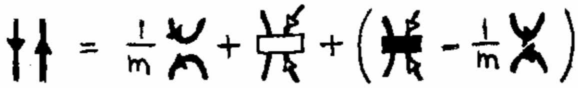


$A_2(8)$ (15.49)

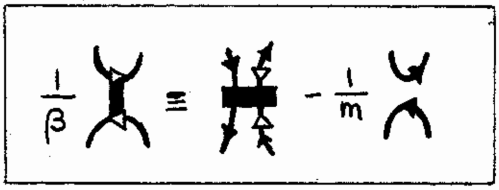
(15.50)

with normalization (15.42). The first relation holds for the adjoint representation of *any* semisimple Lie algebra (3.21).

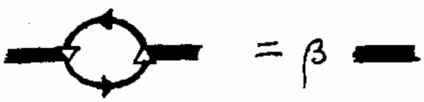
$C_3(14_s)$ algebra. C_3 has two 14 dimensional representations. The symmetric one we are interested in is distinguished by the index $\ell^{-1} = 2$, and appears in the Clebsch-Gordon series for $\bar{V} \otimes V$, where V is the m-dimensional defining space for $Sp(m) = C_{\frac{m}{2}}$ (m) representation of $C_{\frac{m}{2}}$:

 (15.51)

The first term is the singlet (N=1), the second term is the adjoint representation (6.14) and the third term is the symmetric rank 2 tensor representation ($N= \frac{n(n-1)}{2} - 1$). Let us introduce notation

 $\frac{Cm}{2}\left(\frac{m(m-1)}{2}-1\right)_s$ (15.52)

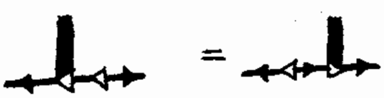
Normalization:

 (15.53)

(I shall usually set $\beta=1$). By construction this representation is traceless

 $= 0$ (15.54)

and symmetric in the sense that it satisfies

 (15.55)

n, the dimension of this <u>defining</u> representation is related to the underlying Sp(m) by

 (15.56)

Define a series of symmetric invariants d_{abc}, d_{abcd}, ... by

(15.57)

by (15.55)

(15.58)

Expansion (15.52) enables us to perform the following reductions

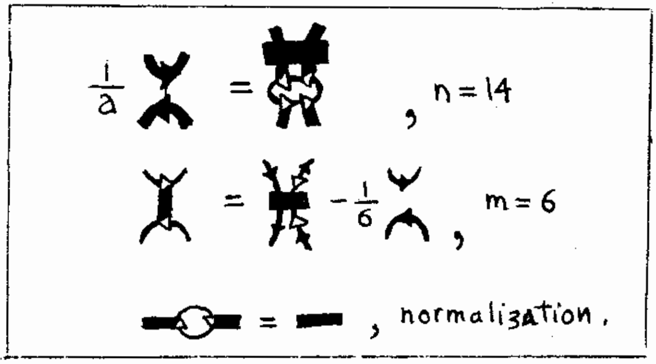
 $= \left(\frac{m-2}{4} - \frac{2}{m} \right)$  (15.59)

 $= \left(\frac{m-4}{8} - \frac{3}{m} \right)$  (15.60)

and again obtain (15.34). As before, we look for the solution of (15.35) and obtain

$$
\begin{aligned}
m - 2 - \frac{8}{m} &= [m(m-1) + 2]\, A \\
\frac{m-3}{2} - \frac{8}{m} &= 2A
\end{aligned}
\qquad (15.61)
$$

yielding $(m+3)(m-6) = 0$, so $\underline{m = 6,\ A = \frac{1}{12}}$, $\underline{n = 14}$ is the only solution. From (15.3) and (15.59) $\alpha = -\frac{2}{3}$, and substituting (15.52) into (15.20) we obtain for the projector



$C_3(14_s)$ (15.62)

(15.63)

(15.64)

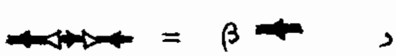
It is easily checked that (15.62) is a projection operator, and that $N = 21$, as it should be for a C_3 algebra.

$F_4(26)$ algebra The above sequence of SO(3), SU(3), Sp(6) reflects the connection of the rows of the magic square with real, complex, quaternion and octonion normed algebras. I shall later show how in Jordan algebra analogues of (15.53) and (15.57) appear, but my diagrammatic notation is not suited to the underlying non-associative algebra. Furthermore, because of non-associativity, there are no reduction expansions analogous to (15.52), and one has to rely on identities like (15.14) to obtain reduction formulas of type (15.23) and (15.24). I do not have a general reduction algorithm for $F_4(26)$.

16. δ^a_b, f_{ab}, d_{abcd} INVARIANCE → $E_7(56)$, ...

(i) primitives  ; n even. (16.1)

By primitiveness assumption

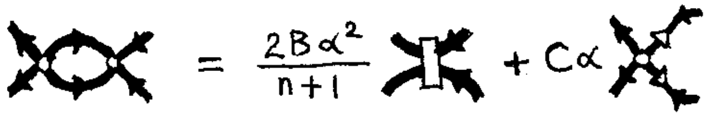
 , $\beta > 0$ (16.2)

(here $\beta = 1$)

 (16.3)

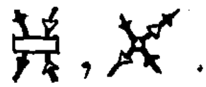
(usually $\alpha = 1$)

(16.4)

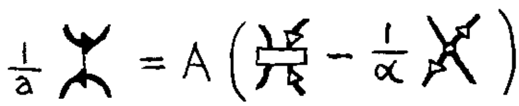


(16.5)

(B and C will be fixed by the invariance condition).

