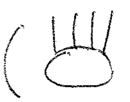
Princeton (IAS), dec 6, 1996

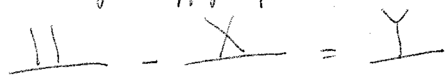
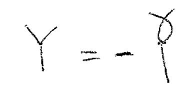
Dear professor Cvitanović,

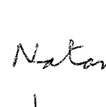
As you can expect, I am most intrigued by your "magical" triangle on p 9 of your Group theory book, especially the bottom line, and frustrated that the relevant section 24 is only to appear.

I like to think to G of the exceptional serie as depending on a parameter μ, which for each G is $6/h^\vee$ ($h^\vee$ = dual Coxeter) — or if one prefers $-1-\mu$.
One has at one disposal the adjoint representation •, the bracket , the Killing form  =  (tr ad x ad y) dualising to $\cap : 1 \to g \otimes g$, and the basic identity

$\underline{\underline{s}}$ () = $\frac{\mu(\mu+1)}{24}$ $\underline{\underline{s}}$ (U U) (1)

where $\underline{\underline{s}} = \frac{1}{4!}\sum_{\sigma \in S_4}$.

I would be very happy if these rules (plus some obvious ones: Jacobi:  [X is not a ~~vertex~~ vertex]
antisymmetry ,
semi-simplicity  = zero†;  = $\frac{1}{2}$ )
(rather: Trace adjoint = 0)
where enough to compute the value (as a function of μ) of any trivalent graph [each trivalent vertex oriented; no external leg].
I asked Bar Natan if this was plausible. He gave me

† More generally: $\mathrm{Hom}(g, 1) = 0$:  (1 external leg) = zero

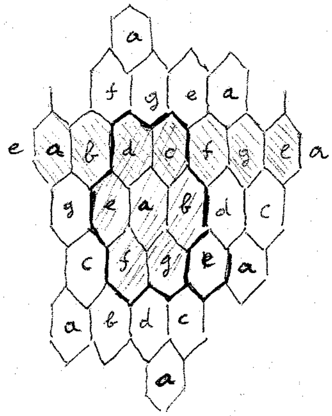
half computational / half heuristic arguments against, expecting the following trivalent graph <u>not</u> to be computable from the above :

Start with the hexagonal paving, viewing the center of the hexagons as forming the lattice of 3-cyclotomic integers $\mathbb{Z}\left[\frac{1+i\sqrt{3}}{2}\right]$. Divide by the sublattice Λ,

$$\Lambda = \text{ideal } ((2+i\sqrt{3})).$$

As $N(2+i\sqrt{3}) = 7$, a fundamental domain consist of 7 hexagons. One can take the central one and its 6 neighboors. Here is the gluing pattern

description as trace, see below, using the fundamental domain ////



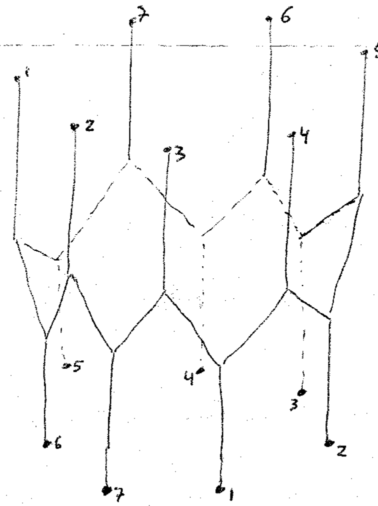
//// : fundamental domain

[makes a $\mathbb{Z}/6$ symmetry clear]

The graph has 14 vertices, 21 edges.

It can also be viewed as giving the trace of a map $g^{\otimes 7} \to g^{\otimes 7}$ (g = adjoint)

Tr (



)

[makes a $\mathbb{Z}/7$ symmetry clear]

Please tell me where I can find information on your magical triangle. Sincerely

P. Del

P. DELIGNE

P.S. forgetting the vertex orientation, I expect the full symmetry group of this graph to be $PGL(2, \mathbb{F}_7)$