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Spinors in Negative Dimensions

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Abstract

We study the properties of spinors in n dimensions and the consequences of taking Dirac matrices to be Grassmann valued. We find that this yields representations of the sympletic group in n dimensions, which can be interpreted as representations of the orthogonal group in $-n$ dimensions. In particular we find the sympletic analogues of spinorial representations. We also prove that the relation Sp $(n) \simeq$ SO $(-n)$ holds in general.

1. Introduction

The subject of this paper is some aspects of the representation theory of the classical groups. However, it is not written in the conventional tensor notation but instead in terms of an equivalent diagrammatic notation. These diagrams are sometimes called "birdtracks" [1]. The advantages of this notation will become self-evident, we hope, but two of the principal benefits are that it eliminates "dummy indices", and that it does not force group-theoretic expressions into the one-dimensional tensor format (both being means whereby identical tensor expressions can be made to look totally different). Similar diagrammatic techniques have been used many times before ([2] and references in [3, 4]).

Section 2 reviews previous results [5] about SO (n) spinor diagrams, which have been reexpressed in a more symmetric notation which emphasizes the relationship between Fierz coefficients on the one hand, and $3-j$ and $6-j$ coefficients on the other. The reduction methods described in this section provide an efficient (polynomial) algorithm for reducing arbitrary spinor traces into sums of terms involving Fierz coefficients, $3-j$ and $6-j$ coefficients for antisymmetric tensor representations of SO (n) groups. We also derive explicit expressions for these coefficients by simple combinatorial arguments: the Fierz [5, 6] and $3-j$ [7] expressions were known before, but the $6-j$ ones are apparently new.

In Section 3 we investigate the consequences of taking γ-matrices to be Grassmann valued. We are led to a new family of objects, which we call spinsters, which play a rôle for symplectic groups analogous to that played by spinors for orthogonal groups. With the aid of spinsters we are able to compute, for example, all the $3-j$ and $6-j$ coefficients for the symmetric representations of Sp (n). We find the surprising result that these coefficients are identical with those obtained for SO (n) if we interchange the roles of symmetrization and antisymmetrization and simultaneously replace the dimension n by $-n$. In Section 4 we investigate this further, and we prove that under the interchange of symmetrizers and antisymmetrizers SO$(-n) \simeq$ Sp (n) and SU $(-n) \simeq$ SU (n) for any scalar quantity made out of arbitrary tensor representations.

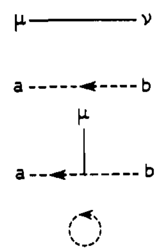
Section 5 makes use of the fact that Sp $(2) \simeq$ Su (2) to show how the formulas for SU (2) $3-j$ and $6-j$ coefficients [8] are special cases of the general expressions for these quantities we derived earlier. The observation that SU (2) can be viewed as SO (-2) was first made by Penrose [2], who used it to compute SU (2) invariants using "binors". His method does not generalize to SO (n), for which spinors are needed to project onto totally antisymmetric representations (for the case $n = 2$ this is not necessary as there are no other representations).

Finally, in Section 6 we discuss various interesting questions concerning the consequences of taking the spinsters seriously, and the extension of our analysis to orthosymplectic groups.

2. Spinors

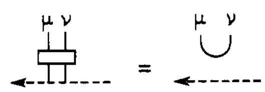
In this section we give a brief review of the diagrammatic notation and results of [5]. The basic notations for spinor and defining representations of SO (n) are

$g^{\mu\nu}$ = , $\mathbb{1}_{ab}$ = , $(\gamma_\mu)_{ab}$ = , tr $\mathbb{1}$ = ; $\mu, \nu = 1, 2, \ldots, n$



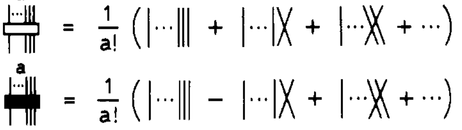
(2.1)

For simplicity we shall omit arrows on defining representation lines, and take all the corresponding indices to be lower. The n-dimensional Dirac matrices are defined by the condition that they satisfy the Clifford algebra



$$\tfrac{1}{2}\{\gamma_\mu, \gamma_\nu\} = g_{\mu\nu}\, \mathbb{1} \tag{2.2}$$

We use the following notation for (anti)symmetric projection operators



(2.3)

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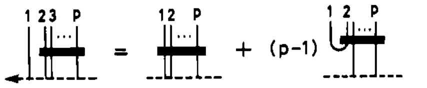
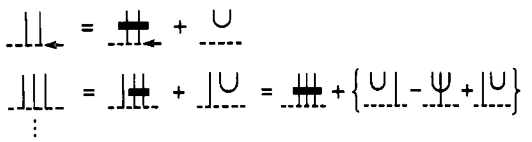
† Present address.

For antisymmetrized products of gamma matrices the Clifford algebra leads to (cf. Appendix A)

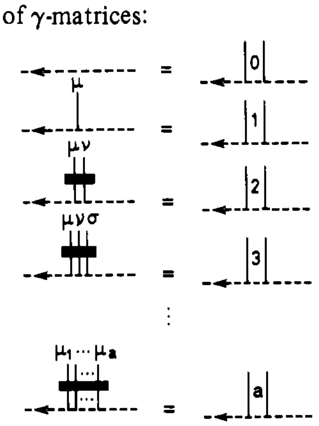

(2.4)

Using this relation recursively we can express any product of γ-matrices as a sum over antisymmetrized products of γ-matrices. For example

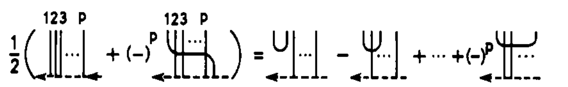

(2.5)

Hence the antizymmetrized tensors $\Gamma^{(k)}$ provide a complete basis for expanding products of γ-matrices:

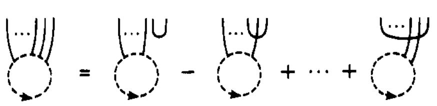
$\Gamma^{(0)} = 1 =$
$\Gamma^{(1)}_{\mu} = \gamma_\mu =$
$\Gamma^{(2)}_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]} =$
$\Gamma^{(3)}_{\mu\nu\sigma} = \gamma_{[\mu}\gamma_\nu\gamma_{\sigma]} =$
$\Gamma^{(k)}_{\mu_1..\mu_a} = \gamma_{[\mu_1}\ldots\gamma_{\mu_a]} =$


(2.6)

Applying the anti-commutator (2.2) to a product of γ-marices we can move the first γ-matrix all the way to the right and obtain

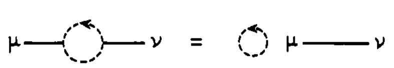

(2.7)

For an even number of γ-matrices this yields a recursive rule for evaluation of spinor traces:

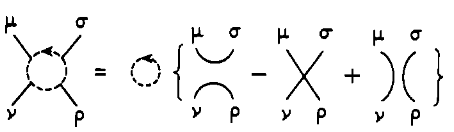


$$\mathrm{tr}\,[\gamma_{\mu_1}\ldots\gamma_{\mu_{p-2}}\gamma_{\mu_{p-1}}\gamma_{\mu_p}] = \sum_{j=1}^{p-1} \mathrm{tr}\,[\gamma_{\mu_1}\ldots\gamma_{\mu_{j-1}}\gamma_{\mu_{j+1}}\ldots \gamma_{\mu_p}]\,(-)^{j+1}\, g_{\mu_j\mu_p}. \qquad (p \text{ even}) \qquad (2.8)$$

