Group Theory

Birdtracks, Lie's, and Exceptional Groups

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Once the projection operators are known, all interesting spectroscopic numbers can be evaluated.

The foregoing run through the basic concepts was inevitably obscure. Perhaps working through the next two examples will make things clearer. The first example illustrates computations with classical groups. The second example is more interesting; it is a sketch of construction of irreducible reps of E_6.

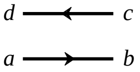
2.2 FIRST EXAMPLE: $SU(n)$

How do we describe the invariance group that preserves the norm of a complex vector? The *list of primitives* consists of a single primitive invariant,

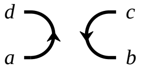
$$m(p,q) = \delta^a_b p^b q_a = \sum_{a=1}^{n} (p_a)^* q_a \,.$$

The Kronecker δ^a_b is the only primitive invariant tensor. We can immediately write down the two *invariant matrices* on the tensor product of the defining space and its conjugate,

$$\text{identity} : \mathbf{1}_{d,b}^{a\,c} = \delta^a_b \delta^c_d =$$



$$\text{trace} : T_{d,b}^{a\,c} = \delta^a_d \delta^c_b =$$



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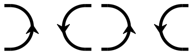
The *characteristic equation* for T written out in the matrix, tensor, and birdtrack notations is

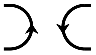
$$T^2 = nT$$

$$T_{d,e}^{a\,f} T_{f,b}^{e\,c} = \delta^a_d \delta^f_e \delta^e_f \delta^c_b = n\, T_{d,b}^{a\,c}$$

$$=$$



$$= n$$



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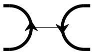
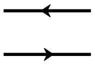
Here we have used $\delta^e_e = n$, the dimension of the defining vector space. The roots are $\lambda_1 = 0$, $\lambda_2 = n$, and the corresponding *projection operators* are

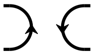
$$SU(n)\text{ adjoint rep:} \qquad \mathbf{P}_1 = \frac{T - n\mathbf{1}}{0-n} = \mathbf{1} - \frac{1}{n}T$$



$$=$$

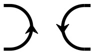


$$- \frac{1}{n}$$



(2.5)

$$U(n)\text{ singlet:} \qquad \mathbf{P}_2 = \frac{T - 0\cdot\mathbf{1}}{n-0} = \frac{1}{n}T = \frac{1}{n}$$



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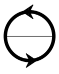
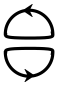
Now we can evaluate any number associated with the $SU(n)$ adjoint rep, such as its dimension and various casimirs.

The *dimensions* of the two reps are computed by tracing the corresponding projection operators (see section 3.5):

$$SU(n)\text{ adjoint:} \quad d_1 = \operatorname{tr} \mathbf{P}_1 =$$



$$=$$



$$- \frac{1}{n}$$

$$= \delta^b_b \delta^a_a - \frac{1}{n} \delta^b_a \delta^a_b$$

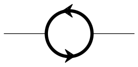
$$= n^2 - 1$$

$$\text{singlet:} \quad d_2 = \operatorname{tr} \mathbf{P}_2 = \frac{1}{n}$$

$$= 1 \,.$$

To evaluate *casimirs*, we need to fix the overall normalization of the generators T_i of $SU(n)$. Our convention is to take

$$\delta_{ij} = \mathrm{tr}\, T_i T_j =$$



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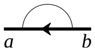
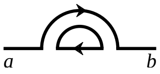
The value of the quadratic casimir for the defining rep is computed by substituting the adjoint projection operator:

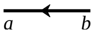
$$SU(n): \quad C_F \delta_a^b = (T_i T_i)_a^b =$$



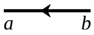
$$=$$



$$- \frac{1}{n}$$



$$= \frac{n^2 - 1}{n}$$



$$= \frac{n^2 - 1}{n} \delta_a^b \, . \qquad (2.6)$$

Chapter Six

Permutations

The simplest example of invariant tensors is the products of Kronecker deltas. On tensor spaces they represent index permutations. This is the way in which the symmetric group S_p, the group of permutations of p objects, enters into the theory of tensor reps. In this chapter, I introduce birdtracks notation for permutations, symmetrizations and antisymmetrizations and collect a few results that will be useful later on. These are the (anti)symmetrization expansion formulas (6.10) and (6.19), Levi-Civita tensor relations (6.28) and (6.30), the characteristic equations (6.50), and the invariance conditions (6.54) and (6.56). The theory of Young tableaux (or plethysms) is developed in chapter 9.

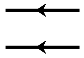
6.1 SYMMETRIZATION

Operation of permuting tensor indices is a linear operation, and we can represent it by a $[d \times d]$ matrix:

$$\sigma_\alpha^\beta = \sigma_{b_1\ldots b_p}^{a_1a_2\ldots a_q}{}_{,c_q\ldots c_2c_1}^{d_p\ldots d_1} \,. \tag{6.1}$$

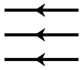
As the covariant and contravariant indices have to be permuted separately, it is sufficient to consider permutations of purely covariant tensors.

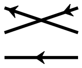
For 2-index tensors, there are two permutations:

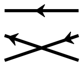
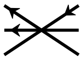
identity: $\mathbf{1}_{ab,}{}^{cd} = \delta_a^d \delta_b^c =$ 

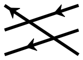
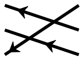
flip: $\sigma_{(12)ab,}{}^{cd} = \delta_a^c \delta_b^d =$  . (6.2)

For 3-index tensors, there are six permutations:

$\mathbf{1}_{a_1a_2a_3,}{}^{b_3b_2b_1} = \delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\delta_{a_3}^{b_3} =$ 

$\sigma_{(12)a_1a_2a_3,}{}^{b_3b_2b_1} = \delta_{a_1}^{b_2}\delta_{a_2}^{b_1}\delta_{a_3}^{b_3} =$ 

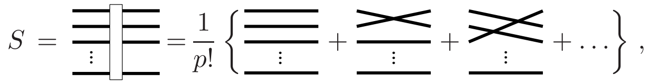
$\sigma_{(23)} =$  , $\sigma_{(13)} =$ 

$\sigma_{(123)} =$  , $\sigma_{(132)} =$  . (6.3)

Subscripts refer to the standard permutation cycles notation. For the remainder of this chapter we shall mostly omit the arrows on the Kronecker delta lines.

