Chapter Thirteen

Negative dimensions

P. Cvitanović and A. D. Kennedy

A cursory examination of the expressions for the dimensions and the Dynkin indices listed in tables 7.3 and 7.5, and in the tables of chapter 9, chapter 10, and chapter 12, reveals intriguing symmetries under substitution $n \to -n$. This kind of symmetry is best illustrated by the reps of $SU(n)$; if λ stands for a Young tableau with p boxes, and $\overline{\lambda}$ for the transposed tableau obtained by flipping λ across the diagonal (*i.e.*, exchanging symmetrizations and antisymmetrizations), then the dimensions of the corresponding $SU(n)$ reps are related by

$$SU(n): \qquad d_\lambda(n) = (-1)^p d_{\overline{\lambda}}(-n) \; . \tag{13.1}$$

Here we shall prove the following:

Negative Dimensionality Theorem 1: For any $SU(n)$ invariant scalar exchanging symmetrizations and antisymmetrizations is equivalent to replacing n by $-n$:

$$SU(n) = \overline{SU}(-n) \; . \tag{13.2}$$

Negative Dimensionality Theorem 2: For any $SO(n)$ invariant scalar there exists the corresponding $Sp(n)$ invariant scalar (and vice versa), obtained by exchanging symmetrizations and antisymmetrizations, replacing the $SO(n)$ symmetric bilinear invariant g_{ab} by the $Sp(n)$ antisymmetric bilinear invariant f_{ab}, and replacing n by $-n$:

$$SO(n) = \overline{Sp}(-n) \; , \qquad Sp(n) = \overline{SO}(-n) \; . \tag{13.3}$$

The bars on $\overline{SU}$, $\overline{Sp}$, $\overline{SO}$ indicate interchange of symmetrizations and antisymmetrizations. In chapter 14 we shall extend the relation (13.3) to spinorial representations of $SO(n)$.

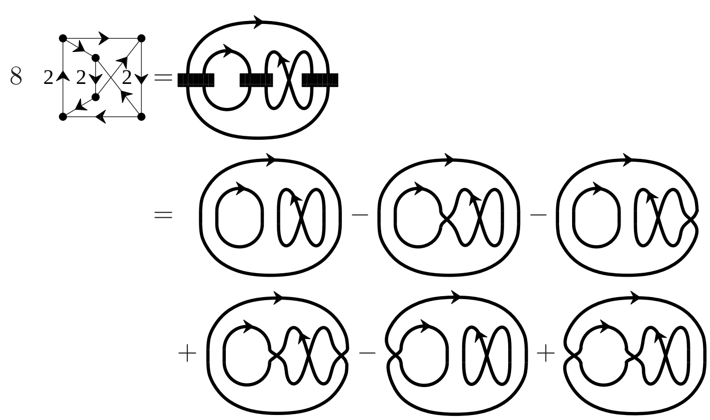
Such relations are frequently noted in literature: Parisi and Sourlas [270] have suggested that a Grassmann vector space of dimension n can be interpreted as an ordinary vector space of dimension $-n$. Penrose [281] has introduced the term "negative dimensions" in his construction of $SU(2) \simeq Sp(2)$ reps as $SO(-2)$. King [191] has proved that the dimension of any irreducible rep of $Sp(n)$ is equal

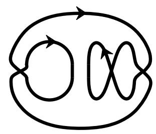
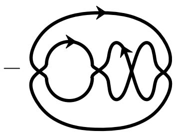
Various examples of $n \to -n$ relations cited in the literature are all special cases of the theorems that we now prove. The birdtrack proof is simpler than the published proofs for the special cases.

13.1 $SU(n) = \underline{SU}(-n)$

As we have argued in section 5.2, all physical consequences of a symmetry (rep dimensions, level splittings, *etc.*) can be expressed in terms of invariant scalars. The primitive invariant tensors of $SU(n)$ are the Kronecker tensor δ^a_b and the Levi-Civita tensor $\epsilon_{a_1 \cdots a_n}$. All other invariants of $SU(n)$ are built from these two objects. A scalar ($3n$-j coefficient, vacuum bubble) is a tensor object with all indices contracted, which in birdtrack notation corresponds to a diagram with no external legs. Thus, in scalars, Levi-Civita tensors can appear only in pairs (the lines must end somewhere), and by (6.28) the Levi-Civita tensors combine to antisymmetrizers. Consequently $SU(n)$ invariant scalars are all built only from symmetrizers and antisymmetrizers. Expanding all symmetry operators in an $SU(n)$ vacuum bubble gives a sum of entangled loops. Each loop is worth n, so each term in the sum is a power of n, and therefore an $SU(n)$ invariant scalar is a polynomial in n.

The idea of the proof is illustrated by the following typical computation: evaluate, for example, the $SU(n)$ 9-j coefficient for recoupling of three antisymmetric rank-2 reps:

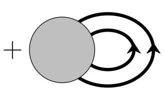
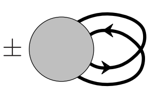


$+$  $-$ 

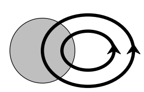
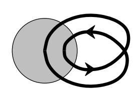
$$
\begin{aligned}
&= n^3 - n^2 - n^2 + n - n^2 + n + n - n^2 \\
&= n(n-1)(n-3) \ . \qquad (13.4)
\end{aligned}
$$

Notice that in the expansion of the symmetry operators the graphs with an odd number of crossings give an even power of n, and vice versa. If we change the three symmetrizers into antisymmetrizers, the terms that change the sign are exactly those with an even number of crossings. The crossing in the original graph that had nothing to do with any symmetry operator, appears in every term of the expansion, and thus does not affect our conclusion; an exchange of symmetrizations and antisymmetrizations amounts to substitution $n \to -n$. The overall sign is only a matter of convention; it depends on how we define the vertices in the $3n$-j's.

The proof for the general $SU(n)$ case is even simpler than the above example: Consider the graph corresponding to an arbitrary $SU(n)$ scalar, and expand all its symmetry operators as in (13.4). The expansion can be arranged (in any of many possible ways) as a sum of pairs of form

$\ldots +$  $\pm$  $+ \ldots \ ,$ (13.5)

with a plus sign if the crossing arises from a symmetrization, and a minus sign if it arises from an antisymmetrization. The gray blobs symbolize the tangle of lines common to the two terms. Each graph consists only of closed loops, *i.e.*, a definite power of n, and thus uncrossing two lines can have one of two consequences. If the two crossed line segments come from the same loop, then uncrossing splits this into two loops, whereas if they come from two loops, it joins them into one loop. The power of n is changed by the uncrossing:

 $= n$  . (13.6)

Hence, the pairs in the expansion (13.5) always differ by $n^{\pm 1}$, and exchanging symmetrizations and antisymmetrizations has the same effect as substituting $n \to -n$ (up to an irrelevant overall sign). This completes the proof of (13.2).

An example of $n \to -n$ relations for $SU(n)$ reps:

Dimensions of the fully symmetric reps (6.13) and the fully antisymmetric reps (6.21) are related by the Beta-function analytic continuation formula

$$\frac{n!}{(n-p)!} = (-1)^p \frac{(-n+p-1)!}{(-n-1)!} \ . \qquad (13.7)$$