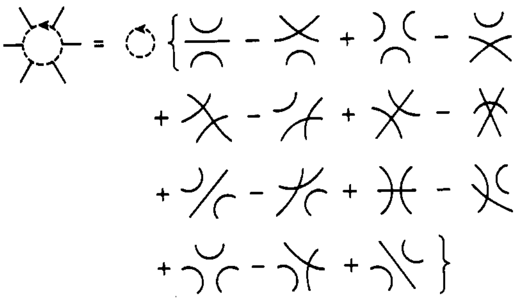
The proof that a trace of an odd number of γ-matrices vanishes for n even is given in Appendix B. Equation (2.8) may be solved recursively, the grapical result being a sum of all $(p-1)!!$ ways of pairing the p external legs with a minus sign for each time two defining-representation lines cross: this is made much clearer from a few examples



$$\mathrm{tr}\,[\gamma_\mu\gamma_\nu] = \mathrm{tr}\,1\; g_{\mu\nu}$$



$$\mathrm{tr}\,[\gamma_\mu\gamma_\sigma\gamma_\rho\gamma_\nu] = \mathrm{tr}\,1\,\{g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma} + g_{\mu\nu}g_{\rho\sigma}\} \qquad (2.9)$$

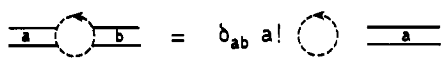


A simple corrolary of this result is that the orientation of a spinor loop is unimportant, i.e., that

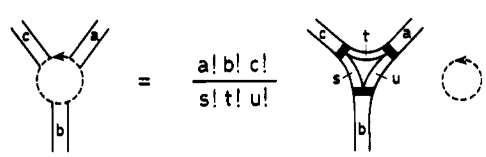
$$\mathrm{tr}\,[\gamma_\mu\ldots\gamma_\nu] = \mathrm{tr}\,[(\gamma_\mu\ldots\gamma_\nu)^T].$$

The preceding identities are in principle a solution to the problem of evaluating spinor traces. In practice they are intractable, because they yield an exponentialy growing number of terms in intermediate steps of trace evaluations. We will therefore describe a more efficient algorithm based on Fierz identities. Fierz identities utilize Γ-matrices as a basis of γ-matrix algebra, and replace traces (2.9) by more compact traces of Γ's.

Evaluation of traces of two or three Γ's is a simple combinatoric exercise using the expansion (2.9). Any term in which a pair of $g_{\mu\nu}$ indices are antisymmetrized vanishes, which implies that $\Gamma^{(k)}$, $k > 0$, is traceless. This is a special case of the orthogonality relation (true for n = even dimensions)


(2.10)

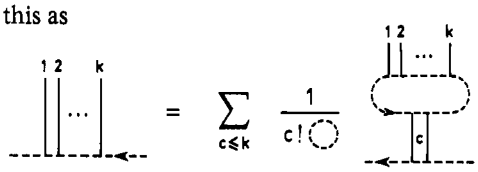
which is in turn a special case of the three-Γ trace


(2.11)

$$s = \tfrac{1}{2}(b+c-a) \quad t = \tfrac{1}{2}(c+a-b) \quad u = \tfrac{1}{2}(a+b-c)$$

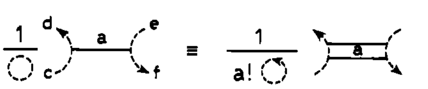
We could continue and compute the four-Γ trace and so on, but fortunately such tedium is unnecessary, as all Γ traces may be reduced to those already calculated using the completeness relation which we will now derive.

As we have shown in eq. (2.4), any product of γ-matrices may be expressed as a sum over the antisymmetric bases $\Gamma^{(k)}$. We write this as


(2.12)

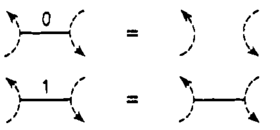
where we have expressed the coupling coefficients as spinor traces; this is a simple consequence of the orthonormality relation (2.10).

For the purposes of reduction of spinor traces to $6-j$ coefficients which we shall carry out here, it is convenient to streamline the notation at this point. The orthogonality of the Γ's eq. (2.10) enables us to introduce projection operators


(2.13)

$$(P_a)_{cd,ef} \equiv \frac{1}{a!\,\mathrm{tr}\,1}\,(\gamma_{[\mu_1}\ldots\gamma_{\mu_k]})_{cd}\,(\gamma^{\mu_k}\ldots\gamma^{\mu_1})_{ef}$$

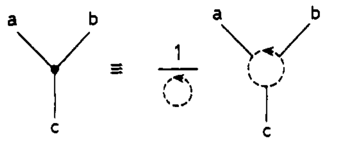
the factor of tr𝟙 is a convenient (but inessential) normalization convention. For the trivial and single γ-matrix representations we shall omit the labels,



(2.14)

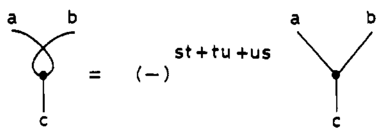
in keeping with the original definitions (2.6).

In terms of this new notation we may define a three-vertex by the three-Γ trace of eq. (2.11)



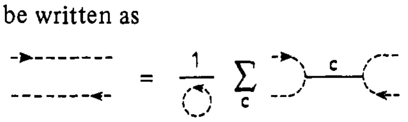
(2.15)

which is non-zero only if $a + b + c$ is even and a, b, and c satisfy the triangle inequalities $|a - b| \leqslant c \leqslant a + b$. It is important to note that the spinor loop runs counterclockwise in this definition because the three-vertex has a non trivial symmetry under the interchange of two legs, the general rule being derived from eq. (2.11)



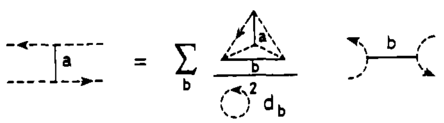
(2.16)

with s, t, u defined in eq. (2.11). The completeness relation (2.12) may be written as



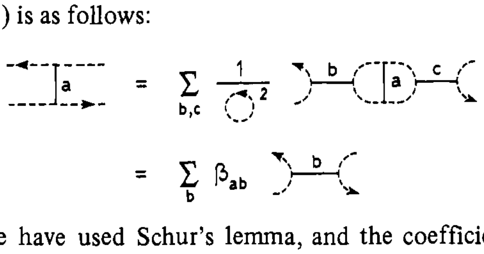
(2.17)

Now that we have the completeness relation we can evaluate spin traces in an efficient way by viewing them as complicated $3n - j$ coefficients and reducing them to a standard set of $3 - j$ and $6 - j$ coefficients. First of all we derive the recoupling relation



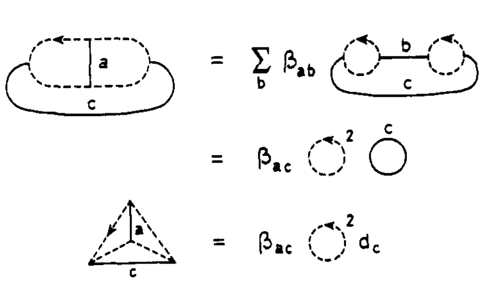
(2.18)

where $d_b = \binom{n}{b}$ is the dimension of the antisymmetric rank b representation, and the $6 - j$ symbol occurring on the right-hand side is called a Fierz coefficient [9]. The derivation of eq. (2.18) is as follows:



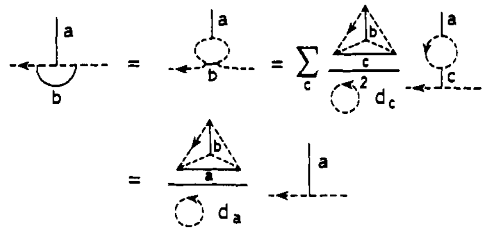
(2.19)

where we have used Schur's lemma, and the coefficients β are found by taking a trace



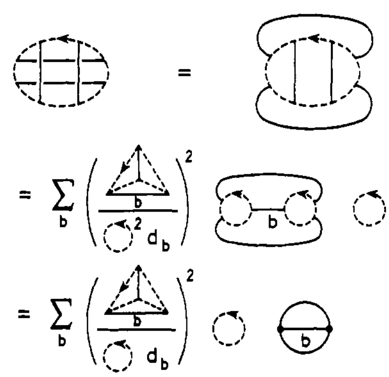
(2.20)

As an example of how eq. (2.18) may be used let us reduce the vertex diagram



(2.21)

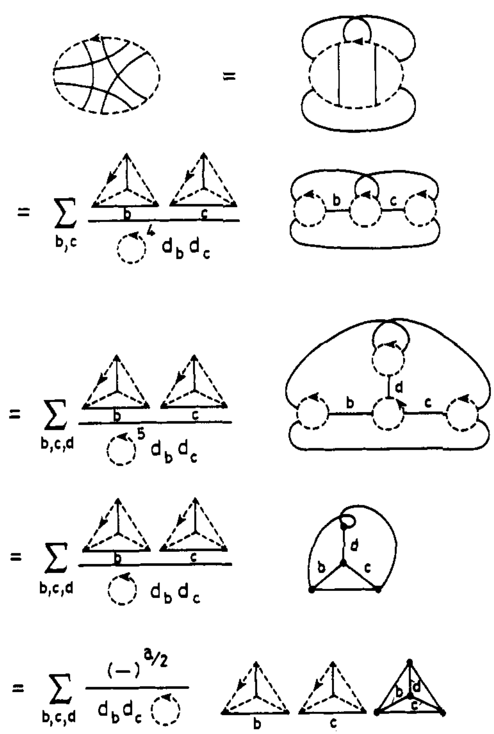
Another example is the reduction of the eight γ-matrix trace ($12 - j$ coefficient) tr $[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma^\nu \gamma^\mu \gamma^\sigma \gamma^\rho]$:



(2.22)

which has been expressed as a sum over Fierz coefficients and $3 - j$ symbols.

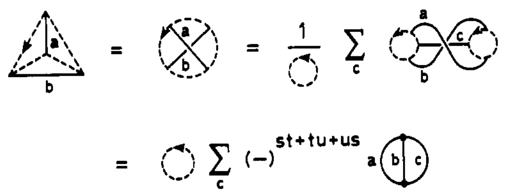
In general the reduction of a spinor trace will also lead to $6 - j$ symbols. As an example consider



(2.23)

It should be clear by now that an arbitrary spinor diagram with external legs may be reduced to a sum of terms each of which consists of a combinatorial factor, expressed as a product of $3 - j$, $6 - j$, and Fierz coefficients, times a tree diagram giving the tensor structure.

All that remains to be done, therefore, is to find explicit expressions for the $3 - j$, $6 - j$, and Fierz coefficients. The Fierz symbols, which are spinorial $6 - j$ symbols, can be related to non-spinorial $3 - j$ symbols using completeness (2.17)



(2.24)

where we have made use of (2.15) and (2.16). The $3-j$ and $6-j$ may be evaluated by simple combinatorial considerations (Appendix C) and are found to be

$$= \frac{n!}{s!t!u!(n-s-t-u)!} \quad (2.25)$$

$$s = \tfrac{1}{2}(b+c-a) \quad t = \tfrac{1}{2}(c+a-b) \quad u = \tfrac{1}{2}(a+b-c)$$

$$= \sum_t \binom{n}{t} \frac{t!}{t_1!t_2!t_3!t_4!t_5!t_6!t_7!} \quad (2.26)$$

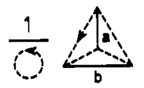
$$t_1 = -\frac{a_1+a_2+a_3}{2}+t \qquad t_5 = \frac{a_1+a_3+a_4+a_6}{2}-t$$

$$t_2 = -\frac{a_1+a_5+a_6}{2}+t \qquad t_6 = \frac{a_1+a_2+a_4+a_5}{2}-t$$

$$t_3 = -\frac{a_2+a_4+a_6}{2}+t \qquad t_7 = \frac{a_2+a_3+a_5+a_6}{2}-t$$

$$t_4 = -\frac{a_3+a_4+a_5}{2}+t$$

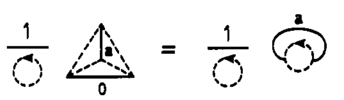
The summation in eq. (2.26) is over all values of t such that all the t_i are non negative integers. Naturally, the $3-j$ (2.25) is a special case of the $6-j$ (2.26). The Fierz symbols (2.24) become



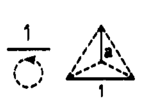
$$= \sum_c (-)^{st+tu+us} \frac{n!}{(n-s-t-u)!s!t!u!}$$

$$= (-)^{ab}\binom{n}{b} \sum_u (-)^u \binom{b}{u}\binom{n-b}{a-u} \quad (2.27)$$

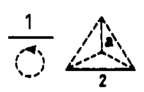
where we have used the identities $t=a-u$, $s=b-u$, and $s+t+u=a+b-u$. Expressions for the cases where b is small are particularly simple,



$$= d_a$$

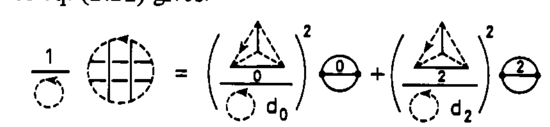


$$= (-)^a(n-2a)\,d_a \quad (2.28)$$



$$= \frac{(n-2a)^2-n}{2} d_a$$

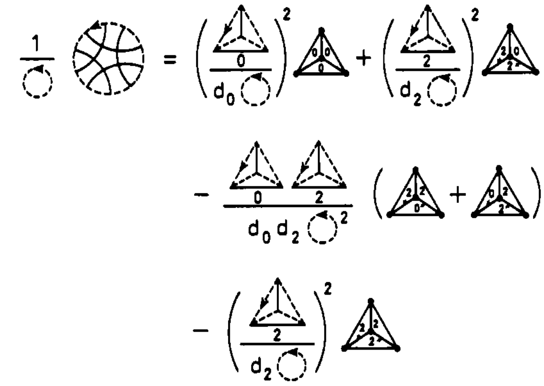
Let us now show how this works by a few examples: the evaluation of eq. (2.22) gives:



(2.29)

$$= n+n(n-1)(n-4)^2$$

because for all terms in the sum other than $b=0$, 2 the $3-j$ coefficients vanish. The result of eq. (2.23) may be written as



(2.30)

$$= n^3+n(n-1)(n-4)^2-2n^2(n-1)(n-4)$$
$$-n(n-1)(n-2)(n-4)^2$$
$$= n^3-n(n-1)(n-4)(n^2-5n+12)$$

To summarize, we have shown that spinor traces in arbitrary dimension can be evaluated quickly and effeciently by means of Fierz relations. The result can be stated in terms of $3-j$ and $6-j$ coefficients for the tensor representations of SO (n). Spinors play a rather peripheral role; in the final results they appear as overall factors of tr$\mathbb{1}$. In the next section we shall repeat the above construction for spinster traces and obtain $3-j$ and $6-j$ coefficients for the fully symmetric representations of Sp (n).

