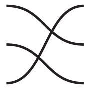
3 Permutations and the symmetric group

Inspecting once more Eqs. (21), (24) and (26) we can come up with a different meaning for the diagrams on the right-hand side. Apparently, we have just found a notation for permutations of 3 objects, i.e. elements of the symmetric group S_3. (We denote by S_n the group of all permutations of n objects, the group multiplication being composition of mappings.) To this end read the diagrams on the r.h.s. of Eq. (21), (24) or (26) as mapping ends of lines from right to left. Recalling two other standard notations for permutations we have, e.g.,

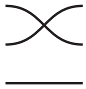
$$\pi = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}_{\text{two-line notation}} = \underbrace{(132)}_{\text{cycle notation}} = \tag{50}$$



birdtrack notation ,

which all mean $\pi(1) = 3$, $\pi(2) = 1$, and $\pi(3) = 2$. Composition with a second permutation, e.g.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (12) = \tag{51}$$



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can be determined in several ways. Say, we are interested in $\pi \circ \sigma$, we can

- determine individual elements

$$\begin{aligned} (\pi \circ \sigma)(1) &= \pi(\sigma(1)) = \pi(2) = 1 \\ (\pi \circ \sigma)(2) &= \pi(\sigma(2)) = \pi(1) = 3 \\ (\pi \circ \sigma)(3) &= \pi(\sigma(3)) = \pi(3) = 2\,, \end{aligned} \tag{52}$$

- multiply cycles (recall that every permutation is a product of disjoint cycles)

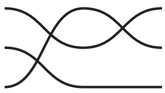
$$\pi \circ \sigma = \underbrace{(132)(12)}_{\text{omit `}\circ\text{'}} \underset{(*)}{=} (1)(23) = \underbrace{(23)}_{\text{omit one-cycles}} \tag{53}$$

(∗) Write '(1', where is it mapped? Thereby read from right to left.
Continue till you'd return to 1, then ')'
Repeat starting with first number not used so far.

or

- compose diagrams,

$$\pi \circ \sigma = \tag{54}$$



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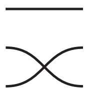

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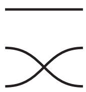
and twist lines at will – it only matters where lines enter and leave.

Finally, we obtain

$$\pi \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (23) = \tag{55}$$



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Exercise 10 *Determine $\sigma \circ \pi$ in three different ways.*

Exercise 11 *Write*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} \in S_5 \tag{56}$$

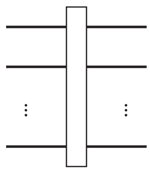
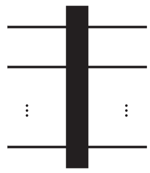
in cycle and birdtrack notation.

Viewing the individual diagrams on the r.h.s. of Eqs. (21), (24) and (26) as permutations, the total expression is not an element of the group S_3 but of the group algebra $\mathscr{A}(S_3)$. Recall that the group algebra $\mathscr{A}(G)$ of a finite group G is the vector space spanned by formal linear combinations of the group elements, with a multiplication induced from the group multiplication.

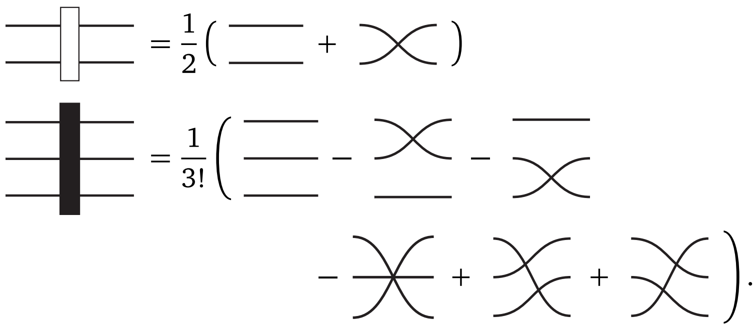
We define symmetrisers S and anti-symmetrisers A by

$$S = \frac{1}{n!} \sum_{\pi \in S_n} \pi \qquad \text{and} \qquad A = \frac{1}{n!} \sum_{\pi \in S_n} \text{sign}(\pi)\, \pi\,, \tag{57}$$

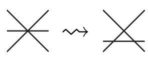
and denote them by open and solid bars, respectively,

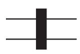
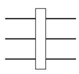
$S =$  and $A =$  . (58)

For instance, see also Eq. (26),

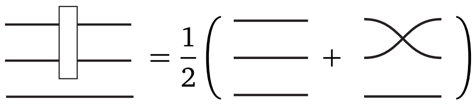


(59)

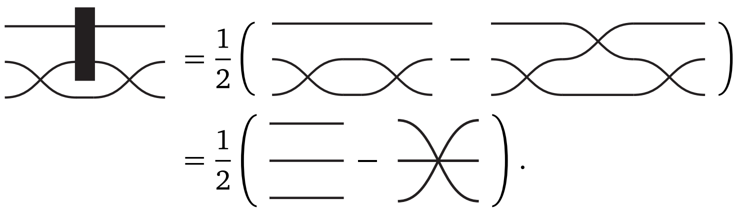
Notice that in birdtrack notation the sign of a permutation, $(-1)^K$, is determined by the number K of line crossings; if more than two lines cross in a point, one should slightly perturb the diagram before counting, e.g.  (K=3).