(ii) projector bases  (16.6)

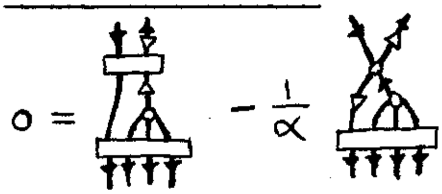
As this algebra must be a subalgebra of the δ_b^a , f_{ab} invariance algebra Sp(n), I have already incorporated the invariance condition (6.12) into bases (16.6). The projector is of form



(16.7)

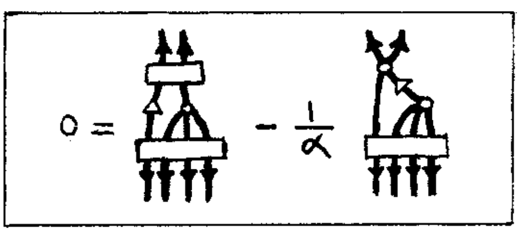
Purely for convenience I have chosen this as the normalization convention for α, rather than (16.3).

(iii) invariance condition



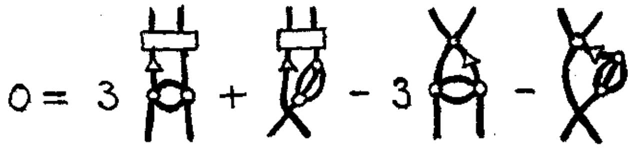
(16.8)

Contracting with  this can be rewritten as



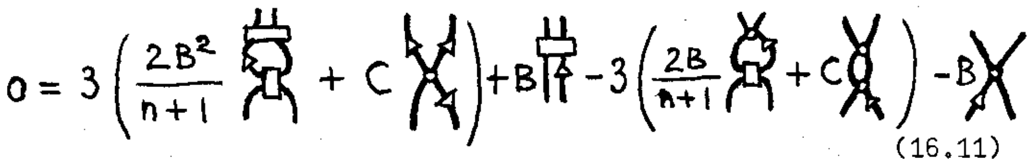
Brown's relation (16.9)

B and C in (16.3) and (16.5) can be computed by contracting Brown's relation with $\ddot{\psi}$



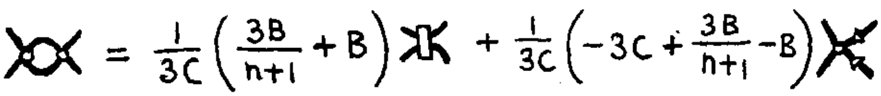
(16.10)

Substituting (16.3) and (16.5)



(16.11)

This gives a new relationship for (16.5)

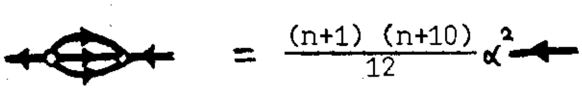


(16.12)

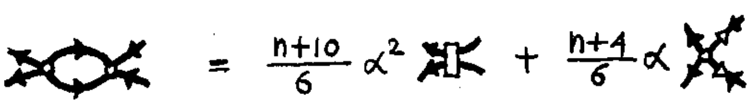
The tensors on the right hand side have different symmetries and are independent. Equating the coefficients in (16.5) and (16.12) yields

$$B = \frac{(n+1)\ (n+10)}{12} \quad , \; C = \frac{n+4}{6} \tag{16.13}$$

Hence invariance condition has brought (16.3) and (16.5) to form



(16.14)



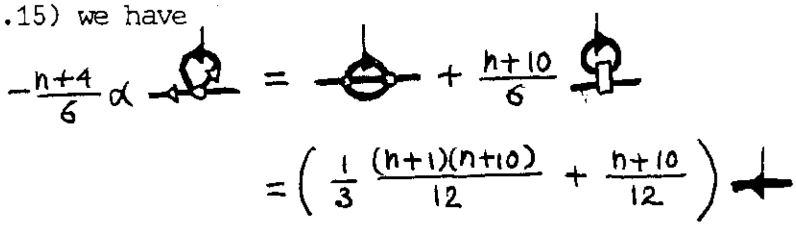
(16.15)

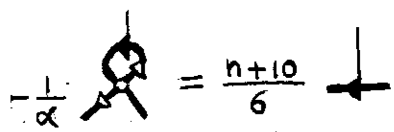
with the normalization α defined by (16.7).

(iv) <u>normalization</u>

(16.17)

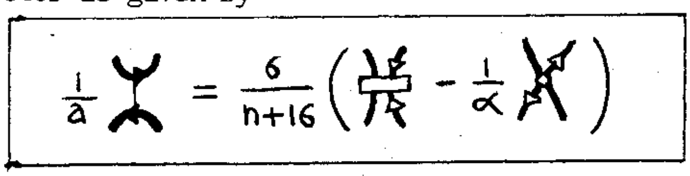
From (16.15) we have





(16.18)

so that the second term in (16.17) is reducible, $A = \frac{6}{n+16}$, and the projector is given by



(16.19)

(v) <u>algebra dimension</u> $N = \frac{3n(n+1)}{n+16}$ (16.20)

(vi) <u>index</u> $\ell^{-1} = \frac{4(n-2)}{n+16}$ (16.21)

<u>Comments</u> Solutions of Diophantine equation (16.20) are summarized in Table I. The identifiable ones include the entire E_7 row of the magic square. Invariance condition for d_{abcd} is simply the Brown's relation, and combined with the primitiveness assumption it yields the dimensional constraints. As for E_6 and F_4, <u>I do not have an algorithm for reduction of arbitrary scalar invariants of E_7.</u>

17. MAGIC SQUARE

The preceding sections have illustrated how one constructs the invariance algebra for a given set of primitives. This effort was only partially successful for higher rank symmetric primitives; we did not derive general reduction algorithms, but we did derive Diophantine equations relating the representation dimension n and the algebra dimension N. They were

$$
\begin{aligned}
&F_4 \text{ series} \qquad && N_1(n) = 3n - 36 + \frac{360}{n+10} \\
&E_6 \text{ series} \qquad && N_2(n) = 4n - 40 + \frac{360}{n+9} \\
&E_7 \text{ series} \qquad && N_3(m) = 6m - 45 + \frac{360}{m+8}, \quad n = 2m
\end{aligned}
\tag{17.1}
$$

They obey a suggestive recursion relation

$$N_2(k) = k-1 + N_1(k-1)$$

$$N_3(k) = 2k-1 + N_2(k-1)$$

which yields as a next sequence

$$
\begin{aligned}
E_8 \text{ series} \qquad N_4(k) &= 4k-1 + N_3(k-1) \\
&= 10k - 52 + \frac{360}{k+7}
\end{aligned}
\tag{17.2}
$$