3. Spinsters

The Clifford algebra (2.2) elements $(\gamma_\mu)_{ab}$ are commuting numbers. In this section we shall investigate consequences of taking γ_μ to be Grassmann valued

$$(\gamma_\mu)_{ab}\,(\gamma_\nu)_{cd} = -(\gamma_\nu)_{cd}\,(\gamma_\mu)_{ab} \quad (3.1)$$

The Grassmann extension of the Clifford algebra (2.2) is

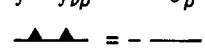
$$\tfrac{1}{2}[\gamma_\mu,\gamma_\nu] = f_{\mu\nu}\,\mathbb{1} \quad \mu,\nu = 1,2,\ldots,n \quad n \text{ even} \quad (3.2)$$

The anticommutator gets replaced by a commutator, and the SO (n) symmetric invariant tensor $g_{\mu\nu}$ by the Sp (n) skew-symmetric invariant tensor $f_{\mu\nu}$. Just as the Dirac gamma matrices lead to spinor representations of SO (n), the Grassmann valued γ_μ give rise to Sp (n) representations, which we shall call spinsters. Introducing a diagramatic notation for the skew symmetric invariant tensor

$$f^{\mu\nu} = -f^{\nu\mu}$$

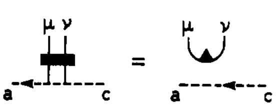
(3.3)

$$f^{\mu\nu}f_{\nu\rho} = -\delta^\mu_\rho$$



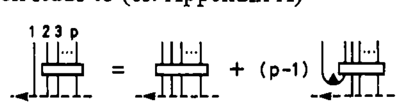
(3.4)

we represent the defining commutation relation (3.2) by



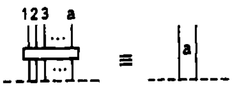
(3.5)

For the symmetrized products of γ-matrices the above commutation relation leads to (cf. Appendix A)



(3.6)

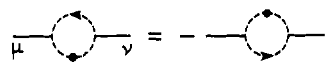
As in the previous section, this gives rise to a complete basis-for expanding products of γ-matrices. Γ's are now the symmetrized product of γ-martices:



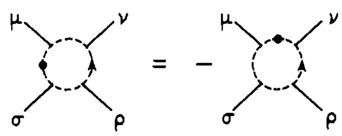
(3.7)

Note that while for spinors the $\Gamma^{(k)}$ vanish by antisymmetry for $k > n$, for spinsters the $\Gamma^{(k)}$'s are non-vanishing for any k, and the number of spinster basis tensors is infinite. However, a reduction of a product of k γ-matrices involves only a finite number of $\Gamma^{(l)}$, $0 \leq l \leq k$. As the components $(\gamma_\mu)_{ab}$ are Grassmann valued, spinster traces of even numbers of γ's are anticyclic

$$\mathrm{tr}\,\gamma_\mu\gamma_\nu = (\gamma_\mu)_{ab}(\gamma_\nu)_{ba} = -\,\mathrm{tr}\,\gamma_\nu\gamma_\mu$$

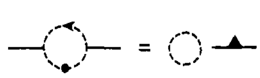


$$\mathrm{tr}\,\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = -\,\mathrm{tr}\,\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\mu \qquad (3.8)$$



In the diagrammatic notation we indicate beginning of a spinster trace by a dot. The dot keeps track of the signs in the same way as the triangle (3.3) for $f_{\mu\nu}$. Indeed, tracing eq. (3.5) we have

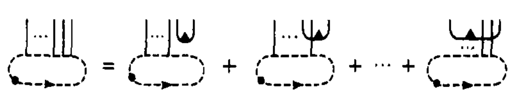
$$\mathrm{tr}\,\gamma_\mu\gamma_\nu = f_{\mu\nu}\,\mathrm{tr}\,\mathbb{1}$$



(3.9)

Moving a dot through a γ-matrix gives a factor -1, as in eq. (3.8).

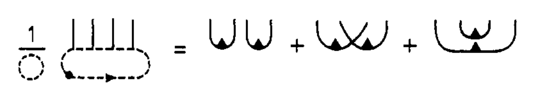
Spinster traces can be evaluated recursively, as in eq. (2.8). For a trace of an even number of γ's we have



(3.10)

A trace of an odd number of γ's vanishes (Appendix B).

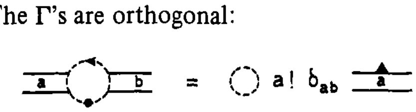
Iteration of the equation (3.10) expresses a spinster trace as a sum of the $(p-1)!! = (p-1)(p-3)\ldots 5.3.1$ ways of connecting the external legs with $f_{\mu\nu}$. The overall sign is fixed uniquely by the position of the dot on the spinster trace:



(3.11)

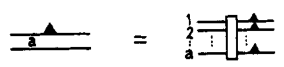
and so on (cf. eq. (2.9)).

Evaluation of traces of several Γ's is again a simple combinatoric exercise. Any term in which a pair of $f_{\mu\nu}$ indices are symmetrized vanishes, which implies that any $\Gamma^{(k)}$ with $k > 0$ is traceless. The Γ's are orthogonal:



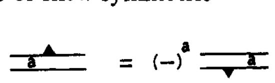
(3.12)

The symmetrized product of a $f_{\mu\nu}$'s, denoted by



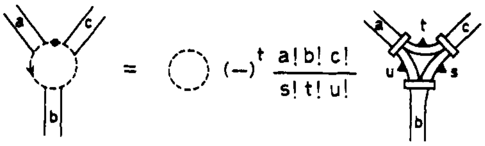
(3.13)

is either symmetric or skew-symmetric



(3.14)

A spinster trace of three symmetric Sp (n) representations defines a 3-vertex:

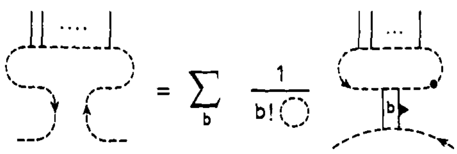


(3.15)

$$= 0 \quad \text{for}\; a+b+c = \text{odd}$$

$$s = \tfrac{1}{2}(b+c-a) \quad t = \tfrac{1}{2}(c+a-b) \quad u = \tfrac{1}{2}(a+b-c)$$

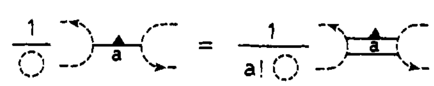
As in eq. (2.12), Γ's provide a complete basis for expanding products of arbitrary numbers of γ-matrices



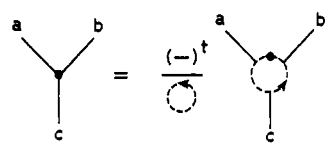
(3.16)

The coupling coefficients in eq. (3.16) are computed as spinster traces using the orthoganality relation (3.12). As only traces of even numbers of γ's are nonvanishing, spinster traces are even Grassmann elements and thus commute with other Γ's and all the signs in the above completeness relation are unambiguous.

