

## Appendix A24

# Deterministic diffusion

**L**ATTICE PROPAGATORS are derived here as sums of all walks on a lattice.

### A24.1 Lattice derivatives

In order to set up continuum field-theoretic equations which describe the evolution of spatial variations of fields, we need to define *lattice derivatives*.

Consider a smooth function  $\phi(x)$  evaluated on a  $d$ -dimensional lattice

$$\phi_\ell = \phi(x), \quad x = a\ell = \text{lattice point}, \quad \ell \in \mathbf{Z}^d, \quad (\text{A24.1})$$

where  $a$  is the lattice spacing. Each set of values of  $\phi(x)$  (a vector  $\phi_\ell$ ) is a possible lattice state (or ‘configuration’). Assume the lattice is hyper-cubic, and let  $\hat{n}_\mu \in \{\hat{n}_1, \hat{n}_2, \dots, \hat{n}_d\}$  be the unit lattice cell vectors pointing along the  $d$  positive directions. The forward *lattice derivative* is then

$$(\partial_\mu \phi)_\ell = \frac{\phi(x + a\hat{n}_\mu) - \phi(x)}{a} = \frac{\phi_{\ell + \hat{n}_\mu} - \phi_\ell}{a}. \quad (\text{A24.2})$$

The backward *lattice derivative* is defined as the transpose

$$(\partial_\mu \phi)_\ell^\top = \frac{\phi(x - a\hat{n}_\mu) - \phi(x)}{a} = \frac{\phi_{\ell - \hat{n}_\mu} - \phi_\ell}{a}. \quad (\text{A24.3})$$

Anything else with the correct  $a \rightarrow 0$  limit would do, but this is the simplest choice. We can rewrite the lattice derivative as a linear operator, by introducing the *stepping operator* in the direction  $\mu$

$$(\sigma_\mu)_{\ell j} = \delta_{\ell + \hat{n}_\mu, j}. \quad (\text{A24.4})$$

As  $\sigma$  will play a central role in what follows, it pays to understand what it does.

In computer discretizations, the lattice will be a finite  $d$ -dimensional hyper-cubic lattice

$$\phi_\ell = \phi(x), \quad x = a\ell = \text{lattice point}, \quad \ell \in (\mathbf{Z}/N)^d, \quad (\text{A24.5})$$

where  $a$  is the lattice spacing and there are  $N^d$  points in all. For a hyper-cubic lattice the translations in different directions commute,  $\sigma_\mu\sigma_\nu = \sigma_\nu\sigma_\mu$ , so it is sufficient to understand the action of (A24.4) on a 1-dimensional lattice.

Let us write down  $\sigma$  for the 1-dimensional case in its full  $[N \times N]$  matrix glory. Writing the finite lattice stepping operator (A24.4) as an ‘upper shift’ matrix,

$$\sigma = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & 0 & 1 \\ 0 & & & & & 0 \end{bmatrix}, \quad (\text{A24.6})$$

is no good, as  $\sigma$  so defined is nilpotent, and after  $N$  steps the particle marches off the lattice edge, and nothing is left,  $\sigma^N = 0$ . A sensible way to approximate an infinite lattice by a finite one is to replace it by a lattice periodic in each  $\hat{n}_\mu$  direction. On a *periodic lattice* every point is equally far from the ‘boundary’  $N/2$  steps away, the ‘surface’ effects are equally negligible for all points, and the stepping operator acts as a cyclic permutation matrix

$$\sigma = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{bmatrix}, \quad (\text{A24.7})$$

with ‘1’ in the lower left corner assuring periodicity.

Applied to the lattice state  $\phi = (\phi_1, \phi_2, \dots, \phi_N)$ , the stepping operator translates the state by one site,  $\sigma\phi = (\phi_2, \phi_3, \dots, \phi_N, \phi_1)$ . Its transpose translates the configuration the other way, so the transpose is also the inverse,  $\sigma^{-1} = \sigma^T$ . The partial lattice derivative (A24.3) can now be written as a multiplication by a matrix:

$$\partial_\mu \phi_\ell = \frac{1}{a} (\sigma_\mu - \mathbf{1})_{\ell j} \phi_j.$$

In the 1-dimensional case the  $[N \times N]$  matrix representation of the lattice derivative is:

$$\partial = \frac{1}{a} \begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \ddots \\ & & & & -1 & 1 \\ 1 & & & & & -1 \end{bmatrix}. \quad (\text{A24.8})$$





## A24.2 Periodic lattices

Our task now is to transform  $M$  into a form suitable to explicit evaluation.