I have not been able to <u>derive</u> this series from invariance analysis because I do not know the primitive invariants of E_8, but I include it here as study of the other identifiable algebras in this series might lead to $E_8(248)$ primitives. The integer solutions of (17.1) and (17.2) can all be simultaneously parametrized by integer m:

$$
\begin{aligned}
N_1 &= 3m-66+360/m, & n_1 &= m-10 \\
N_2 &= 4m-76+360/m, & n_2 &= m-9 \\
N_3 &= 6m-93+360/m, & n_3 &= 2(m-8) \\
N_4 &= 10m-122+360/m, & n_4 &= N_4
\end{aligned}
\tag{17.3}
$$

N is integer if m factors into any combination of $2^3.3^2.5 = 360$. In the same parametrization indices are given by

$$\begin{aligned} \ell_1^{-1} &= 5 - \frac{72}{m} \\ \ell_2^{-1} &= 6 - \frac{72}{m} \\ \ell_3^{-1} &= 4 - \frac{36}{m} \end{aligned} \tag{17.4}$$

while $\ell_4^{-1} = 1$ by definition, because the defining representation is also the adjoint representation. The number of integer solutions is finite and restricted by $m \leqslant 360$, $N \geqslant 1, n \geqslant 1$ and $\ell^{-1} \geqslant 0$. For E_6 condition (10.20) restricts n to $n \leqslant 27$; I have not found analogous conditions for other series of solutions.

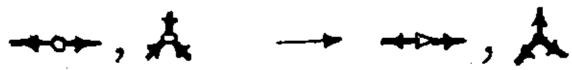
In Table I. I list all the solutions to (17.3) and (17.4), and indicate the ones which can be identified in Cartan's classification. It is quite remarkable that our invariance analysis, which at no time invoked Jordan algebras, yields the complete Freudenthal's magic square[20], as well as Faulkner and Ferrar's extended magic square[7]. Particularly amusing is the series (17.2) which contains all exceptional Lie algebras as well as D_4, which can also be considered exceptional[7].

The extra solutions to the left of the Freudenthal's magic square suggest its extension to a "magic triangle".

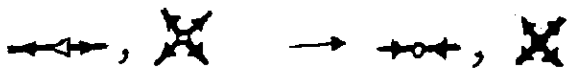
(17.5)

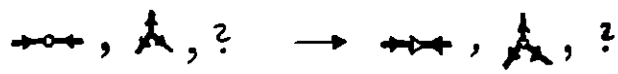
$N_1(-n)$								3	
$N_2(-n)$							1	8	
$N_3(-\frac{n}{2})$							3	14	21
?						2	9	28	
$N_1(n)$					3	8	21	52	
$N_2(n)$				2	8	16	35	78	
$N_3(\frac{n}{2})$		1	3	9	21	35	66	133	
N	3	8	14	28	52	78	133	248	

One can make a guess at their origin by noting the similarity between Secs. 14 and 16; the relations of one section can be obtained from the relations of the other by turning all symmetric objects into antisymmetric objects and vice-versa. We hypothesize that the top four rows of the magic triangle are obtained from the bottom four rows by

F_4 series  (17.6)

E_6 series  (17.7)

E_7 series  (17.8)

E_8 series  (17.9)

and that

$$N(n), \quad \ell(n) \rightarrow N(-n), - \ell(-n). \tag{17.10}$$

All the solutions for (17.8) were already given in (14.35). For (17.6) all the solutions are (here we exclude $N > n^2$ because we are looking at subalgebras of $U(n)$, and $N = n^2 = 16$ because the index is not correct for $U(4)$)

n	1	2	5
N	1	3	21
ℓ^{-1}	3	4	-
		$A_1(2)$	

(17.11)

This set of primitives conflicts with Sec. 6 unless $f_{abc} = 0$ trivially. For (17.7) all the solutions are

n	1	3
N	1	8
ℓ^{-1}	3	6
		$A_2(3)$

(17.12)

In Sec.7 we have shown that $A_2(3)$ is the unique non-trivial solution for this set of primitives. We do not know the primitives for E_8, but we can still guess that $N_4(k) \to N_4(-k)$. That yields four solutions

N	8	28	78	248
	A_2	D_4	E_6	E_8

(17.13)

which are suggestive insofar that D_4 appears, but they do not reproduce the fourth row of the magic triangle.

The above observations are not particularly persuasive; we include them only for their suggestive value, as in the past such numerical tables had played an important role in motivating Freudenthal-Tits constructions [20,15].

In our construction the origin of the parameter m remains mysterious; the underlying normed algebra structure which relates column entries is not explicit. It is apparent from the work of Freudenthal [20,21] that entries along columns form chains of subalgebras, but we have not verified that within our formulation of invariance algebras.

18. RELATION TO OTHER NOTATIONS

In the preceding Sections we have given a self-contained presentation of our method for constructing invariance algebras. However, as the formulation of invariance algebras in terms of invariant tensors δ^a_b, f^{abc}, ... and particularly in terms of their diagrammatic representations  might present a conceptual block to readers accustomed to other notations, we shall use the remainder of this paper to provide a translation between our and the established algebraic notations, and identify those relations which have already been given by other authors. As is customary, we misattribute most of them; Springer's relation had been derived by Freudenthal, Brown's probably first by Springer, and so forth. We chose to call them Springer's and Brown's because those authors were first to use them in the spirit of the present work, as an axiomatic starting point for constructing corresponding exceptional Lie algebras.

Tensor (or matrix) notation is quite standard for classical groups (cf. Hammermash [27] or Gilmore [29]). Let us give an example of translating diagrammatic into tensor notation by rewriting Sec.5. Label the legs by a, b, c, d in counterclockwise order, and use (3.2-5);

(i) <u>primitives</u>: δ^a_b.