The orthogonality of Γ's enables us to introduce projection operators and three-vertices

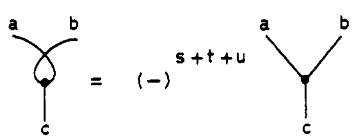


(3.17)



(3.18)

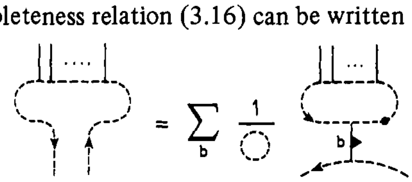
The sign factor $(-)^t$ gives a symmetric definition of the three-vertex, fc. eq. (3.15). It is important to note that the spinster loop runs clockwise in this definition. Because of eq. (3.14) the three-vertex has a non-trivial symmetry under interchange of two legs:



(3.19)

Note that this is different from eq. (2.16) – one of the few instances of spinsters and spinors differing in a way which cannot be immediately understood as $n \to -n$ continuation.

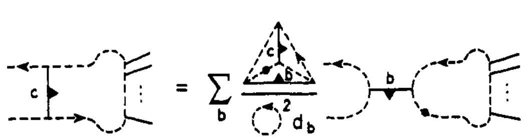
The completeness relation (3.16) can be written



(3.20)

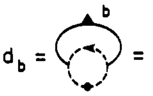
We keep an arbitrary number of γ's to indicate the way in which the spinster trace is to be taken; this keeps track of Grassmann signs.

The recoupling relation is derived as in the spinor case (2.18)



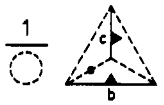
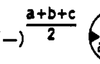
(3.21)

Here d_b is the dimension of the fully symmetrized rank b representation of Sp (n):

$d_b =$  $= \binom{n+b-1}{b} = (-)^b \binom{-n}{b}$ (3.22)

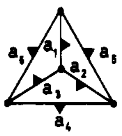
The spinster recoupling coefficients in eq. (3.21) are analogues of the spinor Fierz coefficients in eq. (2.18). Completeness can be used to evaluate spinster traces in the same way as in examples (2.21) to (2.23).

The next step is the evaluation of $3-j$'s, $6-j$'s and spinster recoupling coefficients. The spinster recoupling coefficients can be expressed in terms of $3-j$'s, just as in eq. (2.24):

 $= \sum (-)^{\frac{a+b+c}{2}}$  (2.23)

The evaluation of $3-j$ and $6-j$ coefficients is again a matter of simple combinatorics (Appendix D):

 $= (-)^{s+t+u} \binom{n+s+t+u-1}{s+t+u} \dfrac{(s+t+u)!}{s!t!u!}$ (3.24)

 $= \sum_t \binom{n+t-1}{t} \dfrac{(-)^t t!}{t_1!t_2!t_3!t_4!t_5!t_6!t_7!}$ (3.25)

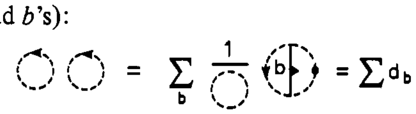
with the t_i defined in eq. (2.26).

We close this section by a comment on the dimensionality of spinster representations. Tracing both sides of the spinor completeness relation (2.17) we determine the dimensionality of spinor representations from the sum rule

$$(\mathrm{tr}\,\mathbb{1})^2 = \sum_{c=0}^{n} \binom{n}{c} = 2^n \tag{3.26}$$

Hence Dirac matrices (in even dimensions) are $[2^{n/2} \times 2^{n/2}]$, and the range of spinor indices in eq. (2.1) is $a, b = 1, 2, \ldots, 2^{n/2}$.

For spinsters tracing the completeness relation (3.20) yields (the string of γ-matrices was indicated only to keep track of signs for odd b's):

 (3.27)

$$(\mathrm{tr}\,\mathbb{1})^2 = \sum_{b=0}^{\infty} \binom{n+b-1}{b}$$

The spinster trace is infinite. This reveals why spinster traces are not to be found in the usual classification of the finite-dimensional irreducible representations of Sp (n). One way of making the traces meaningful is to note that in any spinster trace evaluation only a finite number of Γ's are needed, so we can truncate the completeness relation (3.20) to terms $0 \leqslant b \leqslant b_{\max}$. A more pragmatic attitude is to observe that the final results of the calculation are the $3-j$ and $6-j$ coefficients for the fully symmetric representations of Sp (n), and that the spinster algebra (3.2) is a formal device for projecting only the fully symmetric representations from various Clebsch–Gordan series for Sp (n).

The most striking result of this section is that the $3-j$ and $6-j$ coefficients are just the corresponding SO (n) coefficients evaluated for $n \to -n$. The reason will become clear in the next section.

4. Negative dimensions and SO $(n) \leftrightarrow$ Sp (n) duality

When we took the Grassmann extension of Clifford algebras in Section 3 it was not too surprising that the main effect was to interchange the rôle of symmetrization and antisymmetrization. All the antisymmetric tensor representations of SO (n) correspond to the symmetric representation of Sp (n). What is more surprising is that if we take the expression we derived for the SO (n) $3-j$ and $6-j$ coefficients and replace the dimension n by $-n$ we obtain exactly the corresponding result for Sp (n). The negative dimension arises in these cases through the relation $\binom{-n}{a} = (-)^a \binom{n+a-1}{a}$, which may be justified by defining the binomial coefficients as beta functions.

Such relations between Grassmann extensions and negative dimensions have been noticed before; for example, Parisi and Sourlas [10] have suggested that a Grassmann vector space of dimension n can be interpreted as an ordinary vector space of dimension $-n$. An early example of this property of negative dimensions is Penrose's binors [2], which are representations of SU $(2) \simeq$ Sp (2) constructed as SO (-2). King [11] has proved that the dimension of any irreducible representation of Sp (n) is equal to that of SO (n) with symmetrizations interchanged with antisymmetrizations (i.e., corresponding to the transposed Young tableaux) and n replaced by $-n$. Such relations have also been noted for loop equations, Casimir operators and for exceptional groups [12, 13].

We will show that these observations are not accidental, and that the following general result holds: For *any* scalars constructed from tensor representations of the classical groups (all $3n-j$ coefficients), the interchange of symmetrizations and antisymmetrizations is equivalent to the "analytic continuations" SU $(n) \to$ SU $(-n)$, SO $(n) \to$ SO $(-n) \simeq$ Sp (n), Sp $(n) \to$ Sp $(-n) \simeq$ SO (n).

Let us explain why this is so for a simple example in SU (n) first. SU (n) preserves only two invariant tensors, the Levi-Civita tensor $\epsilon_{\mu_1 \ldots \mu_n}$ and the Kronecker tensor δ^ν_μ, and its birdtracks are made out of these objects:

$\delta^\nu_\mu =$ 

$\epsilon^{\mu_1 \ldots \mu_n} =$  (4.1)

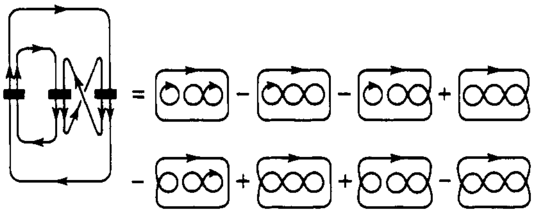
$\epsilon_{\mu_1 \ldots \mu_n} =$ 

A $3n-j$ coefficient is just a number which, in birdtrack notation, corresponds to a graph with no external legs. As the directed lines must end somewhere the Levi–Civita tensors can only be present in pairs, and can thus always be eliminated because of the identity

 (4.2)

$$\epsilon_{\mu_1 \ldots \mu_n} \epsilon^{\nu_1 \ldots \nu_n} \propto \delta^{\nu_1}_{[\mu_1} \delta^{\nu_2}_{\mu_2} \ldots \delta^{\nu_n}_{\mu_n]}$$

