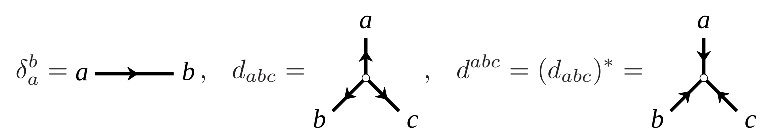
2.3 SECOND EXAMPLE: E_6 FAMILY

What invariance group preserves norms of complex vectors, as well as a symmetric cubic invariant,

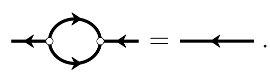
$$D(p,q,r) = d^{abc} p_a q_b r_c = D(q,p,r) = D(p,r,q)\ ?$$

We analyze this case following the steps of the summary of section 2.1:

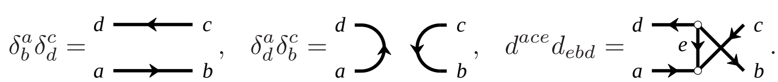
i) *Primitive invariant tensors*



ii) *Primitiveness*. $d_{aef} d^{efb}$ must be proportional to δ_b^a, the only primitive 2-index tensor. We use this to fix the overall normalization of d_{abc}'s:



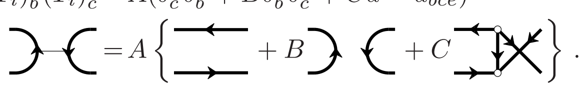
iii) *Invariant hermitian matrices.* We shall construct here the adjoint rep projection operator on the tensor product space of the defining rep and its conjugate. All invariant matrices on this space are



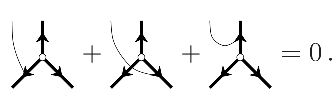
They are hermitian in the sense of being invariant under complex conjugation and transposition of indices (see (3.21)). The crucial step in constructing this basis is the primitiveness assumption: 4-leg diagrams containing loops are not primitive (see section 3.3).

The adjoint rep is always contained in the decomposition of $V \otimes \bar{V} \to V \otimes \bar{V}$ into (ir)reducible reps, so the adjoint projection operator must be expressible in terms of the 4-index invariant tensors listed above:

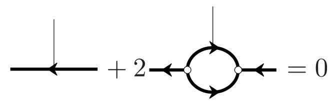
$$(T_i)_b^a (T_i)_c^d = A(\delta_c^a \delta_b^d + B\delta_b^a \delta_c^d + C d^{ade} d_{bce})$$



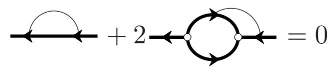
iv) *Invariance*. The cubic invariant tensor satisfies (2.4)



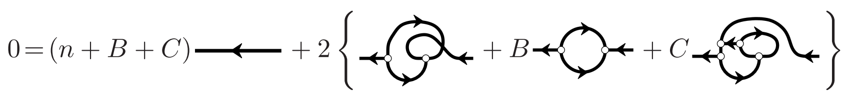
Contracting with d^{abc}, we obtain



Contracting next with $(T_i)_a^{\ b}$, we get an invariance condition on the adjoint projection operator,



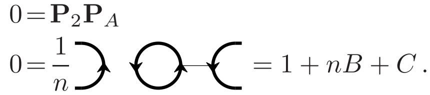
Substituting the adjoint projection operator yields the first relation between the coefficients in its expansion:



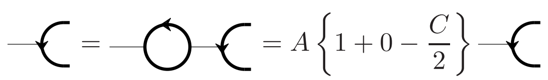
$$0 = B + C + \frac{n+2}{3}\,.$$

v) *The projection operators* should be orthonormal, $\mathbf{P}_\mu \mathbf{P}_\sigma = \mathbf{P}_\mu \delta_{\mu\sigma}$. The adjoint projection operator is orthogonal to (2.5), the singlet projection operator $\mathbf{P}_2$. This yields the second relation on the coefficients:

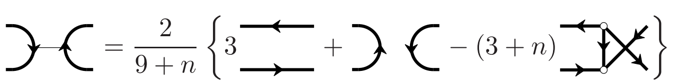
$$0 = \mathbf{P}_2 \mathbf{P}_A$$



Finally, the overall normalization factor A is fixed by $\mathbf{P}_A \mathbf{P}_A = \mathbf{P}_A$:



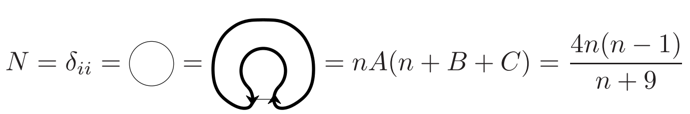
Combining the above three relations, we obtain the adjoint projection operator for the invariance group of a symmetric cubic invariant:



(2.7)

The corresponding *characteristic equation*, mentioned in the point iv) of the summary of section 2.1, is given in (18.10).

The dimension of the adjoint rep is obtained by tracing the projection operator:



This *Diophantine condition* is satisfied by a small family of invariance groups. The most interesting member of this family is the exceptional Lie group E_6, with $n = 27$ and $N = 78$.