Consider the effect of a lattice translation  $\phi \rightarrow \sigma\phi$  on the matrix polynomial

$$S[\sigma\phi] = -\frac{1}{2}\phi^\top (\sigma^\top M^{-1}\sigma)\phi.$$

As  $M^{-1}$  is constructed from  $\sigma$  and its inverse,  $M^{-1}$  and  $\sigma$  commute, and  $S[\phi]$  is invariant under translations,

$$S[\sigma\phi] = S[\phi] = -\frac{1}{2}\phi^\top \cdot \frac{1}{M} \cdot \phi. \quad (\text{A24.16})$$

If a function defined on a vector space commutes with a linear operator  $\sigma$ , then the eigenvalues of  $\sigma$  can be used to decompose the  $\phi$  vector space into invariant subspaces. For a hyper-cubic lattice the translations in different directions commute,  $\sigma_\mu\sigma_\nu = \sigma_\nu\sigma_\mu$ , so it is sufficient to understand the spectrum of the 1-dimensional stepping operator (A24.7). To develop a feeling for how this reduction to invariant subspaces works in practice, let us proceed cautiously, by expanding the scope of our deliberations to a lattice consisting of 2 points.

### A24.2.1 A 2-point lattice diagonalized

The action of the stepping operator  $\sigma$  (A24.7) on a 2-point lattice  $\phi = (\phi_0, \phi_1)$  is to permute the two lattice sites

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

As exchange repeated twice brings us back to the original state,  $\sigma^2 = \mathbf{1}$ , the characteristic polynomial of  $\sigma$  is

$$(\sigma + 1)(\sigma - 1) = 0,$$

with eigenvalues  $\lambda_0 = 1, \lambda_1 = -1$ . The symmetrization, antisymmetrization projection operators are

$$P_0 = \frac{\sigma - \lambda_1 \mathbf{1}}{\lambda_0 - \lambda_1} = \frac{1}{2}(\mathbf{1} + \sigma) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (\text{A24.17})$$

$$P_1 = \frac{\sigma - \lambda_0 \mathbf{1}}{\lambda_1 - \lambda_0} = \frac{1}{2}(\mathbf{1} - \sigma) = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (\text{A24.18})$$

Noting that  $P_0 + P_1 = \mathbf{1}$ , we can project a lattice state  $\phi$  onto the two normalized eigenvectors of  $\sigma$ :

$$\begin{aligned} \phi &= \mathbf{1}\phi = P_0 \cdot \phi + P_1 \cdot \phi, \\ \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} &= \frac{(\phi_0 + \phi_1)}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{(\phi_0 - \phi_1)}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned} \quad (\text{A24.19})$$

$$= \tilde{\phi}_0 \varphi_0 + \tilde{\phi}_1 \varphi_1. \quad (\text{A24.20})$$

As  $P_0P_1 = 0$ , the symmetric and the antisymmetric states transform separately under any linear transformation constructed from  $\sigma$  and its powers.

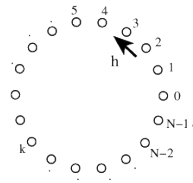
In this way the characteristic equation  $\sigma^2 = \mathbf{1}$  enables us to reduce the 2-dimensional lattice state to two 1-dimensional ones, on which the value of the stepping operator  $\sigma$  is a number,  $\lambda \in \{1, -1\}$ , and the normalized eigenvectors are  $\varphi_0 = \frac{1}{\sqrt{2}}(1, 1)$ ,  $\varphi_1 = \frac{1}{\sqrt{2}}(1, -1)$ . As we shall now see,  $(\tilde{\phi}_0, \tilde{\phi}_1)$  is the 2-site periodic lattice discrete Fourier transform of the field  $(\phi_1, \phi_2)$ .

### A24.3 Discrete Fourier transforms

Let us generalize this reduction to a 1-dimensional periodic lattice with  $N$  sites.