(ii) <u>projector bases</u>: $\delta^a_b\,\delta^c_d$, $\delta^a_d\,\delta^c_b$. $\quad$ (5.1)

$$0 = \delta^a_b\,\delta^c_d + b\,\delta^a_d\,\delta^c_b \tag{5.2}$$

Contracting with δ^b_a and δ^b_c we get $0 = n + b$ and $0 = 1+nb$.

$$\frac{1}{a}(T_i)^a_d\,(T_i)^c_b = A\left(\delta^a_b\,\delta^c_d + b\,\delta^a_d\,\delta^c_b\right) \tag{5.3}$$

$$(T_i)_a^b = A\left[(T_i)_a^b + b\,\delta_a^b\,\mathrm{Tr}(T_i)\right] \tag{5.4}$$

$$P_{d\,b}^{a\,c} = \delta_{b\,d}^{a\,c} \;; \quad U(n) \text{ algebra} \tag{5.5}$$

(This projector is simply identity, because the algebra is entire U(n) algebra. All other projectors, like (5.6), project out subalgebras of U(n)).

$$P_{d\,b}^{a\,c} = \delta_{b\,d}^{a\,c} - \frac{1}{n}\,\delta_d^a\,\delta_b^c \;; \quad SU(n) \text{ algebra} \tag{5.6}$$

(This is sometimes called completeness relation [28]).

Case 1. $\qquad N = P_{ca}^{ac} = \delta_a^a \delta_c^c = n^2 \qquad (5.7)$

Case 2. $\qquad N = \delta_a^a \delta_c^c - \frac{1}{n}\,\delta_c^a \delta_a^c = n^2 - 1 \qquad (5.8)$

$$-\frac{1}{a}\; C_{ijk}\, C_{ji\ell} = 2n\,\delta_{k\ell} - 2\,\mathrm{Tr}\,(T_k)\,\mathrm{Tr}\,(T_\ell) \tag{5.9}$$

Literature discussions of classical groups are considerably longer, because they usually involve an explicit construction of the generators of the group. As we argue in Sec.2, this is superfluous; projectors fully define the invariance algebra.

Rewriting abstract products in tensor notation amounts to introducing a basis for the elements of the underlying algebra A

$$x = x_a e^a$$
$$\bar{x} = x^a e_a, \; a = 1,2, \ldots n \qquad (18.1)$$

$$x = x_a e_a \qquad \text{for real representations} \qquad (18.2)$$

This is simply a notational device, not a choice of a particular basis set; e_a can be any n linearly independent elements of A. In Freudenthal's notation [21] a derivation is a trilinear mapping $A \otimes \bar{A} \otimes A \to A$ denoted by

$$Dz = \langle x, \bar{y} \rangle z \qquad (18.3)$$

In tensor notation this mapping is performed by the projector (3.6). Indeed, if we substitute (18.1) in (18.3) and replace (2.9) by

$$D^b_a = a P^{bd}_{ac} \quad x_d y^c \qquad (18.4)$$

we obtain

$$\langle e^a, e_b \rangle e^d = a P^{ad}_{bc} \; e^c \qquad (18.5)$$

Algebraic notation is more compact than the tensor notation, but not any more compact than the diagrammatic notation;

$$\langle x, \bar{y} \rangle z \qquad \longleftrightarrow$$  (18.6)

In addition, the diagrammatic notation makes explicit symmetries that are not obvious algebraically, such as $D \leftrightarrow y$, $x \leftrightarrow z$ interchange symmetry in (18.6). In the following Sections we illustrate the above by rewriting the literature results on G_2, F_4, E_6 and E_7.

19. HURWITZ'S THEOREM AND $G_2(7)$

<u>Definition</u>[30]: a normed algebra A is an n+1 dimensional vector space over a field F with a product xy such that

i) $x(cy) = (cx)y = c(xy) \qquad c \in F$

ii) $x(y+z) = xy+xz \qquad x,y,z \in A$

$(x+y)z = xz+yz$,

and a nondegenerate quadratic norm which permits composition

$$\text{iii) } N(xy) = N(x)\,N(y)\ ,\quad N(x) \in F \quad . \tag{19.1}$$

Here F will be the field of real numbers. Let

$\{e_o, e_1, \ldots, e_n\}$ be a basis of A over F;

$$x = x_o e_o + x_1 e_1 + \ldots + x_n e_n \ ;\ x_a \in F,\ e_a \in A$$

It is always possible to choose $e_o = \mathbb{1}$ (see Curtis [30]). The product of remaining bases (18.2) must close the algebra

$$e_a e_b = -\, d_{ab}\mathbb{1} + f_{abc}\, e_c \qquad d_{ab}, f_{abc} \quad \epsilon F$$

$$a,\ldots,c = 1,2 \ldots n$$

We define the norm in this basis by

$$N(x) = x_o^2 + d_{ab}\, x_a\, x_b . \tag{19.2}$$

From the symmetry of the associated inner product[2]

$$(x,y) = (y,x) = -\,\frac{N(x+y) - N(x) - N(y)}{2} \tag{19.3}$$

it follows that $-\,d_{ab} = (e_a, e_b) = (e_b, e_a)$ is symmetric, and it is always possible to choose bases $\mathbf{e}_a$ such that

$$e_a e_b = -\,\delta_{ab} + f_{abc}\, e_c \tag{19.4}$$

Furthermore, from

$$-(x\ y,x) = \frac{N(xy+x)-N(x)N(y)}{2} = N(x)\ \frac{N(y+1) - N(y) - 1}{2}$$

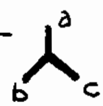
$$= N(x)\ (y,1)$$

it follows that $f_{abc} = (e_a e_b, e_c)$ is fully antisymmetric . (In Tits'[2] notation the multiplication tensor f_{abc} is replaced by a cubic anti-symmetric form (a, a'.a"), his equation (14)). The composition requirement (19.1) expressed in terms of bases (19.4) is

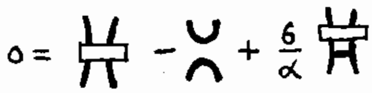
$$0 = N(xy) - N(x)\ N(y)$$

$$= x_a x_b y_c y_d\ (\delta_{ac}\ \delta_{bd} - \delta_{ab}\ \delta_{cd} + f_{ace}\ f_{cbd}) \qquad (19.5)$$

