Chapter 10

Flips, slides and turns

A detour of a thousand pages starts with a single misstep.
—Chairman Miaw

Dynamical systems often come equipped with symmetries, such as the reflection and rotation symmetries of various potentials.

This chapter assumes familiarity with basic group theory, as discussed in appendix A10.1. We find the abstract notions easier to digest by working out the examples; links to these examples are interspersed throughout the chapter. Working through these examples is essential and will facilitate your understanding of various definitions. The erudite reader might prefer to skip the lengthy group-theoretic overture and go directly to $Z_2 = D_1$ example 11.3, example 11.8, and $C_{3v} = D_3$ example 11.5, backtrack as needed.

10.1 Discrete symmetries

We show that a symmetry equates multiplets of equivalent orbits, or 'stratifies' the state space into equivalence classes, each class a 'group orbit'. We start by defining a finite (discrete) group, its state space representations, and what we mean by a *symmetry* (*invariance* or *equivariance*) of a dynamical system. As is always the problem with 'gruppenpest' (read appendix A1.6) way too many abstract notions have to be defined before an intelligent conversation can take place. Perhaps best to skim through this section on the first reading, then return to it later as needed.

Definition: A **group** consists of a set of elements

$$G = \{e, g_2, \dots, g_n, \dots\} \tag{10.1}$$

and a group multiplication rule $g_j \circ g_i$ (often abbreviated as $g_j g_i$), satisfying

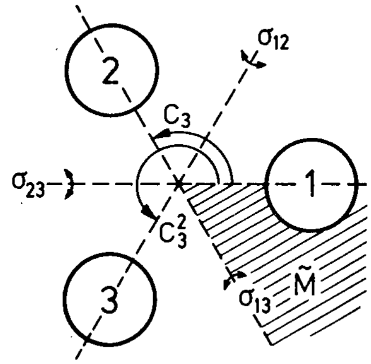

Figure 10.1: The symmetries of three disks on an equilateral triangle. A fundamental domain is indicated by the shaded wedge. Work through example 11.5.

1. Closure: If $g_i, g_j \in G$, then $g_j \circ g_i \in G$
2. Associativity: $g_k \circ (g_j \circ g_i) = (g_k \circ g_j) \circ g_i$
3. Identity e: $g \circ e = e \circ g = g$ for all $g \in G$
4. Inverse g^{-1}: For every $g \in G$, there exists a unique element $h = g^{-1} \in G$ such that $h \circ g = g \circ h = e$.

If the group is finite, the number of elements, $|G| = n$, is called the *order* of the group.

The theory of finite groups is developed on two levels. There is a beautiful theory of *groups* as abstract entities which yields the classification of their structures and their irreducible, orthogonal representations in terms of characters. Then there is the considerably messier matter of *group representations*, in our case the ways in which a given symmetry group acts on and stratifies the particular state space of a problem at hand, the most familiar being the ways in which symmetries reduce and block-diagonalize quantum-mechanical problems. What helps us here is that the symmetries 'commute' with dynamics, i.e., we can first reduce a given state space to its irreducible components, using the symmetry alone, and then study the action of dynamics on these subspaces. As our intuition is based on physical manifestations of group actions, in this brief review we shall freely switch gears between the abstract and the representation levels whenever pedagogically convenient.

Whatever else you must do, do work through example 11.5. Once you understand how this works out for the symmetries of an equilateral triangle, or, equivalently, for the three disk billiard of figure 10.1, you know almost everything you need to know about the general, non-abelian finite groups.

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Definition: Matrix group. The set of $[d\times d]$-dimensional real non-singular matrices $A, B, C, \cdots \in GL(d)$ acting in a d-dimensional vector space $V \in \mathbb{R}^d$ forms the general linear group $GL(d)$ under matrix multiplication. The product of matrices A and B gives the matrix C, $Cx = B(Ax) = (BA)x \in V$, for all $x \in V$. The unit matrix $\mathbf{1}$ is the identity element which leaves all vectors in V unchanged. Every matrix in the group has a unique inverse.