Each application of  $\sigma$  translates the lattice one step; in  $N$  steps the lattice is back in the original state

$$\sigma^N = \mathbf{1}$$



so the eigenvalues of  $\sigma$  are the  $N$  distinct  $N$ -th roots of unity

$$\sigma^N - \mathbf{1} = \prod_{k=0}^{N-1} (\sigma - \omega^k \mathbf{1}) = 0, \quad \omega = e^{i\frac{2\pi}{N}}. \tag{A24.21}$$

As the eigenvalues are all distinct and  $N$  in number, the space is decomposed into  $N$  1-dimensional subspaces. The general theory (expounded in appendix A10.2) associates with the  $k$ -th eigenvalue of  $\sigma$  a projection operator that projects a state  $\phi$  onto  $k$ -th eigenvector of  $\sigma$ ,

$$P_k = \prod_{j \neq k} \frac{\sigma - \lambda_j \mathbf{1}}{\lambda_k - \lambda_j}. \tag{A24.22}$$

A factor  $(\sigma - \lambda_j \mathbf{1})$  kills the  $j$ -th eigenvector  $\varphi_j$  component of an arbitrary vector in expansion  $\phi = \dots + \tilde{\phi}_j \varphi_j + \dots$ . The above product kills everything but the eigen-direction  $\varphi_k$ , and the factor  $\prod_{j \neq k} (\lambda_k - \lambda_j)$  ensures that  $P_k$  is normalized as a projection operator. The set of the projection operators is complete,

$$\sum_k P_k = \mathbf{1}, \tag{A24.23}$$

and orthonormal

$$P_k P_j = \delta_{kj} P_k \quad (\text{no sum on } k). \tag{A24.24}$$

Constructing explicit eigenvectors is usually not a the best way to fritter one's youth away, as choice of basis is largely arbitrary, and all of the content of the theory is in projection operators (see appendix A10.2). However, in case at hand

the eigenvectors are so simple that we can construct the solutions of the eigenvalue condition

$$\sigma \varphi_k = \omega^k \varphi_k \tag{A24.25}$$

by hand:

$$\frac{1}{\sqrt{N}} \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(N-1)k} \end{bmatrix} = \omega^k \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(N-1)k} \end{bmatrix}$$

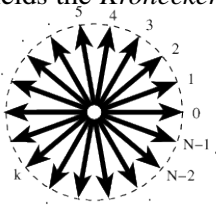
The  $1/\sqrt{N}$  factor is chosen in order that  $\varphi_k$  be normalized complex unit vectors

$$\begin{aligned} \varphi_k^\dagger \cdot \varphi_k &= \frac{1}{N} \sum_{k=0}^{N-1} 1 = 1, \quad (\text{no sum on } k) \\ \varphi_k^\dagger &= \frac{1}{\sqrt{N}} (1, \omega^{-k}, \omega^{-2k}, \dots, \omega^{-(N-1)k}). \end{aligned} \tag{A24.26}$$

The eigenvectors are orthonormal

$$\varphi_k^\dagger \cdot \varphi_j = \delta_{kj}, \tag{A24.27}$$

as the explicit evaluation of  $\varphi_k^\dagger \cdot \varphi_j$  yields the *Kronecker (circular) delta function for a periodic lattice*

$$\delta_{kj} = \frac{1}{N} \sum_{\ell=0}^{N-1} e^{i\frac{2\pi}{N}(k-j)\ell} \tag{A24.28}$$


The sum is over the  $N$  unit vectors pointing at a uniform distribution of points on the complex unit circle; they cancel each other unless  $k = j \pmod{N}$ , in which case each term in the sum equals 1.

The projection operators can be expressed in terms of the eigenvectors (A24.25), (A24.26) as

$$(P_k)_{\ell\ell'} = (\varphi_k)_\ell (\varphi_k^\dagger)_{\ell'} = \frac{1}{N} e^{i\frac{2\pi}{N}(\ell-\ell')k}, \quad (\text{no sum on } k). \tag{A24.29}$$

The completeness (A24.23) follows from (A24.28), and the orthonormality (A24.24) from (A24.27).