The symmetric sum of all permutations,

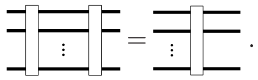
$$S_{a_1 a_2 \ldots a_p},{}^{b_p \ldots b_2 b_1} = \frac{1}{p!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_p}^{b_p} + \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_p}^{b_p} + \ldots\right\}$$



(6.4)

yields the symmetrization operator S. In birdtrack notation, a white bar drawn across p lines will always denote symmetrization of the lines crossed. A factor of $1/p!$ has been introduced in order for S to satisfy the projection operator normalization

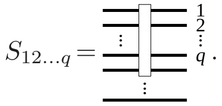
$$S^2 = S$$



(6.5)

A subset of indices $a_1, a_2, \ldots a_q$, $q < p$ can be symmetrized by symmetrization matrix $S_{12\ldots q}$

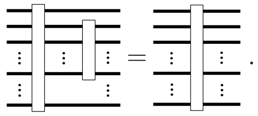
$$(S_{12\ldots q})_{a_1 a_2 \ldots a_q \ldots a_p},{}^{b_p \ldots b_q \ldots b_2 b_1} = \frac{1}{q!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_q}^{b_q} + \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_q}^{b_q} + \ldots\right\}\delta_{a_q+1}^{b_q+1}\ldots\delta_{a_p}^{b_p}$$



(6.6)

Overall symmetrization also symmetrizes any subset of indices:

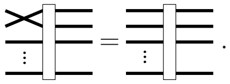
$$SS_{12\ldots q} = S$$



(6.7)

Any permutation has eigenvalue 1 on the symmetric tensor space:

$$\sigma S = S$$



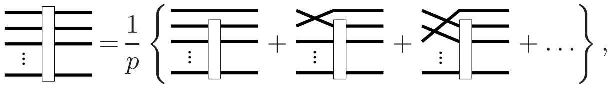
(6.8)

Diagrammatically this means that legs can be crossed and uncrossed at will.

The definition (6.4) of the symmetrization operator as the sum of all $p!$ permutations is inconvenient for explicit calculations; a recursive definition is more useful:

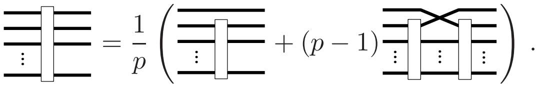
$$S_{a_1 a_2 \ldots a_p},{}^{b_p \ldots b_2 b_1} = \frac{1}{p}\left\{\delta_{a_1}^{b_1} S_{a_2 \ldots a_p},{}^{b_p \ldots b_2} + \delta_{a_2}^{b_1} S_{a_1 a_3 \ldots a_p},{}^{b_p \ldots b_2} + \ldots\right\}$$

$$S = \frac{1}{p}\left(1 + \sigma_{(21)} + \sigma_{(321)} + \ldots + \sigma_{(p\ldots 321)}\right) S_{23\ldots p}$$



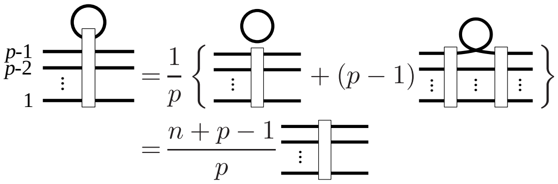
(6.9)

which involves only p terms. This equation says that if we start with the first index, we end up either with the first index, or the second index and so on. The remaining indices are fully symmetric. Multiplying by $S_{23}\ldots p$ from the left, we obtain an even more compact recursion relation with two terms only:



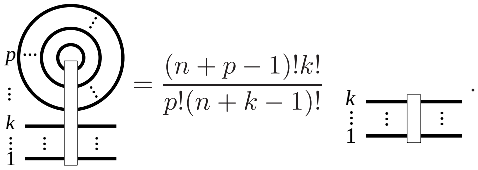
(6.10)

As a simple application, consider computation of a contraction of a single pair of indices:



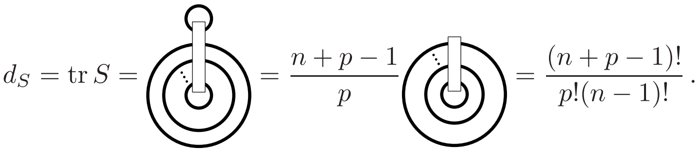
$$S_{a_p a_{p-1}\ldots a_1},{}^{b_1\ldots b_{p-1}a_p} = \frac{n+p-1}{p} S_{a_{p-1}\ldots a_1},{}^{b_1\ldots b_{p-1}} \,. \qquad (6.11)$$

For a contraction in $(p-k)$ pairs of indices, we have



(6.12)

The trace of the symmetrization operator yields the number of independent components of fully symmetric tensors:



(6.13)

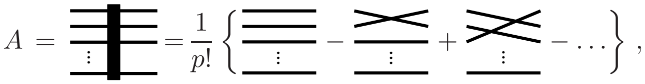
For example, for 2-index symmetric tensors,

$$d_S = n(n+1)/2 \,. \qquad (6.14)$$

6.2 ANTISYMMETRIZATION

The alternating sum of all permutations,

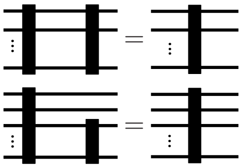
$$A_{a_1 a_2 \ldots a_p},{}^{b_p\ldots b_2 b_1} = \frac{1}{p!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_p}^{b_p} - \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_p}^{b_p} + \ldots\right\}$$



(6.15)

yields the antisymmetrization projection operator A. In birdtrack notation, antisymmetrization of p lines will always be denoted by a black bar drawn across the lines. As in the previous section

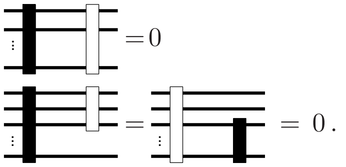
$$A^2 = A$$



(6.16)

and in addition

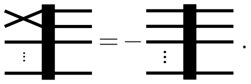
$$SA = 0$$



(6.17)

A transposition has eigenvalue -1 on the antisymmetric tensor space

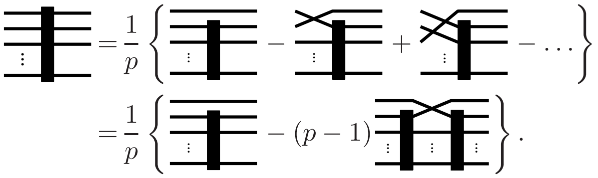
$$\sigma_{(i,i+1)} A = -A$$



(6.18)

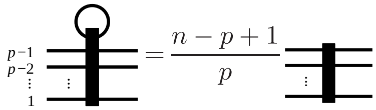
Diagrammatically this means that legs can be crossed and uncrossed at will, but with a factor of -1 for a transposition of any two neighboring legs.