Definition: Matrix representation. Linear action of a group element g on states $x \in \mathcal{M}$ is given by a finite non-singular $[d\times d]$ matrix $D(g)$, the *matrix representation* of element $g \in G$. For brevity we shall often denote by 'g' both the abstract group element and its matrix representation, $D(g)x \to gx$.

However, when dealing simultaneously with several representations of the same group action, the notation $D^{(\mu)}(g)$ is preferable, where μ is a representation label (see appendix A10.1). A linear or matrix representation $\mathsf{D}(G)$ of the abstract group G acting on a *representation space* V is a group of matrices $\mathsf{D}(G)$ such that

1. Any $g \in G$ is mapped to a matrix $D(g) \in \mathsf{D}(G)$.
2. The group product $g_2 \circ g_1$ is mapped onto the matrix product $D(g_2 \circ g_1) = D(g_2)D(g_1)$.
3. The associativity follows from the associativity of matrix multiplication, $D(g_3 \circ (g_2 \circ g_1)) = D(g_3)(D(g_2)D(g_1)) = (D(g_3)(D(g_2))D(g_1)$.
4. The identity element $e \in G$ is mapped onto the unit matrix $D(e) = 1$ and the inverse element $g^{-1} \in G$ is mapped onto the inverse matrix $D(g^{-1}) = D(g)^{-1}$.

Some simple 3D representations of the group order 2 are given in example 10.4.

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example 10.4
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If the coordinate transformation g belongs to a linear non-singular representation of a discrete finite group G, for any element $g \in G$ there exists a number

$m \le |G|$ such that

$$g^m \equiv \underbrace{g \circ g \circ \cdots \circ g}_{m \text{ times}} = e \quad \to \quad |\det D(g)| = 1\,. \tag{10.3}$$

As the modulus of its determinant is unity, $\det g$ is an mth root of 1. This is the reason why all finite groups have unitary representations. [1]

Definition: Symmetry of a dynamical system.

1. A group G is a *symmetry* of the dynamics if for every solution $f(x) \in \mathcal{M}$ and $g \in G$, $gf(x)$ is also a solution.

2. Another way to state this: A dynamical system $(\mathcal{M}, f)$ is *invariant* (or G-*equivariant*) under a symmetry group G if the time evolution $f : \mathcal{M} \to \mathcal{M}$ (a discrete time map f, or the continuous flow f^t map from the d-dimensional manifold $\mathcal{M}$ into itself) commutes with all actions of G,

$$f(gx) = gf(x)\,. \tag{10.4}$$

3. In the language of physicists: The 'law of motion' is invariant, i.e., retains its form in any symmetry-group related coordinate frame (10.2),

$$f(x) = g^{-1} f(gx)\,, \tag{10.5}$$

for $x \in \mathcal{M}$ and any finite non-singular $[d \times d]$ matrix representation g of element $g \in G$. As this are true for any state x, one can state this more compactly as $f \circ g = g \circ f$, or $f = g^{-1} \circ f \circ g$.

Why 'equivariant?' A scalar function $h(x)$ is said to be G-*invariant* if $h(x) = h(gx)$ for all $g \in G$. The group actions map the solution $f : \mathcal{M} \to \mathcal{M}$ into different (but equivalent) solutions $g\,f(x)$, hence the invariance condition $f(x) = g^{-1} f(gx)$ appropriate to vectors (and, more generally, tensors). The full set of such solutions is G-*invariant*, but the flow that generates them is said to be G-equivariant. It is obvious from the context, but for verbal emphasis applied mathematicians like to distinguish the two cases by *in/equi*-variant.

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example 10.6 p. 190
example 10.9 p. 191

10.2 Subgroups, cosets, classes

Normal is just a setting on a washing machine.
—Borgette, Borgo's daughter

Inspection of figure 11.1 indicates that various 3-disk orbits are the same up to a symmetry transformation. Here we set up some group-theoretic notions needed to describe such relations. The reader might prefer to skip to sect. 11.1, backtrack as needed.

Definition: Subgroup. A set of group elements $H = \{e, b_2, b_3, \ldots, b_h\} \subseteq G$ closed under group multiplication forms a subgroup.