Our $3n-j$ coefficient, therefore, corresponds to a diagram made solely of closed loops of directed lines and symmetry projection operators. Consider the following typical example, an SU (n) $9-j$ coefficient for recoupling three antisymmetric rank-two representations:

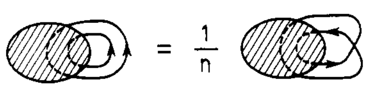

(4.3)

$$= n^3 - n^2 - n^2 + n - n^2 + n + n - n^2$$

$$= n(n-1)(n-3)$$

Notice that in the expansion of the symmetry operators those graphs with an odd number of crossings give an even power of n, and vice versa. Furthermore, if we change all three antisymmetrizers to symmetrizers the terms which change sign are exactly those with an even number of crossings. The two facts show that combining the symmetry–antisymmetry exchange with the replacement of n by $-n$ only has the effect of changing the overall sign of the $9-j$ coefficient. This is in agreement with our claim, because the overall sign is only a matter of convention (it depends on whether we choose to flip two lines entering an antisymmetrizer in the original graph or not). The crossing in the original graph, unconnected with any symmetry operator, also appears in every term in the expansion, and thus does not play any rôle in the argument.

We now present the proof for the general SU (n) case. Consider the graph corresponding to an arbitrary SU (n) scalar, and choose any two terms from the collection obtained by expanding all its symmetry operators. These two graphs can differ only insofar as they come from different terms in the expansion of symmetry operators, so if one graph has ΔC crossings more than the other then, upon exchanging symmetry and antisymmetry, the change in the relative sign of the graphs is $(-)^{\Delta C}$. Each graph consists only of closed loops, i.e., a definite power of n, and thus uncrossing two lines can have one of two consequences. If the two crossed line segments came from the same loop then uncrossing them splits this into two loops, whereas if they came from two loops it joins them into one loop: the power of n is therefore changed by ± 1 by uncrossing one pair of crossed line segments.


(4.4)

The ratio of our two graphs must, therefore, be an even power of n if ΔC is even, and an odd power of n if ΔC is odd. We conclude that the ratio of the graphs is unchanged under the simultaneous exchange of symmetry and antisymmetry and replacement of n by $-n$.

The proof for SO (n) is essentially the same. SO (n) differs from SU (n) by the existence of the symmetric bilinear invariant tensor $g_{\mu\nu}$. Instead of using this to eliminate the arrows from lines, as we did previously, we shall now write it and its inverse as open circles

$g^{\mu\nu} =$

(4.5)

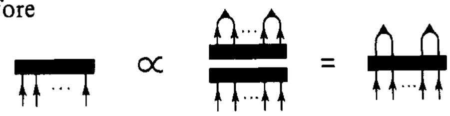
$g_{\mu\nu} =$

Such open circles can also occur in $3n-j$ graphs; the Levi-Civita tensor still cannot, as directed lines starting on an ϵ tensor would have to end on a g, which gives zero by symmetry.

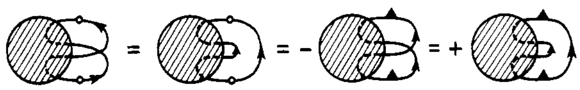
Sp (n) differs from SU (n) by having an additional skew-symmetric bilinear invariant tensor $f_{\mu\nu}$, denoted diagrammatically as before by a black triangle. A Levi-Civita tensor can appear in a $3n-j$ diagram now, but it adds nothing new because

 $\propto \det f = \text{number}$ (4.6)

and therefore


(4.7)

For any SO (n) scalar, there is a corresponding Sp (n) scalar (for even n only, of course) obtained by exchanging symmetrizers and antisymmetrizers *and* $g_{\mu\nu}$'s and $f_{\mu\nu}$'s in the corresponding graphs. The proof that these two scalars are transformed into each other by replacing n by $-n$ (up to an arbitrary overall sigh) is now the same as for SU (n), except that the two line segments at a crossing could come from one new kind of loop containing $g_{\mu\nu}$'s or $f_{\mu\nu}$'s. The required generalization of eq. (4.4) is then


(4.8)

which shows that while uncrossing the lines does not change the number of loops, changing $g_{\mu\nu}$'s to $f_{\mu\nu}$'s does provide the necessary minus sign.

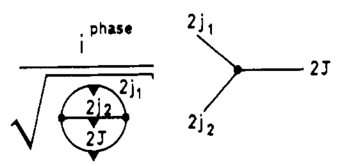
5. Racah coefficients

In Section 3 we have computed the $6-j$ coefficients for fully symmetric representations of Sp (n). Sp (2) plays a special rôle here; the skew symmetric invariant $f^{\mu\nu}$ has only one independent component and it must be proportional to $\epsilon^{\mu\nu}$. Hence Sp (2) $\simeq$ SU (2). For SU (2) *all* representations are fully symmetric (Young tableaux consist of a single row), and our $6-j$'s are all the $6-j$'s needed for computing SU (2) $\simeq$ SO (3) group theoretic factors.* Hence all the Racah and Wigner coefficients familiar from the atomic physics are special cases of our spinor/spinster $6-j$'s.

Wigner's $3-j$ symbol [14]

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} \equiv \frac{(-)^{j_1-j_2+M}}{\sqrt{2J+1}} \langle j_1 j_2 m_1 m_2 | JM \rangle \tag{5.1}$$

is really a Clebsch-Gordon coefficient with our $3-j$ as a normalization factor. This may be expressed more simply in diagramatic form

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} =$$ 

(5.2)

where we have not specified the phase convention on the right-hand side as, in the calculation of physical quantities, such phases cancel. Factors of 2 appear because our integers $a, b, \ldots = 1, 2, \ldots$ count the numbers of SU (2) 2-dimensional representations (SO (3) spinors), while the usual $j_1, j_2, \ldots = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ labels correspond to SO (3) angular momenta.

It is easy to verify (up to a sign) the completeness and orthogonality properties of Wigner's $3-j$ symbols

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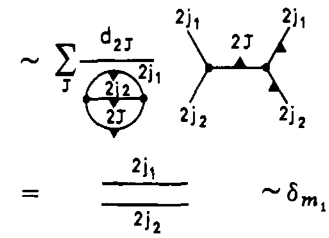
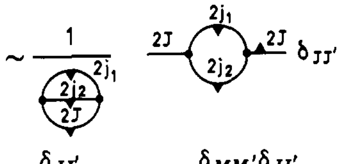
* More pedantically, SU (2) $\simeq$ spin (3) $\simeq \overline{\text{SO (3)}}$.

$$\sum_{J,M}(2J+1)\begin{pmatrix} j_1 & j_2 & J\\ m_1 & m_2 & M\end{pmatrix}\begin{pmatrix} j_1 & j_2 & J\\ m_1' & m_2' & M\end{pmatrix}$$



$$\sim \delta_{m_1 m_1'}\delta_{m_2 m_2'} \qquad (5.3)$$

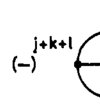
$$\sum_{m_1 m_2}\begin{pmatrix} j_1 & j_2 & J\\ m_1 & m_2 & M\end{pmatrix}\begin{pmatrix} j_1 & j_2 & J'\\ m_1 & m_2 & M'\end{pmatrix}$$



$$= \frac{\delta_{JJ'}}{d_{2J}} \quad 2J \sim \frac{\delta_{MM'}\delta_{JJ'}}{2J+1} \qquad (5.4)$$