As in the case of symmetrization operators, the recursive definition is often computationally convenient



(6.19)

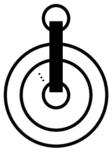
This is useful for computing contractions such as



$$A_{aa_{p-1}\ldots a_1},^{b_1\ldots b_{p-1}a} = \frac{n-p+1}{p} A_{a_{p-1}\ldots a_1},^{b_1\ldots b_{p-1}} \ .$$ (6.20)

The number of independent components of fully antisymmetric tensors is given by

$$d_A = \mathrm{tr}\, A \ =$$



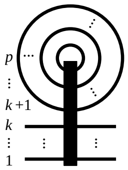
$$= \ \frac{n-p+1}{p}\frac{n-p+2}{p-1}\cdots\frac{n}{1}$$

$$= \left\{ \begin{array}{ll} \frac{n!}{p!(n-p)!}\,, & n \geq p \\ 0\,, & n < p \end{array} \right. \ .$$ (6.21)

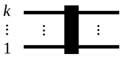
For example, for 2-index antisymmetric tensors the number of independent components is

$$d_A = \frac{n(n-1)}{2} \,. \tag{6.22}$$

Tracing $(p-k)$ pairs of indices yields

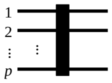


$$= \frac{k!(n-k)!}{p!(n-p)!}$$



$$. \tag{6.23}$$

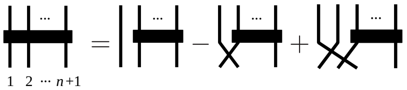
The antisymmetrization tensor $A_{a_1 a_2 \ldots,}{}^{b_p \ldots b_2 b_1}$ has nonvanishing components, only if all lower (or upper) indices differ from each other. If the defining dimension is smaller than the number of indices, the tensor A has no nonvanishing components:



$$= 0 \qquad \text{if } p > n \,. \tag{6.24}$$

This identity implies that for $p > n$, not all combinations of p Kronecker deltas are linearly independent. A typical relation is the $p = n+1$ case

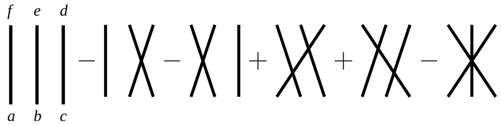
$$0 =$$



$$- \ldots \,. \tag{6.25}$$

For example, for $n = 2$ we have

$$n = 2: \quad 0 =$$



$$(6.26)$$

$$0 = \delta_a^f \delta_b^e \delta_c^d - \delta_a^f \delta_c^e \delta_b^d - \delta_b^f \delta_a^e \delta_c^d + \delta_b^f \delta_c^e \delta_a^d + \delta_c^f \delta_a^e \delta_b^d - \delta_c^f \delta_b^e \delta_a^d \,.$$

Chapter Nine

Unitary groups

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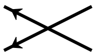
$U(n)$ is the group of all transformations that leave invariant the norm $\overline{q}q = \delta^a_b q^b q_a$ of a complex vector q. For $U(n)$ there are no other invariant tensors beyond those constructed of products of Kronecker deltas. They can be used to decompose the tensor reps of $U(n)$. For purely covariant or contravariant tensors, the symmetric group can be used to construct the Young projection operators. In sections. 9.1–9.2 we show how to do this for 2- and 3-index tensors by constructing the appropriate characteristic equations.

For tensors with more indices it is easier to construct the Young projection operators directly from the Young tableaux. In section 9.3 we review the Young tableaux, and in section 9.4 we show how to construct Young projection operators for tensors with any number of indices. As examples, 3- and 4-index tensors are decomposed in section 9.5. We use the projection operators to evaluate $3n$-j coefficients and characters of $U(n)$ in sections. 9.6–9.9, and we derive new sum rules for $U(n)$ 3-j and 6-j symbols in section 9.7. In section 9.8 we consider the consequences of the Levi-Civita tensor being an extra invariant for $SU(n)$.

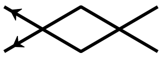
For mixed tensors the reduction also involves index contractions and the symmetric group methods alone do not suffice. In sections. 9.10–9.12 the mixed $SU(n)$ tensors are decomposed by the projection operator techniques introduced in chapter 3. $SU(2)$, $SU(3)$, $SU(4)$, and $SU(n)$ are discussed from the “invariance group" perspective in chapter 15.

9.1 TWO-INDEX TENSORS

Consider 2-index tensors $q^{(1)} \otimes q^{(2)} \in \otimes V^2$. According to (6.1), all permutations are represented by invariant matrices. Here there are only two permutations, the identity and the flip (6.2),

$$\sigma =$$  .

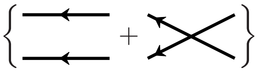
The flip satisfies

$$\sigma^2 =$$  $= \mathbf{1}\,,$

$$(\sigma + \mathbf{1})(\sigma - \mathbf{1}) = 0\,. \tag{9.1}$$

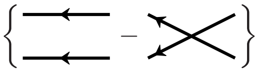
The eigenvalues are $\lambda_1 = 1, \lambda_2 = -1$, and the corresponding projection operators (3.48) are

$$\mathbf{P}_1 = \frac{\sigma - (-1)\mathbf{1}}{1-(-1)} = \frac{1}{2}(\mathbf{1}+\sigma) = \frac{1}{2}$$



$, \qquad (9.2)$

$$\mathbf{P}_2 = \frac{\sigma - \mathbf{1}}{-1-1} = \frac{1}{2}(\mathbf{1}-\sigma) = \frac{1}{2}$$

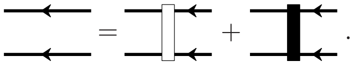


$. \qquad (9.3)$

We recognize the symmetrization, antisymmetrization operators (6.4), (6.15); $\mathbf{P}_1 = \mathbf{S}$, $\mathbf{P}_2 = \mathbf{A}$, with subspace dimensions $d_1 = n(n+1)/2$, $d_2 = n(n-1)/2$. In other words, under general linear transformations the symmetric and the antisymmetric parts of a tensor x_{ab} transform separately:

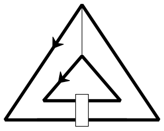
$$x = \mathbf{S}x + \mathbf{A}x\,,$$

$$x_{ab} = \frac{1}{2}(x_{ab}+x_{ba}) + \frac{1}{2}(x_{ab}-x_{ba})$$

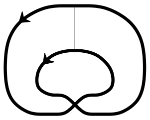


(9.4)

The Dynkin indices for the two reps follow by (7.29) from $6j's$:



$$= \frac{1}{2}(0) + \frac{1}{2}$$



$$= \frac{N}{2}$$

$$\ell_1 = \frac{2\ell}{n}\cdot d_1 + \frac{2\ell}{N}\cdot\frac{N}{2}$$
$$= \ell(n+2)\,. \qquad (9.5)$$

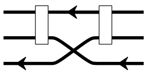
Substituting the defining rep Dynkin index $\ell^{-1} = C_A = 2n$, computed in section 2.2, we obtain the two Dynkin indices

$$\ell_1 = \frac{n+2}{2n}\,, \qquad \ell_2 = \frac{n-2}{2n}\,. \qquad (9.6)$$

9.2 THREE-INDEX TENSORS

Three-index tensors can be reduced to irreducible subspaces by adding the third index to each of the 2-index subspaces, the symmetric and the antisymmetric. The results of this section are summarized in figure 9.1 and table 9.1. We mix the third index into the symmetric 2-index subspace using the invariant matrix

$$\mathbf{Q} = \mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12} =$$

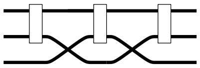


$. \qquad (9.7)$

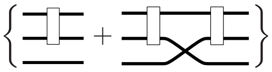
Here projection operators $\mathbf{S}_{12}$ ensure the restriction to the 2-index symmetric subspace, and the transposition $\sigma_{(23)}$ mixes in the third index. To find the characteristic equation for $\mathbf{Q}$, we compute $\mathbf{Q}^2$:

$$\mathbf{Q}^2 = \mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12} = \frac{1}{2}\left\{\mathbf{S}_{12} + \mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12}\right\} = \frac{1}{2}\mathbf{S}_{12} + \frac{1}{2}\mathbf{Q}$$