Definition: Coset. Let $H = \{e, b_2, b_3, \ldots, b_h\} \subseteq G$ be a subgroup of order $h = |H|$. The set of h elements $\{c, cb_2, cb_3, \ldots, cb_h\}$, $c \in G$ but not in H, is called left *coset* cH. For a given subgroup H the group elements are partitioned into H and $m-1$ cosets, where $m = |G|/|H|$. The cosets *cannot be* subgroups, since they do not include the identity element. A nontrival subgroup can exist only if $|G|$, the order of the group, is divisible by $|H|$, the order of the subgroup, i.e., only if $|G|$ is not a prime number.

example 10.7
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Next we need a notion that will, for example, identify the three 3-disk 2-cycles in figure 11.1 as belonging to the same class.

Definition: Class. An element $b \in G$ is *conjugate* to a if $b = c\,a\,c^{-1}$ where c is some other group element. If b and c are both conjugate to a, they are conjugate to each other. Application of all conjugations separates the set of group elements into mutually not-conjugate subsets called *classes*, *types* or *conjugacy classes*. The identity e is always in the class $\{e\}$ of its own. This is the only class which is a subgroup, all other classes lack the identity element.

example 10.8
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The geometrical significance of classes is clear from (10.5); it is the way coordinate transformations act on mappings. The action, such as a reflection or rotation, of an element is equivalent to redefining the coordinate frame.

Definition: Conjugate symmetry subgroups. The splitting of a group G into a symmetry group G_p of orbit $\mathcal{M}_p$ and $m_p - 1$ cosets cG_p relates the orbit $\mathcal{M}_p$ to $m_p{-}1$ other distinct orbits $c\mathcal{M}_p$. All of them have equivalent symmetry subgroups, or, more precisely, the points on the same group orbit have *conjugate symmetry subgroups* (or *conjugate stabilizers*):

$$G_{c\,p} = c\,G_p\,c^{-1}\,, \tag{10.6}$$

i.e., if G_p is the symmetry of orbit $\mathcal{M}_p$, elements of the coset space $c \in G/G_p$ generate the $m_p - 1$ distinct copies of $\mathcal{M}_p$.

Definition: Reducibility. If state space $\mathcal{M}$ on which G acts can be written as a direct sum of irreducible subspaces, then the representation of G on state space $\mathcal{M}$ is completely reducible.

This being group theory, definitions could go on forever. But we stop here, hopefully having defined everything that we need at the moment, and we pile on a few more definitions in sect. 11.1, chapter 12, chapter 25 and chapter 26. There are also chapter 30, appendix A10, and beyond that the $n \to \infty$ group theory textbooks, if you thirst for more.

Commentary

Remark 10.1. Literature. We found Tinkham [12] the most enjoyable as a no-nonsense, the user friendliest introduction to the basic concepts. Slightly longer, but perhaps student-friendlier is *Part I Basic Mathematics* of Dresselhaus *et al.* [4]. Byron and Fuller [1], the last chapter of volume two, offers an introduction even more compact than Tinkham's. For a summary of the theory of discrete groups see, for example, Johnson [9]. Chapter 3 of Rebecca Hoyle [8] is a very student-friendly overview of the group theory a nonlinear dynamicist might need, with exception of the quotienting, reduction of dynamics to a fundamental domain, which is not discussed at all. For that, *Fundamental domain* wiki is very clear. We also found *Quotient group* wiki helpful. Curiously, we have not read any of the group theory books that Hoyle recommends as background reading, which just confirms that there are way too many group theory books out there. For example, one that you will not find useful at all is ref. [3]. The reason is presumably that in the 20th century physics (which motivated much of the work on the modern group theory) the focus was on the linear representations used in quantum mechanics, crystallography and quantum field theory. We shall need these techniques in Chapter 25, where we reduce the linear action of evolution operators to irreducible subspaces. However, in ChaosBook we are looking at nonlinear dynamics, and the emphasis is on the symmetries of orbits, their reduced state space sisters, and the isotypic decomposition of their linear stability matrices.

References

[1] F. W. Byron and R. W. Fuller, *Mathematics of Classical and Quantum Physics* (Dover, New York, 1992).

[2] P. Chossat and R. Lauterbach, *Methods in Equivariant Bifurcations and Dynamical Systems* (World Scientific, Singapore, 2000).

[3] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2004).