To make a contact with Sec. 13 we introduce diagrammatic notation (factor $i\sqrt{\frac{6}{\alpha}}$ adjusts the normalization)

$f_{abc} = i\sqrt{\frac{6}{\alpha}}$  (19.6)

Diagrammatically (19.5) is given by

 (19.7)

This is precisely the relation (13.15) which we have proven to be nontrivially realizable only in 3 and 7 dimensions. The trivial realizations are n = 0 and n = 1, $f_{abc} = 0$. So in Sec. 13 we have proven Hurwitz's theorem[30]: n + 1 dimensional normed algebras over reals exist only for n = 0, 1, 3, 7 (real, complex, quaternion, octonion). We call (19.7) the alternativity relation because it can also be obtained by substituting (19.4) into the alternativity condition for octonions [31]

$$[xyz] \equiv (xy)z - x(yz) \qquad (19.8)$$

$$[xyz] = [zxy] = [yzx] = -[yxz]$$

Cartan [1] was first to note that $G_2(7)$ is the isomorphism group of octonions, i.e. the set of transformations of octonion bases

$$e'_a = (\delta_{ab} + i\, D_{ab})\, e_b$$

which preserve the octionionic multiplication rule (19.4). The reduction identity (13.16) was first derived by Behrends et al. [32] and independently by Tits [2] (in very different notation, his equation (16)). Tits also constructed the projector for $G_2(7)$ by defining the derivation on an octionion algebra

$$Dz = \langle x, y \rangle z$$

$$= -\tfrac{1}{2}((x.y).z) + \tfrac{3}{2}[(y,z)x-(x,z)y] \qquad \text{Tits (23)}$$

where $\quad e_a . e_b \equiv \; f_{abc}\, e_c$ (19.9)

$$(e_a, e_b) \equiv -\, \delta_{ab} \qquad (19.10)$$

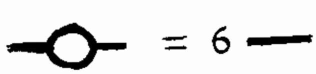
Substituting (18.2) we find

$$(Dz)_d = -\,3\, x_a y_b \left[\tfrac{1}{2}\,(\delta_{ad}\,\delta_{bc} - \delta_{ac}\,\delta_{bd}) + \tfrac{1}{6}\, f_{abe}\, f_{ecd} \right] z_c$$

The term in the brackets is just the G_2 (7) projector (13.31), with normalization $a = -3$ in (18.4). The non-hermitean normalization $\alpha = -6$ in (13.2) does not follow our convention (3.11-13) and we prefer to replace (19.9) by

$$(e_a . e_b) \equiv i\, f_{abc}\, e_c \qquad (19.11)$$

analogous to (2.13). In that case the normalization



(19.12)

has a simple combinatorial interpretation; if we colour the diagram (19.12) with seven colours, and require that colouring around the vertex is such that for any two given colours there is a unique third colour allowed (triality!), then there are 6 possible colourations.

20. JORDAN ALGEBRA AND $F_4(26)$

Consider the exceptional simple Jordan algebra of traceless Hermitian [3x3] matrices x with octonion matrix elements [20,31]. The non-associative multiplication rule for elements x can be written in basis (18.2) as

$$e_a e_b = e_b e_a = \frac{\delta ab}{3} \mathbb{1} + d_{abc} e_c \tag{20.1}$$

a, b, c = 1, 2 ...26.

where $\mathrm{Tr}\,(e_a) = 0$, and $\mathbb{1}$ is the [3x3] unit matrix. Traceless [3x3] matrices satisfy a characteristic equation

$$x^3 - \tfrac{1}{2}\,\mathrm{Tr}\,(x^2)x - \tfrac{1}{3}\,\mathrm{Tr}(x^3)\mathbb{1} = 0 \tag{20.2}$$

Substituting (20.1) we obtain (15.14), with normalization $\alpha = \frac{7}{3}$.

Substituting (20.1) into the Jordan identity

$$(xy)x^2 = x(yx^2) \tag{20.3}$$

we obtain (15.15). It is interesting to note that the Jordan identity (which defines Jordan algebra in the way Jacobi identity defines Lie algebra) is a trivial consequence of (15.14).

$F_4(26)$ is the group of isomorphisms which leave forms $\mathrm{Tr}(xy) = \delta_{ab}\, x_a\, x_b$ and $\mathrm{Tr}\,(xyz) = d_{abc}\, x_a y_b z_c$ invariant. The derivation is given by Tits [2] as

$Dz = (xz)y - x(zy).$ Tits (28)

Substituting (20.1) we obtain the projector (15.20)

$$(Dz)_d = -3\, x_a y_b \left(\frac{\delta_{ad}\ \delta_{bc} - \delta_{ac}\ \delta_{bd}}{9} + \frac{d_{bce} d_{ead} - d_{ace}\ d_{ebd}}{3}\right) z_c \tag{20.4}$$

Note the definition $d_{abc} = \mathrm{Tr}\,(e_a\, e_b\, e_c)$ is analogous to (15.30), (15.40), and (15.57); the crucial difference between those invariants and the $F_4(26)$ invariant is that the underlying algebras are associative.

This enables us to give reduction algorithms in terms of projection operators for $B_1(5)$, $A_2(8)$ and $C_3(14)$ representations, but not for $F_4(26)$.