$=$



$$= \frac{1}{2}$$



$.$

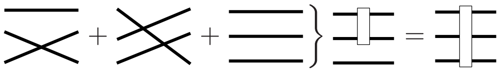
Hence, $\mathbf{Q}$ satisfies

$$(\mathbf{Q}-1)(\mathbf{Q}+1/2)\mathbf{S}_{12}=0\,, \tag{9.8}$$

and the corresponding projection operators (3.48) are

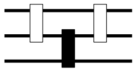
$$\mathbf{P}_1=\frac{\mathbf{Q}+\frac{1}{2}\mathbf{1}}{1+\frac{1}{2}}\mathbf{S}_{12}=\frac{1}{3}\left\{\sigma_{(23)}+\sigma_{(123)}+\mathbf{1}\right\}\mathbf{S}_{12}=\mathbf{S}$$

$$=\frac{1}{3}\Big\{$$



(9.9)

$$\mathbf{P}_2=\frac{\mathbf{Q}-\mathbf{1}}{-\frac{1}{2}-1}\mathbf{S}_{12}=\frac{4}{3}\mathbf{S}_{12}A_{23}\mathbf{S}_{12}=\frac{4}{3}$$

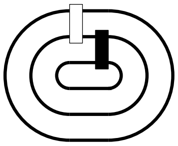


. (9.10)

Hence, the symmetric 2-index subspace combines with the third index into a symmetric 3-index subspace (6.13) and a mixed symmetry subspace with dimensions

$$d_1=\mathrm{tr}\,\mathbf{P}_1\;=\;n(n+1)(n+2)/3! \tag{9.11}$$

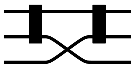
$$d_2=\mathrm{tr}\,\mathbf{P}_2\;=\;\frac{4}{3}$$



$$=\;n(n^2-1)/3\,. \tag{9.12}$$

The antisymmetric 2-index subspace can be treated in the same way using the invariant matrix

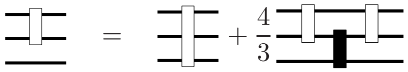
$$\mathbf{Q}=\mathbf{A}_{12}\sigma_{(23)}\mathbf{A}_{12}=$$



. (9.13)

The resulting projection operators for the antisymmetric and mixed symmetry 3-index tensors are given in figure 9.1. Symmetries of the subspace are indicated by the corresponding Young tableaux, table 9.2. For example, we have just constructed

$$\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\otimes\begin{array}{|c|}\hline 3\\\hline\end{array}=\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\oplus\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$$



$$\frac{n^2(n+1)}{2}=\frac{n(n+1)(n+2)}{3!}+\frac{n(n^2-1)}{3}\,. \tag{9.14}$$

The projection operators for tensors with up to 4 indices are shown in figure 9.1, and in figure 9.2 the corresponding stepwise reduction of the irreps is given in terms of Young standard tableaux (defined in section 9.3.1).

9.3 YOUNG TABLEAUX

We have seen in the examples of sections. 9.1–9.2 that the projection operators for 2-index and 3-index tensors can be constructed using characteristic equations. For tensors with more than three indices this method is cumbersome, and it is much simpler to construct the projection operators directly from the Young tableaux. In this section we review the Young tableaux and some aspects of symmetric group representations that will be important for our construction of the projection operators in section 9.4.

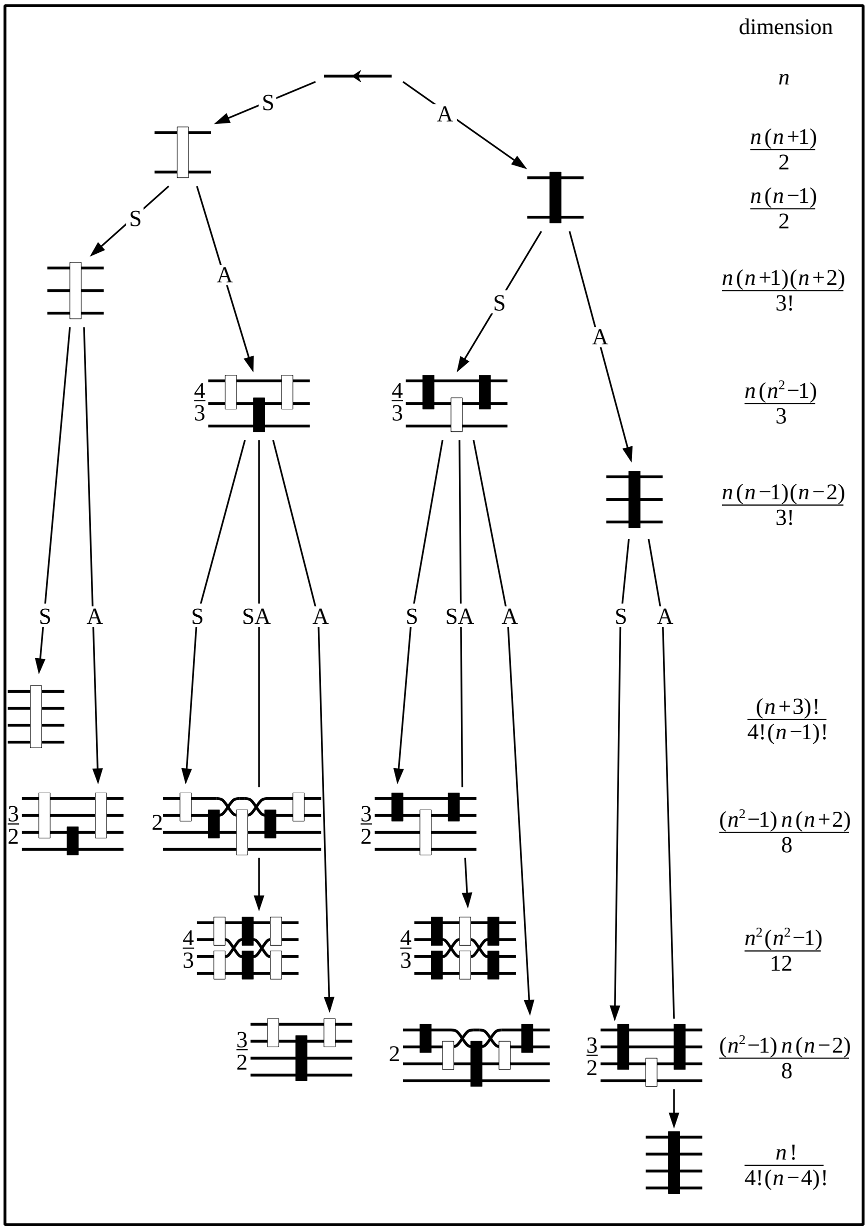


Figure 9.1 Projection operators for 2-, 3-, and 4-index tensors in $U(n)$, $SU(n)$, $n \geq p =$ number of indices.