[4] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

[5] M. Golubitsky and I. Stewart, *The Symmetry Perspective* (Birkhäuser, Boston, 2002).

[6] M. Golubitsky, I. Stewart, and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory*, Vol. 2 (Springer, New York, 1988).

[7] W. J. F. Govaerts, *Numerical Methods for Bifurcations of Dynamical Equilibria* (SIAM, Philadelphia, 2000).

[8] R. Hoyle, *Pattern Formation: An Introduction to Methods* (Cambridge Univ. Press, Cambridge, 2006).

[9] S. G. Johnson, Topics in applied mathematics, MIT course 18.325, 2005.

[10] L. D. Landau, "On the problem of turbulence", Dokl. Akad. Nauk SSSR **44**, 339 (1944).

[11] G. Sartori, "Geometric invariant theory: a model-independent approach to spontaneous symmetry and/or supersymmetry breaking", Rivista Nuovo Cim. **14**, 1–120 (1991).

[12] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

10.4 Examples

Example 10.1. Finite groups. Some finite groups that frequently arise in applications:

- C_n (also denoted Z_n): the *cyclic group* of order n.
- D_n: the *dihedral group* of order $2n$, rotations and reflections in plane that preserve a regular n-gon.
- S_n: the *symmetric group* of all permutations of n symbols, order $n!$.

Example 10.2. Cyclic and dihedral groups. The cyclic group $C_n \subset SO(2)$ of order n is generated by one element. For example, this element can be rotation through $2\pi/n$.
The dihedral group $D_n \subset O(2)$, $n > 2$, can be generated by two elements one at least of which must reverse orientation. For example, take σ corresponding to reflection in the x-axis. $\sigma^2 = e$; such operation σ is called an *involution*. C to rotation through $2\pi/n$, then $D_n = \langle\sigma, C\rangle$, and the defining relations are $\sigma^2 = C^n = e$, $(C\sigma)^2 = e$.

Example 10.7. Subgroups, cosets of D_3. (Continued from example 11.6)
The 3-disks symmetry group, the D_3 dihedral group (11.8) has six subgroups

$$\{e\}, \quad \{e, \sigma_{12}\},\ \{e, \sigma_{13}\},\ \{e, \sigma_{23}\}, \quad \{e, C^{1/3}, C^{2/3}\}, \quad D_3\,. \tag{10.16}$$

The left cosets of subgroup $D_1 = \{e, \sigma_{12}\}$ are $\{\sigma_{13}, C^{1/3}\}$, $\{\sigma_{23}, C^{2/3}\}$. The coset of subgroup $C_3 = \{e, C^{1/3}, C^{2/3}\}$ is $\{\sigma_{12}, \sigma_{13}, \sigma_{23}\}$. The significance of the coset is that if a solution has a symmetry H, for example the symmetry of a 3-cycle $\overline{123}$ is D_3, then all elements in a coset act on it the same way, for example $\{\sigma_{12}, \sigma_{13}, \sigma_{23}\}\overline{123} = \overline{132}$.

The nontrivial subgroups of D_3 are $D_1 = \{e, \sigma\}$, consisting of the identity and any one of the reflections, of order 2, and $C_3 = \{e, C^{1/3}, C^{2/3}\}$, of order 3, so possible cycle multiplicities are $|G|/|G_p| = 1$, 2, 3 or 6. Only the fixed point at the origin has full symmetry $G_p = G$. Such equilibria exist for smooth potentials, but not for the 3-disk billiard. Examples of other multiplicities are given in figure 11.1 and figure 11.6. (continued in example 10.8)

Example 10.8. Classes of D_3. (Continued from example 10.7)
The three classes of the 3-disk symmetry group $D_3 = \{e, C^{1/3}, C^{2/3}, \sigma, \sigma C^{1/3}, \sigma C^{2/3}\}$, are the identity, any one of the reflections, and the two rotations,

$$\{e\}\,, \quad \left\{ \begin{array}{c} \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{array} \right\}, \quad \left\{ \begin{array}{c} C^{1/3} \\ C^{2/3} \end{array} \right\}. \tag{10.17}$$

In other words, the group actions either flip or rotate. (continued in example 11.7)