21. SPRINGER'S CONSTRUCTION OF $E_6(27)$

Consider the exceptional simple Jordan algebra A of Hermitian [3x3] matrices x with octonion matrix elements [20,21,3], and its dual $\bar{A}$ (complex conjugate of A). Following Springer [3] define products

$$(\bar{x},y) = \mathrm{Tr}\,(\bar{x}y) \tag{21.1}$$

$$x \times y = \bar{z} \tag{21.2}$$

$$3\,\langle x,y,z\rangle = (x \times y,\ z)$$

and assume they satisfy

$$(x \times x) \times (x \times x) = \langle x,x,x\rangle\, x \tag{21.3}$$

Expanding x, $\bar{x}$ in bases (18.1), n = 27, we chose a normalization

$$(e_a, e^b\rangle = \delta_a^b \tag{21.4}$$

and define

$$e_a \times e_b \;\; = d_{abc} \;\; e^c \tag{21.5}$$

Substituting into (21.3) we obtain (10.10), with $\alpha = \frac{5}{2}$. Freudenthal [21] and Springer [3] prove that (21.3) is satisfied if d_{abc} is related to the usual Jordan product

$$e_a \circ e_b = \hat{d}_{abc} e_c \tag{21.6}$$

by

$$\begin{aligned} d_{abc} \equiv \hat{d}_{abc} &- \tfrac{1}{2} (\delta_{ab} \operatorname{Tr}(e_c) + \delta_{ac} \operatorname{Tr}(e_b) + \delta_{bc} \operatorname{Tr}(e_a)) \\ &+ \tfrac{1}{2} \operatorname{Tr}(e_a) \,.\, \operatorname{Tr}(e_b) .\, \operatorname{Tr}(e_c) \end{aligned} \tag{21.7}$$

$E_6(27)$ is the group of isomorphisms which leave $(\bar{x},y) = \delta^b_a\, x^a y_b$ and $\langle x,y,z\rangle = d^{abc}\, x_a\, y_b\, z_c$ invariant. The derivation was constructed by Freudenthal [21] (his equation (1.21)):

$$Dz \equiv \langle x,\bar{y}\rangle\ z = 2\bar{y} \times (x \times z) - \tfrac{1}{2} \langle \bar{y},z\rangle x - \tfrac{1}{6}\langle x,\bar{y}\rangle\ z$$

Substituting (21.4-5) we obtain the projector (10.14), n = 27;

$$(Dz)_d = -\ 3\ x_a y^b\ \ P^{ac}_{bd}\ \ z_c \tag{21.8}$$

The object $\langle x,\bar{y}\rangle$ considered by Freudenthal is in our notation  and the above factor -3 is the normalization (3.19), Freudenthal's (1.26). The invariance of x-product is given by Freudenthal as

$$\langle\, x,\ x \times x\, \rangle = 0$$

Substituting (21.5) we obtain (4.1) for .

22. BROWN'S CONSTRUCTION OF $E_7(56)$

Brown [4] considers a finite dimensional complex vector space A with following properties

i) A possesses a non-degenerate skew-symmetric bilinear form $\{x,y\}$.

ii) A possesses a symmetric four-linear form $q\,(x,y,z,w)$.

iii) If the ternary product $T(x,y,z)$ is defined on A by $\{T(x,y,z),w\} = q\,(x,y,z,w)$, then

$$3\ \{T(x,x,y),\ T(y,y,y)\} = \{x,y\}\ \ q(x,y,y,y) \tag{22.1}$$

Substituting (18.1) into (22.1) and defining

$$\{e^a, e^b\} = f^{ab} \tag{22.2}$$

$$q\,(e^a, e^b, e^c, e^d) = d^{abcd} \tag{22.3}$$

we obtain Brown's relation (16.9) with normalization $\alpha = \frac{1}{3}$. The derivation was constructed by Freudenthal* (θP on p.225, Ref. 21), but not in a form that can be readily compared with (16.19).

Another variant of the above axiomatization is a symplectic triple system [33,34] defined by a ternary product $[xyz] \in A$ which satisfies

$$[xyz] = [yxz] \tag{22.4}$$

$$[xyz] - [xzy] + \{x,y\}\, z - \{x,z\}\, y - 2\,\{y,z\}\, x = 0 \tag{22.5}$$

$$[xy[uvw]] = [[xyu]vw] + [u[xyv]w] + [uv[xyw]] \tag{22.6}$$

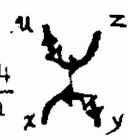
From the last relation it is clear that this product is a derivation

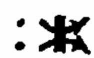
$$Dz = [xyz]$$

Indeed, if we substitute (18.1) and define

$$[e_a e_b e_c] = e^d \; 24\, P_{ac}^{ef}\, f_{eb}\, f_{fd},$$

or diagrammatically

$$\{[xyz], u\} \longleftrightarrow \frac{24}{a}$$  (22.7)

(the extra f_{ab} factors make this an invariant of A alone; A can be considered pseudo-real), it is easily checked that (22.4-6) are satisfied. In particular, (22.5) follows from application of  to the projector (16.19), n = 56;

$\frac{1}{a}$  $= \frac{1}{24}$  (22.8)

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* We thank T. Springer for bringing this to our attention.

23. SO(4)

In Sec. 14 we have shown how $D_2(6)$ decomposes into $A_1(3) + A_1(3)$. This has been used in the study of the classical solutions of Yang-Mills theories, in the following form; 't Hooft [38-40] maps the vectors of one SO(3) subalgebra on self-dual SO(4) tensors $A_{\mu\nu}$ by

$$A_{\mu\nu} = \eta_{a\mu\nu} \; A_a \qquad \begin{array}{c} a = 1,2,3 \\ \mu,\nu = 1,2,3,4 \end{array} \tag{23.1}$$

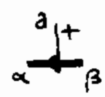
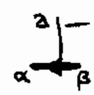
and the vectors of the other SO(3) subalgebra on to anti-self-dual SO(4) tensors $B_{\mu\nu}$ by

$$B_{\mu\nu} = \bar{\eta}_{a\mu\nu} \; B_a \tag{23.2}$$

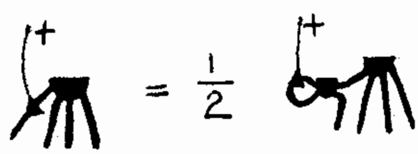
We do not need his explicit representation for $\eta,\bar{\eta}$ SO(3) generators, as our projectors (14.18) and (14.19) already describe the subalgebras. They are related to $\eta,\bar{\eta}$ of 't Hooft by (2.21)

$$P^{(+)}{}_{\alpha\beta,\mu\nu} = -\frac{1}{4} \; \eta_{a\alpha\beta} \, \eta_{a\mu\nu} \tag{23.3}$$

$$P^{(-)}{}_{\alpha\beta,\mu\nu} = -\frac{1}{4} \; \bar{\eta}_{a\alpha\beta} \, \bar{\eta}_{a\mu\nu} \tag{23.4}$$

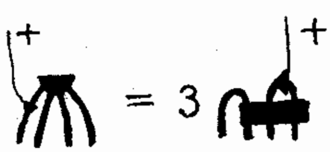
Diagrammatically $\eta_{a\alpha\beta}$ = , $\bar{\eta}_{a\alpha\beta}$ =  and various identities [38-40] for $\eta,\bar{\eta}$ follow from the relations of Sec.14.