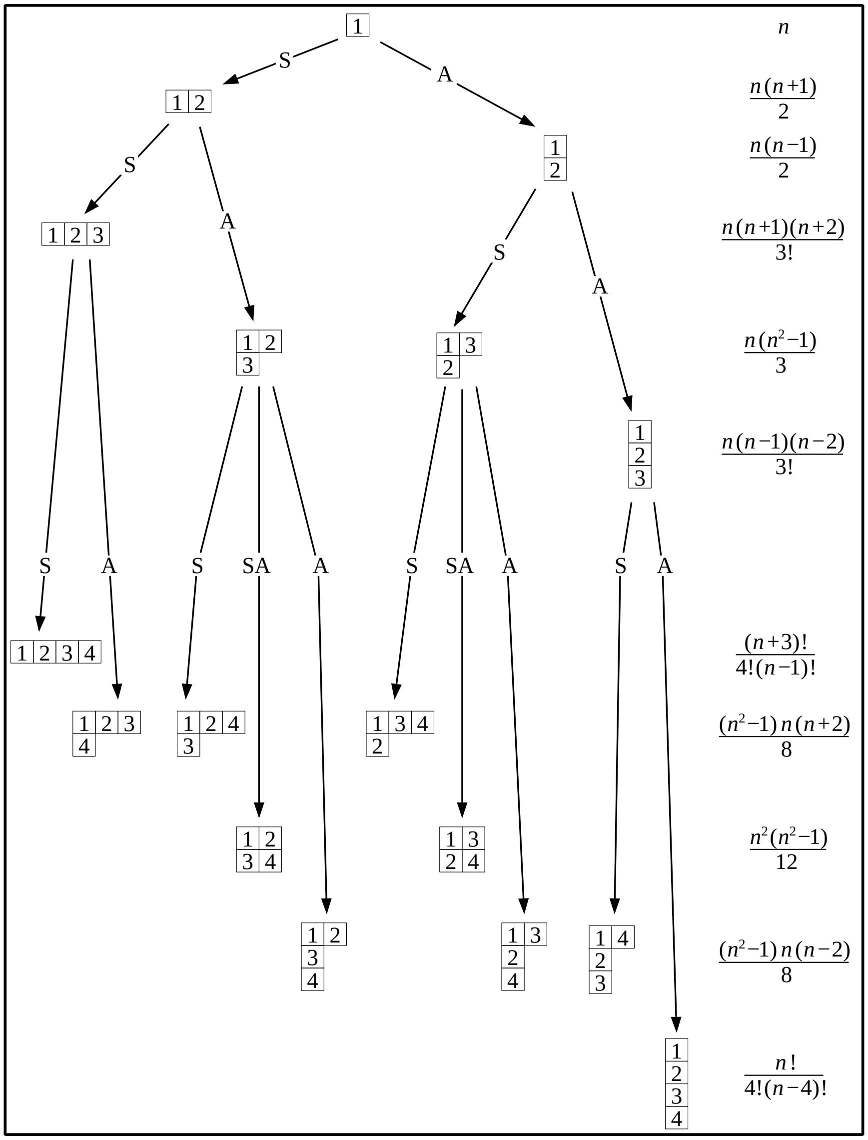
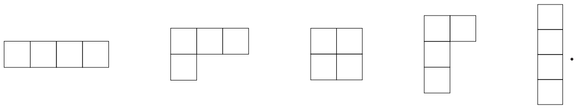


Figure 9.2 Young tableaux for the irreps of the symmetric group for 2-, 3-, and 4-index tensors. Rows correspond to symmetrizations, columns to antisymmetrizations. The reduction procedure is not unique, as it depends on the order in which the indices are combined; this order is indicated by labels 1, 2, 3 , ..., p in the boxes of Young tableaux.

9.3.1 Definitions

Partition k identical boxes into D subsets, and let λ_m, $m = 1, 2, \ldots, D$, be the number of boxes in the subsets ordered so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_D \geq 1$. Then the partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_D]$ fulfills $\sum_{m=1}^{D} \lambda_m = k$. The diagram obtained by drawing the D rows of boxes on top of each other, left aligned, starting with λ_1 at the top, is called a *Young diagram* Y.

Examples:
The ordered partitions for $k = 4$ are $[4], [3, 1], [2, 2], [2, 1, 1]$ and $[1, 1, 1, 1]$. The corresponding Young diagrams are



Inserting a number from the set $\{1, \ldots, n\}$ into every box of a Young diagram Y_λ in such a way that numbers increase when reading a column from top to bottom, and numbers do not decrease when reading a row from left to right, yields a *Young tableau* Y_a. The subscript a labels different tableaux derived from a given Young diagram, *i.e.*, different admissible ways of inserting the numbers into the boxes.

A *standard tableau* is a k-box Young tableau constructed by inserting the numbers $1, \ldots, k$ according to the above rules, but using each number exactly once. For example, the 4-box Young diagram with partition $\lambda = [2, 1, 1]$ yields three distinct standard tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \;, \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \;, \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \;. \tag{9.15}$$

An alternative labeling of a Young diagram are Dynkin labels, the list of numbers b_m of columns with m boxes: $(b_1 b_2 \ldots)$. Having k boxes we must have $\sum_{m=1}^{k} m b_m = k$. For example, the partition $[4, 2, 1]$ and the labels $(21100 \cdots)$ give rise to the same Young diagram, and so do the partition $[2, 2]$ and the labels $(020 \cdots)$.

We define the *transpose* diagram Y^t as the Young diagram obtained from Y by interchanging rows and columns. For example, the transpose of $[3, 1]$ is $[2, 1, 1]$,

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}^{\,t} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \;,$$

or, in terms of Dynkin labels, the transpose of $(210\ldots)$ is $(1010\ldots)$.

The Young tableaux are useful for labeling irreps of various groups. We shall use the following facts (see for instance ref. [153]):

1. The k-box *Young diagrams* label all irreps of the symmetric group S_k.
2. The *standard tableaux* of k-box Young diagrams with no more than n rows label the irreps of $GL(n)$, in particular they label the irreps of $U(n)$.

3. The *standard tableaux* of k-box Young diagrams with no more than $n - 1$ rows label the irreps of $SL(n)$, in particular they label the irreps of $SU(n)$.

In this section, we consider the Young tableaux for reps of S_k and $U(n)$, while the case of $SU(n)$ is postponed to section 9.8.

9.3.2 Symmetric group S_k

The irreps of the symmetric group S_k are labeled by the k-box Young diagrams. For a given Young diagram, the basis vectors of the corresponding irrep can be labeled by the standard tableaux of Y; consequently the dimension Δ_{Y} of the irrep is the number of standard tableaux that can be constructed from the Young diagram Y. The example (9.15) shows that the irrep $\lambda = [2, 1, 1]$ of S_4 is 3-dimensional.