$\tilde{\phi}_k$ , the projection of the  $\phi$  state on the  $k$ -th subspace is given by

$$\begin{aligned} (P_k \cdot \phi)_\ell &= \tilde{\phi}_k (\varphi_k)_\ell, \quad (\text{no sum on } k) \\ \tilde{\phi}_k &= \varphi_k^\dagger \cdot \phi = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} e^{-i\frac{2\pi}{N}k\ell} \phi_\ell \end{aligned} \tag{A24.30}$$

We recognize  $\tilde{\phi}_k$  as the *discrete Fourier transform* of  $\phi_\ell$ . Hopefully rediscovering it this way helps you a little toward understanding why Fourier transforms are full of  $e^{ix \cdot p}$  factors (they are eigenvalues of the generator of translations) and when are they the natural set of basis functions (only if the theory is translationally invariant).

### A24.3.1 Fourier transform of the propagator

Now insert the identity  $\sum P_k = \mathbf{1}$  wherever profitable:

$$\mathbf{M} = \mathbf{1M1} = \sum_{kk'} P_k \mathbf{M} P_{k'} = \sum_{kk'} \varphi_k (\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) \varphi_{k'}^\dagger.$$

The matrix

$$\tilde{M}_{kk'} = (\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) \quad (\text{A24.31})$$

is the Fourier space representation of  $\mathbf{M}$ . According to (A24.27) the matrix  $U_{k\ell} = (\varphi_k)_\ell = \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N} k\ell}$  is a unitary matrix, so the Fourier transform is a linear, unitary transformation,  $UU^\dagger = \sum P_k = \mathbf{1}$ , with Jacobian  $\det U = 1$ . The form of the invariant function (A24.16) does not change under  $\phi \rightarrow \tilde{\phi}_k$  transformation, and from the formal point of view, it does not matter whether we compute in the Fourier space or in the configuration space that we started out with. For example, the trace of  $\mathbf{M}$  is the trace in either representation

$$\begin{aligned} \text{tr } \mathbf{M} &= \sum_{\ell} M_{\ell\ell} = \sum_{kk'} \sum_{\ell} (P_k \mathbf{M} P_{k'})_{\ell\ell} \\ &= \sum_{kk'} \sum_{\ell} (\varphi_k)_\ell (\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'})_\ell = \sum_{kk'} \delta_{kk'} \tilde{M}_{kk'} = \text{tr } \tilde{\mathbf{M}}. \end{aligned}$$

From this it follows that  $\text{tr } \mathbf{M}^n = \text{tr } \tilde{\mathbf{M}}^n$ , and from the  $\text{tr } \ln = \ln \text{tr}$  relation that  $\det \mathbf{M} = \det \tilde{\mathbf{M}}$ . In fact, any scalar combination of  $\phi$ 's,  $J$ 's and couplings, such as the partition function  $Z[J]$ , has exactly the same form in the configuration and the Fourier space.

OK, a dizzying quantity of indices. But what's the payback?

### A24.3.2 Lattice Laplacian diagonalized

Now use the eigenvalue equation (A24.25) to convert  $\sigma$  matrices into scalars. If  $\mathbf{M}$  commutes with  $\sigma$ , then  $(\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) = \tilde{M}_k \delta_{kk'}$ , and the matrix  $\mathbf{M}$  acts as a multiplication by the scalar  $\tilde{M}_k$  on the  $k$ th subspace. For example, for the 1-dimensional version of the lattice Laplacian (A24.9) the eigenvalue condition



(A24.25) yields the projection on the  $k$ -th subspace,

$$\begin{aligned} (\varphi_k^\dagger \cdot \square \cdot \varphi_{k'}) &= \frac{2}{a^2} \left( \frac{1}{2}(\omega^{-k} + \omega^k) - 1 \right) (\varphi_k^\dagger \cdot \varphi_{k'}) \\ &= \frac{2}{a^2} \left( \cos\left(\frac{2\pi}{N}k\right) - 1 \right) \delta_{kk'}. \end{aligned} \tag{A24.32}$$

In the  $k$ -th subspace the bare propagator is simply a number, and, in contrast to the mess generated by (A24.14), there is nothing to inverting  $M^{-1}$ :

$$(\varphi_{\mathbf{k}}^\dagger \cdot M \cdot \varphi_{\mathbf{k}'}) = \frac{\delta_{\mathbf{k}\mathbf{k}'}}{m^2 - 2 \sum_{\mu=1}^d \left( \cos\left(\frac{2\pi}{N}k_\mu\right) - 1 \right)}, \tag{A24.33}$$

where  $\mathbf{k} = (k_1, k_2, \dots, k_d)$  is a  $d$ -dimensional vector in the  $N^d$ -dimensional dual lattice.