For example, substitute (14.33) in



(23.5)

and use (8.7) to obtain



(23.6)

In 't Hooft's notation this is

$$\delta_{\kappa} \quad \eta_{a\mu\nu} + \delta_{\kappa\nu}\, \eta_{a\lambda\mu} + \delta_{\kappa\mu}\, \eta_{a\nu\lambda} + \eta_{a\sigma\kappa}\, \varepsilon_{\lambda\mu\nu\sigma} = 0 \qquad (23.7)$$

APPENDIX

Here we list some identities for F_4 series which illustrate how the projectors and the characteristic identities satisfied by the primitives are used to reduce complicated invariants. $F_4(26)$ identities are given by $n = 26$.

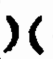
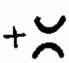
 $= 0$ simplicity (15.2)

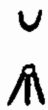
 $= \alpha$ normalization (15.3)

 $= \frac{2\alpha}{n+2}$  characteristic equation (15.14)

From these follow:

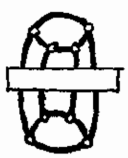
 $= -\frac{\alpha}{2}\frac{n-2}{n+2}$  (15.23)

$\frac{2(n+2)^2}{\alpha^2}$  $= (3n+2)\Big($  $+$  $\Big) - (n+6)$  $- (n-6)(n+2)$  (15.24)

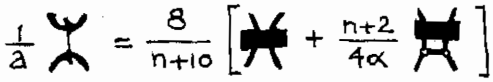
 $= \frac{\alpha}{28}\Big(3$  $-$  $+$  $-$  $\Big)$, $n = 26$ (A.1)

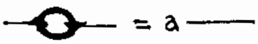
$\frac{1}{n\alpha^3}$  $= -\frac{n^2-10n-16}{2(n+2)^2} \Big|_{n=26} = -\frac{5^2}{2\cdot 7^2}$ (A.2)

$\frac{1}{n\alpha^4}$  $= \frac{n^3-3n^2+80n+100}{4(n+2)^3} = \frac{277}{4\cdot 7^3}$ (A.3)

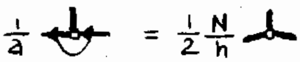
$\frac{1}{n\alpha^4}$  $= \frac{(3n+10)^2}{12(n+2)^3} = \frac{11^2}{12\cdot 7^3}$ (A.4)

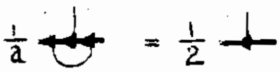
From the definition of the projector

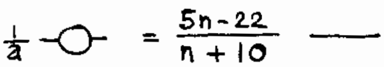
 (15.20)

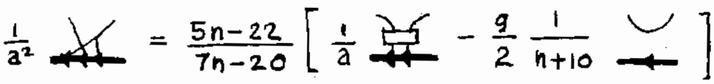
 normalization (2.14)

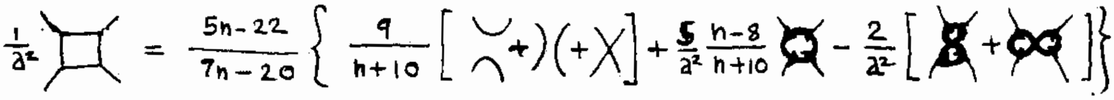
 (15.21)

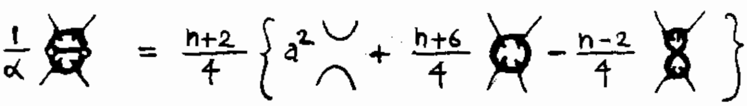
 (A.5)

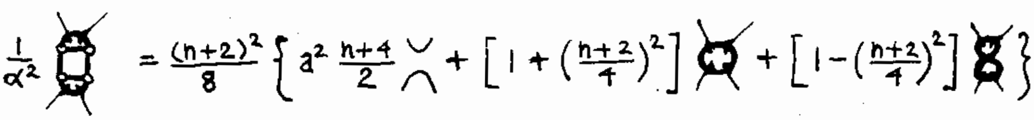
 (A.6)

 (15.22)

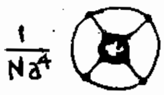
 (A.7)

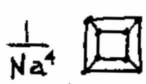
 (A.8)

 (A.9)

 (A.10)

 $= \frac{21n^3+225n^2-1700n+6620}{12(n+10)^3} = \Big|_{n=26} \frac{7\cdot 13\cdot 17\cdot 127}{2\cdot 3^7}$ (A.11)

 $= \frac{6}{8}\,\frac{(5n-22)^2(5n^2-41n+170)}{(n+10)^3(7n-20)}$ (A.12)

 $= \frac{1}{4}\,\frac{(5n-22)^3(85n^3-501n^2+10650n-50000)}{(n+10)^4(7n-20)^2} = \Big|_{n=26} \frac{13\cdot 79}{2}$ (A.13)

 $= \frac{5}{12}\,\frac{(5n-22)^3(2n^3+195n^2+4758n-28240)}{(n+10)^4(7n-20)^2} = \Big|_{n=26} \frac{13\cdot 5^2}{2}$ (A.14)