As an alternative to counting standard tableaux, the dimension Δ_{Y} of the irrep of S_k corresponding to the Young diagram Y can be computed easily as

$$\Delta_{\mathrm{Y}} = \frac{k!}{|\mathrm{Y}|}\,, \tag{9.16}$$

where the number $|\mathrm{Y}|$ is computed using a “hook” rule: Enter into each box of the Young diagram the number of boxes below and to the right of the box, including the box itself. Then $|\mathrm{Y}|$ is the product of the numbers in all the boxes. For instance,

$$\mathrm{Y} = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \\ \cline{1-3} & \\ \cline{1-2} \end{array} \qquad \longrightarrow \qquad |\mathrm{Y}| = \begin{array}{|c|c|c|c|}\hline 6 & 5 & 3 & 1 \\ \hline 4 & 3 & 1 \\ \cline{1-3} 2 & 1 \\ \cline{1-2} \end{array} = 6!\,3\,. \tag{9.17}$$

The hook rule (9.16) was first proven by Frame, de B. Robinson, and Thrall [123]. Various proofs can be found in the literature [296, 170, 133, 142, 21]; see also Sagan [303] and references therein.

We now discuss the regular representation of the symmetric group. The elements $\sigma \in S_k$ of the symmetric group S_k form a basis of a $k!$-dimensional vector space V of elements

$$s = \sum_{\sigma \in S_k} s_\sigma\, \sigma \;\in V\,, \tag{9.18}$$

where s_σ are the components of a vector s in the given basis. If $s \in V$ has components (s_σ) and $\tau \in S_k$, then τs is an element in V with components $(\tau s)_\sigma = s_{\tau^{-1}\sigma}$. This action of the group elements on the vector space V defines an $k!$-dimensional matrix representation of the group S_k, the *regular representation*.

The regular representation is reducible, and each irrep λ appears Δ_λ times in the reduction; Δ_λ is the dimension of the subspace V_λ corresponding to the irrep λ. This gives the well-known relation between the order of the symmetric group $|S_k| = k!$ (the dimension of the regular representation) and the dimensions of the irreps,

$$|S_k| = \sum_{\text{all irreps } \lambda} \Delta_\lambda^2\,.$$

Using (9.16) and the fact that the Young diagrams label the irreps of S_k, we have

$$1 = k! \sum_{(k)} \frac{1}{|Y|^2}\,, \tag{9.19}$$

where the sum is over all Young diagrams with k boxes. We shall use this relation to determine the normalization of Young projection operators in appendix B.3.

The reduction of the regular representation of S_k gives a completeness relation,

$$\mathbf{1} = \sum_{(k)} \mathbf{P}_{\mathrm{Y}} \,,$$

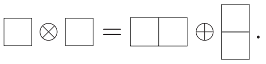
in terms of projection operators

$$\mathbf{P}_{\mathrm{Y}} = \sum_{\mathrm{Y}_a \in \mathrm{Y}} \mathbf{P}_{\mathrm{Y}_a} \,.$$

The sum is over all standard tableaux derived from the Young diagram Y. Each $\mathbf{P}_{\mathrm{Y}_a}$ projects onto a corresponding invariant subspace V_{Y_a}: for each Y there are Δ_{Y} such projection operators (corresponding to the Δ_{Y} possible standard tableaux of the diagram), and each of these project onto one of the Δ_{Y} invariant subspaces V_{Y} of the reduction of the regular representation. It follows that the projection operators are orthogonal and that they constitute a complete set.

9.3.3 Unitary group $U(n)$

The irreps of $U(n)$ are labeled by the k-box Young standard tableaux with no more than n rows. A k-index tensor is represented by a Young diagram with k boxes — one typically thinks of this as a k-particle state. For $U(n)$, a 1-index tensor has n-components, so there are n 1-particle states available, and this corresponds to the n-dimensional fundamental rep labeled by a 1-box Young diagram. There are n^2 2-particle states for $U(n)$, and as we have seen in section 9.1 these split into two irreps: the symmetric and the antisymmetric. Using Young diagrams, we write the reduction of the 2-particle system as



(9.20)

Except for the fully symmetric and the fully antisymmetric irreps, the irreps of the k-index tensors of $U(n)$ have mixed symmetry. Boxes in a row correspond to indices that are symmetric under interchanges (symmetric multiparticle states), and boxes in a column correspond to indices antisymmetric under interchanges (antisymmetric multiparticle states). Since there are only n labels for the particles, no more than n particles can be antisymmetrized, and hence only standard tableaux with up to n rows correspond to irreps of $U(n)$.

The number of standard tableaux Δ_{Y} derived from a Young diagram Y is given in (9.16). In terms of irreducible tensors, the Young diagram determines the symmetries of the indices, and the Δ_{Y} distinct standard tableaux correspond to the independent ways of combining the indices under these symmetries. This is illustrated in figure 9.2.

For a given $U(n)$ irrep labeled by some standard tableau of the Young diagram Y, the basis vectors are labeled by the Young tableaux Y_a obtained by inserting the numbers $1, 2, \ldots, n$ into Y in the manner described in section 9.3.1. Thus the dimension of an irrep of $U(n)$ equals the number of such Young tableaux, and we note that all irreps with the same Young diagram have the same dimension. For $U(2)$, the $k = 2$ Young tableaux of the symmetric and antisymmetric irreps are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\,, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array}\,, \quad \text{and} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\,,$$

so the symmetric state of $U(2)$ is 3-dimensional and the antisymmetric state is 1-dimensional, in agreement with the formulas (6.4) and (6.15) for the dimensions of the symmetry operators. For $U(3)$, the counting of Young tableaux shows that the symmetric 2-particle irrep is 6-dimensional and the antisymmetric 2-particle irrep is 3-dimensional, again in agreement with (6.4) and (6.15). In section 9.4.3 we state and prove a dimension formula for a general irrep of $U(n)$.

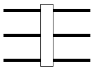
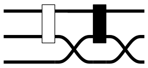
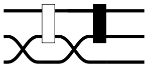
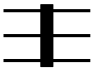
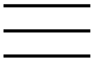
Y_a	$\mathbf{P}_{\mathrm{Y}_a}$	d_{Y_a}
$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline\end{array}$		$\frac{n(n+1)(n+2)}{6}$
$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$	$\frac{4}{3}$ 	$\frac{n(n^2-1)}{3}$
$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$	$\frac{4}{3}$ 	
$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$		$\frac{(n-2)(n-1)n}{6}$
$\boxed{1}\otimes\boxed{2}\otimes\boxed{3}$		n^3

Table 9.1 Reduction of 3-index tensor. The last row shows the direct sum of the Young tableaux, the sum of the dimensions of the irreps adding up to n^3, and the sum of the projection operators adding up to the identity as verification of completeness (3.51).

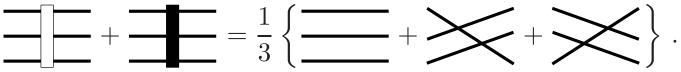
of direct products stated below, in section 9.5.1. We have already treated the decomposition of the 2-index tensor into the symmetric and the antisymmetric tensors, but we shall reconsider the 3-index tensor, since the projection operators are different from those derived from the characteristic equations in section 9.2.