Exercise 13 *Expand*  *and*  *as in Eq.* (59).

We use the corresponding notation for partial (anti-)symmetrisation over a subset of lines, e.g.

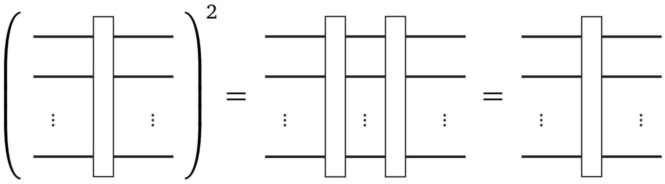


or

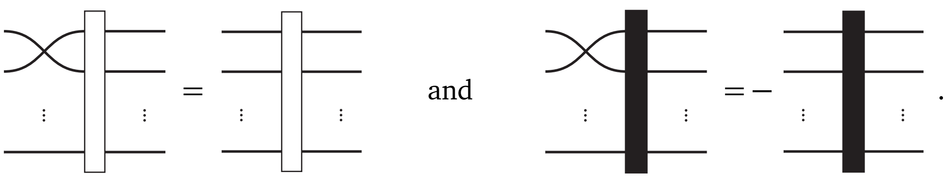

(60)

The prefactor $1/n! = 1/|S_n|$ in Eq. (57) is chosen such that $S^2 = S$ and $A^2 = A$.

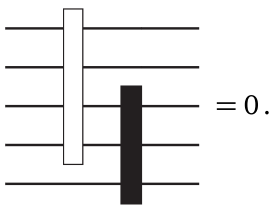
Exercise 14 *Convince yourself that*


and $A^2 = A$. (61)

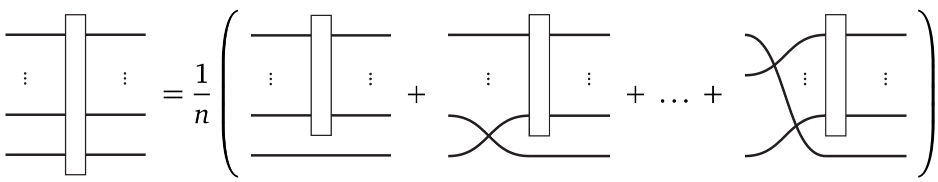
It follows directly from the definition of S and A that when intertwining any two lines S remains invariant and A changes by a factor of (-1), i.e.

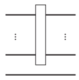

(62)

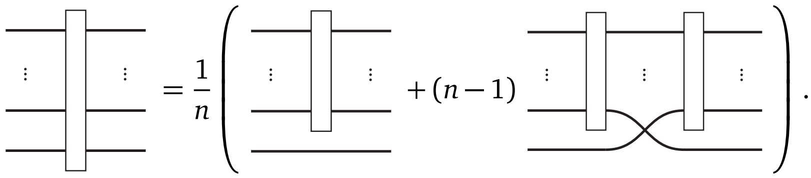
This immediately implies that whenever two (or more) lines connect a symmetriser to an anti-symmetrizer the whole expression vanishes, e.g.


(63)

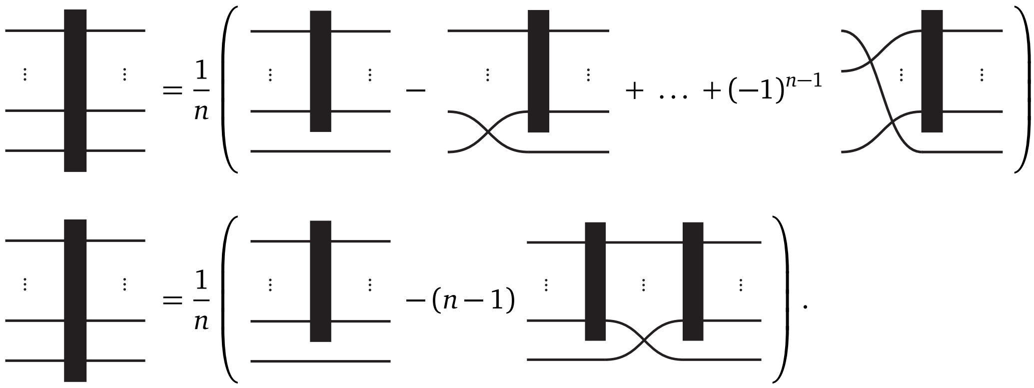
Symmetrisers and anti-symmetrisers can by built recursively. To this end notice that on the r.h.s. of


(64)

we have sorted the terms according to where the last line is mapped – to the nth, to the (n−1)th, …, to the first line line. Multiplying with  from the left and disentangling lines we obtain the compact relation

 (65)

Similarly for anti-symmetrisers:

 (66)

Exercise 15 *Convince yourself that the signs in Eq.* (66) *are correct.*

Birdtracks for SU(N)

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