REFERENCES

1. É. Cartan, Oeuvres Completes (Gauthier-Villars, Paris, 1952).
2. J. Tits, Ned. Akad. Wetensch. Proc. A69, 223 (1966) and Math. Reviews 36, # 2658.
3. T.A. Springer, Ned. Akad. Wetensch. Proc. A62, 254 (1959); ibid. A65, 259 (1962).
4. R.B. Brown, J. Reine Angew. Math. 236, 79 (1969).
5. H. Weyl, The Classical Groups (Princeton Univ. Press, Princeton, N.J., 1946).
6. P. Cvitanović, Phys. Rev. D14, 1536 (1976).
7. J.R. Faulkner and J.C. Ferrar, Bull. London Math. Soc. 9, 1 (1977).
8. I.L. Kantor, Soviet Math. Dokl. 14, 254 (1973).
9. C. Chevalley, Mat. Reviews 14, 948 (1953).
10. A.K. Bose, Computer Phys. Comm. 11, 1 (1976).
11. P. Ramond, "Introduction to Exceptional Lie Groups and Algebras", Caltech preprint CALT-68-577 (Dec. 1976).
12. B.G. Wybourne and M.J. Bowick, "Basic Properties of Exceptional Lie Groups", Univ. of Canterbury, New Zealand preprint (Feb. 1977).
13. J. Patera, R.T. Sharp and P. Winternitz, J. Math. Phys. 17, 1972 (1977).
14. W. McKay, J. Patera and R.T. Sharp, J. Math. Phys. 17, 1371 (1977).
15. B.A. Rosenfeld in Algebraical and Topological Foundations of Geometry, ed. by J.L. Tits (Pergamon, Oxford, 1962).
16. B.A. Rosenfeld, Dokl. Akad. Nauk SSSR 106, 600 (1956) (In Russian).
17. P. Jordan, J. von Neumann and E.P. Wigner, Ann. Math. 35, 29 (1934).
18. G.E. Stedman, J. Phys. A8, 1021 (1975) and 9, 1999 (1976).

19. R. Penrose, in <u>Quantum Theory and Beyond,</u> ed. by Ted Bastin (Cambridge Univ. Press, 1971).

20. H. Freudenthal, Advan. Math. <u>1</u>, 145 (1964).

21. H. Freudenthal, Ned. Akad. Wetensch, Proc. <u>A57</u>, 218 (1954).

22. E.B. Dynkin, Trans. Amer. Math. Soc. (2) <u>6</u>, 111 (1957).

23. E.M. Andreev, É.B. Vinberg and A.G. Élashvili, Funct.Analysis and Appl.<u>1</u>, 257 (1967)

24. J. Patera and D. Sankoff, <u>Tables of Branching Rules for Representations of Simple Lie Algebras</u> (Univ. de Montreal, Montreal, 1973).

25. R. Penrose, in <u>Battelle Rencontres</u>, ed. by C.M. de Witt and J.A. Wheeler (Benjamin, New York, 1968).

26. R. Penrose, in <u>Combinatorial Mathematics and its Applications</u>, ed. by D.J.A. Welsh (Academic, New York, 1971).

27. M. Hamermesh. <u>Group Theory</u> (Addison-Wesley, Reading, Mass., 1962).

28. A.J. Macfarlane, A. Sudbery and P.M. Weisz, Commun. Math. Phys. <u>11</u>, 77 (1968); Proc. R. Soc. London <u>A314,</u> 217 (1970).

29. R. Gilmore, <u>Lie Groups, Lie Algebras and Some of Their Applications</u> (Wiley, New York, 1974).

30. C.W. Curtis, in <u>Studies in Modern Algebra, vol. 2</u>, ed. by A.A. Albert (Prentice-Hall, New York, 1963).

31. R.D. Schafer, <u>Introduction to Nonassociative Algebras</u> (Academic, New York, 1966).

32. R.E. Behrends, J. Dreitlein, C. Fronsdal and W. Lee, Rev. Mod. Phys. <u>34</u>, 1 (1962).

33. K. Yamaguti and H. Asano, Proc. Japan Acad. <u>51</u>, 253 (1975).

34. K. Meyberg, Ned. Akad. Wetensch. Proc. <u>A71</u>,162 (1960).

35. G. 't Hooft, Nucl. Phys. <u>B72</u>, 461 (1974).

36. C.G. Callan, N. Coote and D.J. Gross, Phys. Rev. D13, 1649 (1976).

37. T.P. Cheng, E. Eichten and Ling-Fong Li, Phys. Rev. D9, 2259 (1974).

38. G.'t Hooft, Phys. Rev. Letters 37,8 (1976); Phys. Rev. D14,3432 (1976).

39. E. Corrigan and D.B. Fairlie, Phys. Lett. 67B, 69 (1977).

40. E. Corrigan, D.B. Fairlie and R.G. Yates, "SU(2) gauge potentials and their strengths", U. of Durham preprint (April 1977).

m	8	9	10	12	15	18	20	24	30	36	40	45	60	72	90	120	180	360

representation dimension n

n_1					5	8	10	14	20	26	30	35	50	62	80	110	170	350
n_2				3	6	9	11	15	21	27								
n_3		2	4	8	14	20	24	32	44	56	64	74	104	128	164	224	344	704
n_4	3	8	14	28	52	78	96	133	190	248	287	336	484	603	782	1081	1680	3479

algebra dimension N

N_1					3	8	12	21	36	52	63	77	120	155	208	297	476	1015
N_2				2	8	16	22	35	56	78								
N_3		1	3	9	21	35	45	66	99	133	156	185	273	344	451	630	989	2068
N_4	3	8	14	28	52	78	96	133	190	248	287	336	484	603	782	1081	1680	3479

representation index ℓ^{-1}

ℓ_1^{-1}						1		2		3				4				
ℓ_2^{-1}				0		2		3		4								
ℓ_3^{-1}		0		1		2				3								
ℓ_4^{-1}	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Cartan classification

δ_{ab}, d_{abc}					B_1	A_2		C_3		F_4
δ^a_b, d^{abc}				u	A_2	$A_2 A_2$		A_5		E_6
f^{ab}, d^{abcd}			A_1	$3A_1$	C_3	A_5		D_6		E_7
E_8 series	B_1	A_2	G_2	D_4	F_4	E_6		E_7		E_8

Table I.

All solutions of the Diophantine equations of Sec. 17. The solutions which cannot be realized are given in small print. Algebras within the solid box form the Freudenthal's magic square; the dotted box is the extension of magic square of Faulkner and Ferrar.