The 3-index tensor reduces to

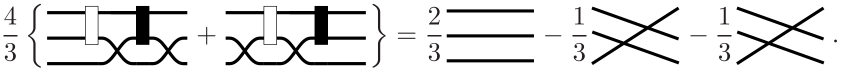
$$\begin{aligned}\boxed{1}\otimes\boxed{2}\otimes\boxed{3} &= \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\oplus\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right)\otimes\boxed{3}\\ &= \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\oplus\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}\oplus\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\,.\end{aligned}\tag{9.32}$$

The corresponding dimensions and Young projection operators are given in table 9.1. For simplicity, we neglect the arrows on the lines where this leads to no confusion.

The Young projection operators are orthogonal by inspection. We check completeness by a computation. In the sum of the fully symmetric and the fully antisymmetric tensors, all the odd permutations cancel, and we are left with



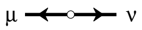
Expanding the two tensors of mixed symmetry, we obtain



Chapter Ten

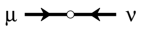
Orthogonal groups

Orthogonal group $SO(n)$ is the group of transformations that leaves invariant a symmetric quadratic form $(q,q) = g_{\mu\nu}q^\mu q^\nu$:

$g_{\mu\nu} = g_{\nu\mu} =$ 

$\mu,\nu = 1,2,\ldots,n\,.$ (10.1)

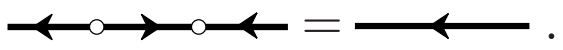
If (q,q) is an invariant, so is its complex conjugate $(q,q)^* = g^{\mu\nu}q_\mu q_\nu$, and

$g^{\mu\nu} = g^{\nu\mu} =$ 

(10.2)

is also an invariant tensor. The matrix $A_\mu^\nu = g_{\mu\sigma}g^{\sigma\nu}$ must be proportional to unity, as otherwise its characteristic equation would decompose the defining n-dimensional rep. A convenient normalization is

$$g_{\mu\sigma}g^{\sigma\nu} = \delta_\mu^\nu$$

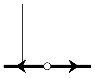
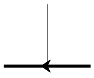

(10.3)

As the indices can be raised and lowered at will, nothing is gained by keeping the arrows. Our convention will be to perform all contractions with metric tensors with upper indices and omit the arrows and the open dots:

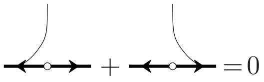
$g^{\mu\nu} \equiv$ 

(10.4)

All other tensors will have lower indices. For example, Lie group generators $(T_i)_\mu{}^\nu$ from (4.31) will be replaced by

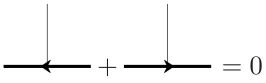
$(T_i)_\mu{}^\nu =$  $\to (T_i)_{\mu\nu} =$  .

The invariance condition (4.36) for the metric tensor



$$(T_i)_\mu{}^\sigma g_{\sigma\nu} + (T_i)_\nu{}^\sigma g_{\mu\sigma} = 0 \qquad (10.5)$$

becomes, in this convention, a statement that the $SO(n)$ generators are antisymmetric:



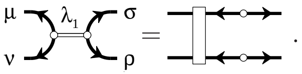
$$(T_i)_{\mu\nu} = -\,(T_i)_{\nu\mu}\,. \qquad (10.6)$$

Our analysis of the reps of $SO(n)$ will depend only on the existence of a symmetric metric tensor and its invertability, and not on its eigenvalues. The resulting Clebsch-Gordan series applies both to the compact $SO(n)$ and noncompact orthogonal groups, such as the Minkowski group $SO(1,3)$. In this chapter, we outline the construction of $SO(n)$ tensor reps. Spinor reps will be taken up in chapter 11.

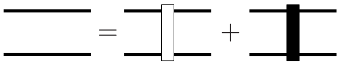
10.1 TWO-INDEX TENSORS

In section 9.1 we have decomposed the $SU(n)$ 2-index tensors into symmetric and antisymmetric parts. For $SO(n)$, the rule is to lower all indices on all tensors, and the symmetric state projection operator (9.2) is replaced by

$$\begin{aligned} S_{\mu\nu,\rho\sigma} &= g_{\rho\rho'} g_{\sigma\sigma'} S_{\mu\nu,}{}^{\rho'\sigma'} \\ &= \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}\right) \end{aligned}$$



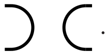
From now on, we drop all arrows and $g^{\mu\nu}$'s and write (9.4) as



$$g_{\mu\sigma}g_{\nu\rho} = \frac{1}{2}(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}) + \frac{1}{2}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) \,. \qquad (10.7)$$

The new invariant, specific to $SO(n)$, is the index contraction:

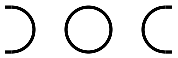
$$\mathbf{T}_{\mu\nu,\rho\sigma} = g_{\mu\nu}g_{\rho\sigma}\,, \qquad \mathbf{T} = $$



$$(10.8)$$

The characteristic equation for the trace invariant

$$\mathbf{T}^2 = $$

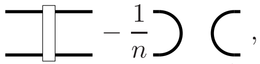


$$= n\mathbf{T} \qquad (10.9)$$

yields the trace and the traceless part projection operators (9.53), (9.54). As $\mathbf{T}$ is symmetric, $S\mathbf{T} = \mathbf{T}$, only the symmetric subspace is resolved by this invariant. The final decomposition of $SO(n)$ 2-index tensors is

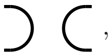
traceless symmetric:

$$(P_2)_{\mu\nu,\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}\right) - \frac{1}{n}g_{\mu\nu}g_{\rho\sigma} = $$



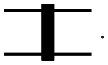
$$(10.10)$$

$$\text{singlet:}\quad (P_1)_{\mu\nu,\rho\sigma} = \frac{1}{n}g_{\mu\nu}g_{\rho\sigma} = \frac{1}{n}$$



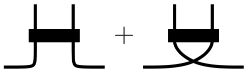
$$(10.11)$$

$$\text{antisymmetric:}\quad (P_3)_{\mu\nu,\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}\right) = $$



$$(10.12)$$

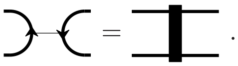
The adjoint rep (9.57) of $SU(n)$ is decomposed into the traceless symmetric and the antisymmetric parts. To determine which of them is the new adjoint rep, we substitute them into the invariance condition (10.5). Only the antisymmetric projection operator satisfies the invariance condition



$$= 0\,,$$

so the adjoint rep projection operator for $SO(n)$ is

$$\frac{1}{a}$$



$$(10.13)$$

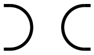
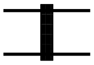
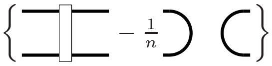
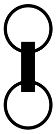
Young tableaux	☐ × ☐	=	•	+		+	☐☐
Dynkin labels	$(10\ldots) \times (10\ldots)$	=	$(00\ldots)$	+	$(010\ldots)$	+	$(20\ldots)$
Dimensions	n^2	=	1	+	$\frac{n(n-1)}{2}$	+	$\frac{(n+2)(n-1)}{2}$
Dynkin indices	$2n\frac{1}{n-2}$	=	0	+	1	+	$\frac{n+2}{n-2}$
Projectors		=	$\frac{1}{n}$ 	+		+	

Table 10.1 $SO(n)$ Clebsch-Gordan series for $V \otimes V$.