The expression (3.24) for our $3-j$ coefficient with $n=2$ gives the expression usually written as Δ in Racah's formula for $\left(\begin{smallmatrix} j & k & l\\ \alpha & \gamma & \gamma\end{smallmatrix}\right)$,

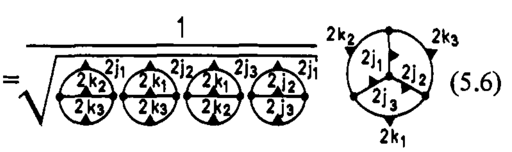
$$\frac{1}{\Delta(j,k,l)} = (-)^{j+k+l}$$



$$= \frac{(j+k+l+1)!}{(j+k-l)!(k+l-j)!(l+j-k)!} \qquad (5.5)$$

Wigner's $6-j$ coefficients are the same as ours, except that the 3-vertices are normalized as in eq. (5.2)

$$\begin{Bmatrix} j_1 & j_2 & j_3\\ k_1 & k_2 & k_3\end{Bmatrix} =$$



(5.6)

which gives Racah's formula using eq. (3.25) with $n = 2$

$$\begin{Bmatrix} j_1 & j_2 & j_3\\ k_1 & k_2 & k_3\end{Bmatrix} = [\Delta(j_1k_2k_3)\Delta(k_1j_2k_3)\Delta(k_1k_2j_3)\Delta(j_1j_2j_3)]^{1/2}$$

$$\times\sum_t \frac{(-)^t(t+1)!}{(t-j_1-j_2-j_3)!(t-j_1-k_2-k_3)!(t-k_1-j_2-k_3)!(t-k_1-k_2-j_3)!(j_1+j_2+k_1+k_2-t)!(j_2+j_3+k_2+k_3-t)!(j_3+j_1+k_3+k_1-t)!} \qquad (5.7)$$

6. Conclusions

The main practical result of this paper is a complete algorithm for the evaluation of spinor traces in n-dimensions. We believe this algorithm to be the most efficient algorithm available, in the sense that the usual algorithm (2.9) requires $(2p-1)!!$ evaluation steps, while the present algorithm requires only about p^2 steps. (The Kahne [15] algorithm applies only for $n=4$.)

The most interesting question raised by this paper is what are spinsters? A sceptic would answer that they are merely a trick for relating SO (n) antisymmetric representations to Sp (n) symmetric representations. That can be achieved without spinsters: indeed, Penrose [2, 7] had observed already some 29 years ago that SO (-2) yields Racah coefficients in a much more elegant manner than the usual angular momentum manipulations. In this paper we have also proved that for any scalar constructed from tensor invariants, SO $(-n) \simeq$ Sp (n). (Various examples of such relations cited in literature are all special cases of our theorem.) This theorem is based on elementary properties of permutations, and establishes the equivalence between $6-j$ coefficients for SO $(-n)$ and Sp (n) without reference to spinsters or any other Grassmann extensions.

Nevertheless, we hope that the spinsters are the natural supersymmetric extension of spinors, and that they might be of interest for superfield formulations of supersymmetric field theories. They do not appear in the usual classifications, because they are infinite dimensional representations. However, they are not as unfamiliar as they might seem; if we write the Grassmannian γ-matrices for Sp (2ω) as $\gamma_\mu = (p_1, p_2, \ldots p_\omega, x_1, x_2 \ldots x_\omega)$ and choose $f_{\mu\nu}$ of form

$$f = \begin{pmatrix} 0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix} \qquad (6.1)$$

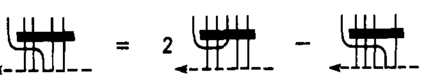
the defining commutator relation (3.2) is like the Heisenberg algebra except for a missing factor of i,

$$[p_i, x_j] = \delta_{ij}\,\mathbb{1} \qquad i,j = 1,2,\ldots\omega \qquad (6.2)$$

It is well known that Heisenberg algebras have infinite dimensional representations, so the infinite dimensionality of spinsters is no surprise. If we include an extra factor of i into the definition of the "momenta" above we finde that spinsters resemble an antiunitary Grassmann-valued representation of the usual Heisenberg algebra. If there is any significance in these observations, which is not clear, it is intriguing to consider the relationship between superspace and the spinor/spinster representations of the orthosymplectic groups.

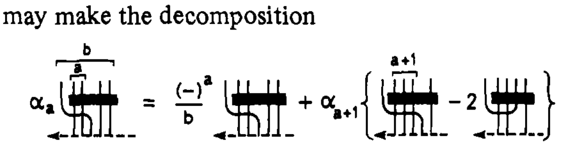
Appendix A. Completeness of the Γ tensor basis

We need to establish the relation (2.4) which we used to demonstrate the completeness of the basis tensors $\Gamma^{(a)}$. We shall set out to prove (2.4) using only the defining anticommutation relation (2.2). As the following identity holds



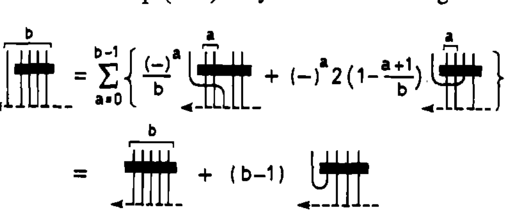
(A.1)

we may make the decomposition



(A.2)

where $\alpha_{a+1} = -\alpha_a + (-)^a/b$. This recurrence relation has the solution $\alpha_a = (-)^a(\alpha_0 - a/b)$, so that taking $\alpha_0 = 1$ we have $\alpha_b = 0$. This means eq. (A.2) may be iterrated to give

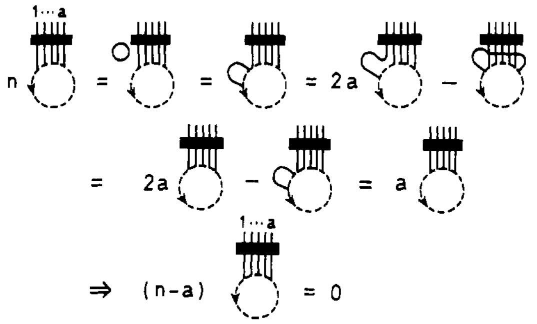


(A.3)

as desired. The corresponding relation for spinsters may be derived in the same way.

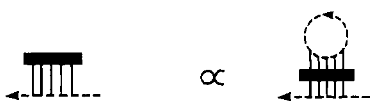
Appendix B. Trace of an odd number of γ matrices

We have shown in Appendix A that any γ-matrix product may be reduced to a sum of terms involving $g_{\mu\nu}$'s and the antisymmetric basis tensors $\Gamma^{(a)}$, so it suffices to consider the trace of such tensors for odd values of a. This may be done as follows:



(B.1)

where we have made use of the relation (2.7) and the fact that a is odd in the third step. The result shows that tr $\Gamma^{(a)}$ vanishes for all odd a $\neq n$, and hence that in even dimensions all traces of an odd number of γ-matrices must vanish, just as was promised earlier. In odd dimensions the situation is a little more complicated, as tr $\Gamma^{(n)}$ does not vanish in general. It is easy to see why this is so, for we can construct the (pseudo) scalar quantity γ^* (the n-dimensional analogue of γ_5),

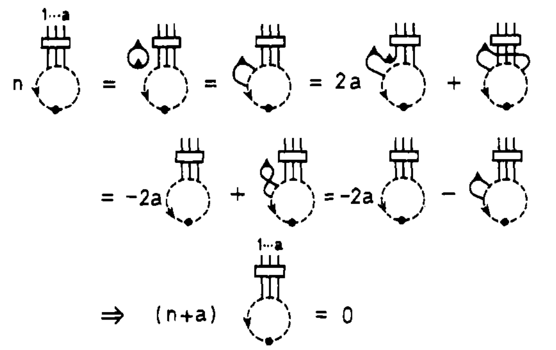


(B.2)

$$\gamma^* = \epsilon^{\mu_1\ldots\mu_n}\gamma_{\mu_1}\ldots\gamma_{\mu_n}$$

which commutes with all the γ-matrices for n odd. It follows by Schur's lemma that γ^* must be a multiple of the unit matrix and hence is not traceless.