Going back to the partition function and sticking in the factors of  $\mathbf{1}$  into the bilinear part of the interaction, we replace the spatial  $J_\ell$  by its Fourier transform  $\tilde{J}_k$ , and the spatial propagator  $(M)_{\ell\ell'}$  by the diagonalized Fourier transformed  $(\tilde{G}_0)_k$

$$J^\top \cdot M \cdot J = \sum_{k,k'} (J^\top \cdot \varphi_k) (\varphi_k^\dagger \cdot M \cdot \varphi_{k'}) (\varphi_{k'}^\dagger \cdot J) = \sum_k \tilde{J}_k^\dagger (\tilde{G}_0)_k \tilde{J}_k. \tag{A24.34}$$

## A24.4 Diffusion in sawtooth and cat maps

(R. Artuso)



**I**N THIS SECTION we will deal with the prototype example of chaotic Hamiltonian maps, hyperbolic toral automorphisms. Diffusive properties will arise in considering such maps acting on the cylinder or over  $\mathbb{R}^2$ , while the dynamics restricted to the fundamental domain involves maps on  $\mathbf{T}^2$  (two-dimensional torus). An Anosov map thus corresponds to the action of a matrix in  $SL_2(\mathbb{N})$  with unit determinant and absolute value of the trace bigger than 2.

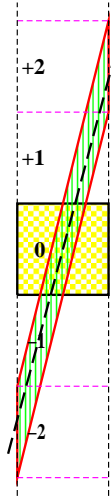
Maps of this kind are as examples of genuine Hamiltonian chaotic evolution. They admit simple finite Markov partitions, which paves the way to a good symbolic dynamics. Within the framework of Hamiltonian dynamical systems the role of hyperbolic linear automorphisms is analogous to piecewise linear Markov maps: their symbolic dynamics can be encoded in a grammatically simple way, and their linearity leads to uniformity of cycle stabilities.

chapter 28

We will consider the “two-coordinates” representation for them

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}$$

**Figure A24.1:** The elementary cell for the torus map  $[-1/2, 1/2]$  (checkered yellow) together with its image, in green ( $K = 2$ ): symbols refer to the linear code. The dashed line through the origin gives the direction of the unstable manifold. Though hardly understandable from the scale of the picture the unstable manifold is not parallel to the image sides.



with

$$M = \begin{bmatrix} 0 & 1 \\ -1 & K + 2 \end{bmatrix}$$

which allows considering their extension on a cylinder phase space  $([-1/2, 1/2] \times \mathbb{R})$  in a natural way. So it is natural to study diffusion properties along the  $y$  direction.

Though Markov partitions encode the symbolic dynamics in the simplest possible way, they are not well suited to deal with diffusion, as the jumping factor is not related in a simple way to the induced symbol sequence. To this end the following linear code is quite natural: before describing it let us fix the notations:  $\chi$  will denote the trace of the map ( $\chi = K + 2$ ): the leading eigenvalue will be denoted by  $\lambda = (\chi + \sqrt{D})/2$ , where  $D = \chi^2 - 4$ . In principle the code (and the problem of diffusion) can be also considered for real values of  $K$  (thus losing continuity of the torus map when  $K$  is not an integer): we will remark in what follows that results which are exact for  $K \in \mathbb{N}$  are only approximate for generic  $K$ .

The cardinality of the alphabet is determined by the parameter  $K$ : the letters are integer numbers, whose absolute values does not exceed  $\text{Int}(1 + \chi/2)$  (see figure A24.1 for the case  $K = 2$ ). The code is linear, as, given a bi-infinite sequence  $\{x_i\}_{i \in \mathbb{N}}$

$$b_t \stackrel{\text{def}}{=} \left\lfloor (K + 2)x_t - x_{t-1} + \frac{1}{2} \right\rfloor, \tag{A24.35}$$

[...] denoting the integer part, while the inversion formula (once a bi-infinite symbolic string  $\{b_i\}_{i \in \mathbb{N}}$  is given), reads

$$x_t = \frac{1}{\sqrt{D}} \sum_{s \in \mathbb{N}} \lambda^{-|t-s|} b_s, \tag{A24.36}$$