The dimension of $SO(n)$ is given by the trace of the adjoint projection operator:

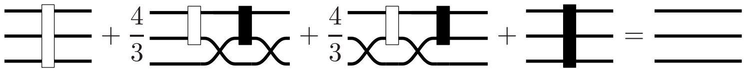
$$N = \mathrm{tr}\,\mathbf{P}_A =$$



$$= \frac{n(n-1)}{2} \,. \tag{10.14}$$

Dimensions of the other reps and the Dynkin indices (see section 7.5) are listed in table 10.1.

Adding the two equations we get



,(9.33)

verifying the completeness relation.

For 4-index tensors the decomposition is performed as in the 3-index case, resulting in table 9.2.

Acting with any permutation on the fully symmetric or antisymmetric projection operators gives ±1 times the projection operator (see (6.8) and (6.18)). For projection operators of mixed symmetry the action of a permutation is not as simple, because the permutations will mix the spaces corresponding to the distinct tableaux. Here we shall need only the action of a permutation within a $3n$-j symbol, and, as we shall show below, in this case the result will again be simple, a factor ±1 or 0.

Chapter Twenty

E_7 family and its negative-dimensional cousins

Parisi and Sourlas [270] have suggested that a Grassmann vector space of dimension n can be interpreted as an ordinary vector space of dimension $-n$. As we have seen in chapter 13, semisimple Lie groups abound with examples in which an $n \to -n$ substitution can be interpreted in this way. An early example was Penrose's binors [281], reps of $SU(2) = Sp(2)$ constructed as $SO(-2)$, and discussed here in chapter 14. This is a special case of a general relation between $SO(n)$ and $Sp(-n)$ established in chapter 13; if symmetrizations and antisymmetrizations are interchanged, reps of $SO(n)$ become $Sp(-n)$ reps. Here we work out in detail a 1977 example of a negative-dimensions relation [74], subsequently made even more intriguing [78] by Cremmer and Julia's discovery of a global E_7 symmetry in supergravity [68].

We extend the Minkowski space into Grassmann dimensions by requiring that the invariant length and volume that characterize the Lorentz group ($SO(3,1)$ or $SO(4)$ — compactness plays no role in this analysis) become a quadratic and a quartic supersymmetric invariant. The symmetry group of the Grassmann sector will turn out to be one of $SO(2)$, $SU(2)$, $SU(2) \times SU(2) \times SU(2)$, $Sp(6)$, $SU(6)$, $SO(12)$, or E_7, which also happens to be the list of possible global symmetries of extended supergravities.

As shown in chapter 10, $SO(4)$ is the invariance group of the Kronecker delta $g_{\mu\nu}$ and the Levi-Civita tensor $\varepsilon_{\mu\nu\sigma\rho}$; hence, we are looking for the invariance group of the supersymmetric invariants

$$\begin{aligned} (x, y) &= g_{\mu\nu} x^\mu y^\nu , \\ (x, y, z, w) &= e_{\mu\nu\sigma\rho} x^\mu y^\nu z^\sigma w^\rho , \end{aligned} \tag{20.1}$$

where $\mu, \nu, \ldots = 4, 3, 2, 1, -1, -2, \ldots, -n$. Our motive for thinking of the Grassmann dimensions as $-n$ is that we define the dimension as a trace (3.52), $n = \delta^\mu_\mu$, and in a Grassmann (or fermionic) world each trace carries a minus sign. For the quadratic invariant $g_{\mu\nu}$ alone, the invariance group is the orthosymplectic $OSp(4, n)$. This group [177] is orthogonal in the bosonic dimensions and symplectic in the Grassmann dimensions, because if $g_{\mu\nu}$ is symmetric in the $\nu, \mu > 0$ indices, it must be antisymmetric in the $\nu, \mu < 0$ indices. In this way the supersymmetry ties in with the $SO(n) \sim Sp(-n)$ equivalence developed in chapter 13.

Following this line of reasoning, a quartic invariant tensor $e_{\mu\nu\sigma\rho}$, antisymmetric in ordinary dimensions, is symmetric in the Grassmann dimensions. Our task is then to determine all groups that admit an antisymmetric quadratic invariant, together with a symmetric quartic invariant.

20.3.1 $SO(4)$ or $A_1 + A_1$ algebra

The first solution, $d = 4$, is not a surprise; it was $SO(4)$, Minkowski or euclidean version, that motivated the whole project. The quartic invariant is the Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$. Even so, the projectors constructed are interesting. Taking

$$Q^{\mu\,\delta}_{\nu\,\rho} = g^{\mu\varepsilon} g^{\delta\rho} \varepsilon_{\varepsilon\sigma\nu\gamma} \,, \tag{20.25}$$

one can immediately calculate (20.6):

$$\mathbf{Q}^2 = 4\mathbf{P}_3 \,. \tag{20.26}$$

The projectors (20.14) become

$$\mathbf{P}_A = \frac{1}{2}\mathbf{P}_3 + \frac{1}{4}\mathbf{Q}, \quad \mathbf{P}_7 = \frac{1}{2}\mathbf{P}_3 - \frac{1}{4}\mathbf{Q} \,, \tag{20.27}$$

and the dimensions are $N = d_7 = 3$. Also both $\mathbf{P}_A$ and $\mathbf{P}_7$ satisfy the invariance condition, the adjoint rep splits into two invariant subspaces. In this way, one shows that the Lie algebra of $SO(4)$ is the semisimple $SU(2) + SU(2) = A_1 + A_1$. Furthermore, the projection operators are precisely the $\eta, \overline{\eta}$ symbols used by 't Hooft [164] to map the self-dual and self-antidual $SO(4)$ antisymmetric tensors onto $SU(2)$ gauge group:

$$\begin{aligned}
(\mathbf{P}_A)^{\mu\delta}_{\nu\rho} &= \frac{1}{4}\left(\delta^\mu_\rho \delta^\delta_\nu - g^{\mu\delta} g_{\nu\rho} + \varepsilon^{\mu\delta}{}_{\nu\rho}\right) = -\frac{1}{4}\, \eta_a{}^\mu_\nu \, \eta_a{}^\delta_\rho \,, \\
(\mathbf{P}_7)^{\mu\delta}_{\nu\rho} &= \frac{1}{4}\left(\delta^\mu_\rho \delta^\delta_\nu - g^{\mu\delta} g_{\nu\rho} - \varepsilon^{\mu\delta}{}_{\nu\rho}\right) = -\frac{1}{4}\, \overline{\eta}_a{}^\mu_\nu \, \overline{\eta}_a{}^\delta_\rho \,.
\end{aligned} \tag{20.28}$$

The only difference is that instead of using an index pair ${}^\mu_\nu$, 't Hooft indexes the adjoint spaces by $a = 1, 2, 3$. All identities, listed in the appendix of ref. [164], now follow from the relations of section 20.1.