For spinsters the argument is quite similar:

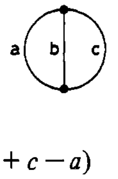


(B.3)

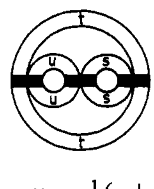
which in this case implies that all traces of an odd number of γ-matrices must vanish for all (positive) values of n.

Appendix C. Evaluation of $3n-j$ coefficients for spinors

Substituting eqs. (2.15), (2.13) and (2.11) into the definition of the $3-j$ coefficient leads to a strand-network [7] expression



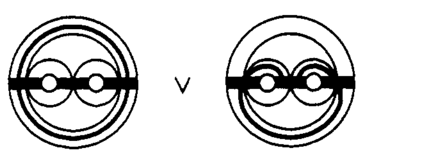
$$= \frac{a!\,b!\,c!}{(s!\,t!\,u!)^2}$$



(C.1)

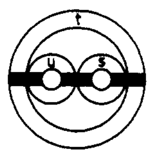
$$s = \tfrac{1}{2}(b+c-a) \quad t = \tfrac{1}{2}(c+a-b) \quad u = \tfrac{1}{2}(a+b-c)$$

Pick out a line in a strand (let us say the lower t strand) and follow it through antisymmetrizations. It can either complete a loop, or end on an antisymmetrizer



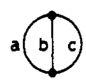
(C.2)

The terms of the latter type all vanish identically. Now one stares* at the terms of first type and writes down the following expression



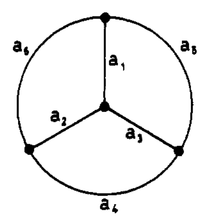
$$= \binom{n}{s+t+u}\frac{(s+t+u)!}{s!\,t!\,u!}\,\frac{(s!\,t!\,u!)^2}{a!\,b!\,c!} \qquad \text{(C.3)}$$

$\binom{n}{s+t+u}$ factor counts the number of ways of colouring $s+t+u$ lines with n different colours. The second factor counts the number of distinct partitions of $s+t+u$ lines into three strands with s, t and u lines respectively. The last factor is due to the projector operator normalization for the antisymmetrization operator (2.3) and the fact that a strand of w lines may be coloured $w!$ different ways with w colours. It cancels against the corresponding factor in eq. (C.1). The value of the $3j$ coefficient is

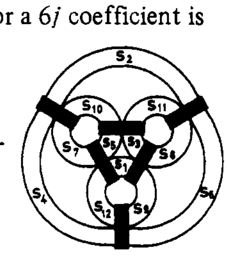


$$= \frac{1}{s!\,t!\,u!}\,\frac{n!}{(n-s-t-u)!} \qquad \text{(C.4)}$$

The strand-network expression for a $6j$ coefficient is



$$= \frac{\prod_{i=1}^{6}(a_i!)}{\prod_{j=1}^{12}(s_j!)}$$



(C.5)

Pick out a line in a strand, and follow its possible routes through the strand network. Seven types of terms give non-vanishing contributions: four "mini tours"

(C.6)

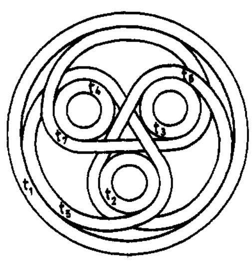
and three "grand tours"

(C.7)

Let the numbers of lines in different tours be t_1, t_2, t_3, t_4, t_5, t_6 and t_7.

A non-vanishing contribution to the $6-j$ coefficient (C.5) corresponds to a partition of twelve strands $s_1, s_2, \ldots, s_{12}$ into seven tours $t_1, t_2, \ldots, t_7$

$M(t_1) =$



(C.8)

Comparing with eq. (C.5) we see that each s_i is a sum of two t_i. $s_1 = t_2 + t_7$, $s_2 = t_1 + t_7$, etc. It is sufficient to specify one

* A well-known theorem states that combinatorial factors are impossible to explain [16].

t_I; this fixes all t_i's. Now, just as for eq. (C.3), one stares at the above figure and writes down

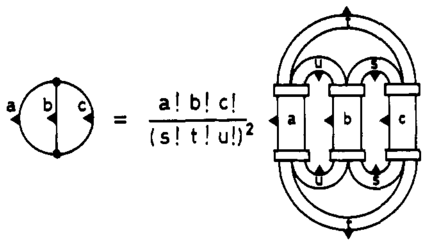
$$M(t_1) = \binom{n}{t} \frac{t!}{\prod_{i=1}^{7} t_i!} \frac{\prod_{i=1}^{12} s_i!}{\prod_{j=1}^{6} a_j!} \tag{C.9}$$

$$t = t_1 + t_2 + \ldots + t_7$$

The $\binom{n}{t}$ factor counts the number of ways of colouring $t_1 + t_2 + \ldots + t_7$ lines with n different colours. The second factor counts the number of distinct partitions of t lines into seven strands $t_1, t_2, \ldots, t_7$. The last factor again comes from the projector operator normalizations (2.3) and the number of ways of colouring each strand, and cancels against the corresponding factor in eq. (C.5). Summing over the allowed partitions (for example, taking $0 \leqslant t_1 \leqslant s_2$) we obtain the $6j$ expression (2.26).

Appendix D. Evaluation of $3n - j$ coefficients for spinsters

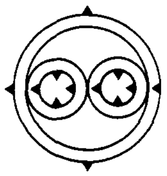
Substituting eqs. (3.15), (3.17) and (3.18) into the definition of the $3 - j$ coefficient yields



(D.1)

$$s = \tfrac{1}{2}(b + c - a) \quad t = \tfrac{1}{2}(c + a - b) \quad u = \tfrac{1}{2}(a + b - c)$$

Again, those strands which pass a symmetrizer twice give a vanishing contribution (they have an odd number of warts on them), and all contributions come from loops of the type



(D.2)

All $f_{\mu\nu}$'s are contracted in such a way that by eq. (3.4) the traces yield various powers of $-n$. This yields an overall factor of $(-1)^{s+t+u}$. The combinatorics is the same as in (C.3) except that $\binom{n}{s+t+u}$ is replaced by $\binom{n+s+t+u+1}{s+t+u}$, the number of symmetric colourings of $s + t + u$ lines. The resulting $3 - j$ is given in eq. (3.24).

The $6 - j$ coefficient is evaluated in the same way as in the Appendix C, except that all antisymmetrizations are replaced by symmetrizations, and each strand consists of $f_{\mu\nu}$ tensors. There are again 7 tours, and it can be checked that $f_{\mu\nu}$'s along each tour give rise to factors of $-n$ by eq. (3.4). The number of coulourings in eq. (C.9) gets replaced by the number of symmetric colourings

$$\binom{n+t-1}{t} = (-)^t \binom{-n}{t}$$

The resulting $6 - j$ is given in eq. (3.25).

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