As the  $x$  coordinate lives in the interval  $[-1/2, 1/2]$ , (A24.36) induces a condition

of allowed symbol sequences:  $\{b_i\}_{i \in \mathbb{N}}$  will be an admissible orbit if

$$\frac{1}{2} \leq \frac{1}{\sqrt{D}} \sum_{s \in \mathbb{N}} \lambda^{-|t-s|} b_s < \frac{1}{2} \quad (\text{A24.37})$$

By (A24.35) and (A24.36) it is easy to observe that periodic orbits and allowed periodic symbol sequences are in one-to-one correspondence. From (A24.37) we get the condition that a  $\{b_i\}_{i=1, \dots, T}$  sequence corresponds to a  $T$ -periodic orbit of the map

$$|A_n b_t + A_{n-1}(b_{t+1} + b_{t-1}) + \dots + A_0(b_{t+n} + b_{t-n})| < \frac{B_n}{2} \quad \forall t = 1, \dots, T$$

when  $T = 2n + 1$ , and

$$|C_n b_t + C_{n-1}(b_{t+1} + b_{t-1}) + \dots + C_0(b_{t+n})| < \frac{D_n}{2} \quad \forall t = 1, \dots, T \quad (\text{A24.38})$$

when  $T = 2n$  where

$$\begin{aligned} B_k &= \lambda^k(\lambda - 1) + \lambda^{-k}(\lambda^{-1} - 1) & A_k &= \frac{\lambda^{k+1} + \lambda^{-k}}{\lambda + 1} \\ D_k &= (\lambda^k - \lambda^{-k})(\lambda - \lambda^{-1}) & C_k &= \lambda^k + \lambda^{-k} \end{aligned} \quad (\text{A24.39})$$

exercise A24.3  
exercise A24.4

The pruning rules (A24.38) admit a simple geometric interpretation: a lattice point  $\mathbf{b} \in \mathbb{N}^T$  identifies a  $T$ -periodic point of the map if  $\mathbf{b} \in \mathcal{P}_T$  where

$$\mathcal{P}_T \stackrel{\text{def}}{=} \{ \mathbf{x} \in \mathbb{R}^T : \begin{cases} |a_1 x_1 + \dots + a_T x_T| < e_T \\ \vdots \\ |a_2 x_1 + \dots + a_1 x_T| < e_T \end{cases} \} \quad (\text{A24.40})$$

and

$$\begin{aligned} a_1 \dots a_T &= A_0 A_1 \dots A_{n-1} A_n A_{n-1} \dots A_0 & e_T &= B_n/2 \\ a_1 \dots a_T &= C_1 \dots C_{n-1} C_n C_{n-1} \dots C_1 C_0 & e_T &= D_n/2 \end{aligned} \quad (\text{A24.41})$$

for  $T = 2n+1$  or  $T = 2n$ , respectively. Thus  $\mathcal{P}_T$  is a measure polytope [7], obtained by deforming a  $T$ -cube. This is the key issue of this appendix: though the map is endorsed with a most remarkable symbolic dynamics, the same is hardly fit to deal with transport properties, as the rectangles that define the partition are not directly connected to translations once the map is unfolded to the cylinder. The partition connected to the linear code (see figure A24.1) on the other side is most natural when dealing with transport, though its not being directly related to invariant manifolds leads to a multitude of pruning rules (which in the present example bear a remarkable geometric interpretation, which is not to be expected as a generic feature).

exercise A24.5

We will denote by  $\mathcal{N}_{n,s}$  the number of periodic points of period  $n$  with jumping number  $s$ . A way to compute  $D$  for cat maps is provided by

$$D = \lim_{n \rightarrow \infty} D_n \quad D_n = \frac{1}{n \mathcal{N}_n} \sum_{k=1}^{p(n)} k^2 \mathcal{N}_{n,k} \quad (\text{A24.42})$$

where  $\mathcal{N}_n$  is the number of periodic points of period  $n$ ,  $p(n)$  is the highest jumping number of  $n$ -periodic orbits and we employed

$$\left| \det \left( \mathbf{1} - \mathbf{J}_x^{(n)} \right) \right| = (\lambda^n - 1)(1 - \lambda^{-n}) = \mathcal{N}_n$$

which is valid for cat maps.

Sums can be converted into integrals by using Poisson summation formula: we define

$$f_T(n) = \begin{cases} (n_1 + \dots + n_T)^2 & n \in \mathcal{P}_T \cap \mathbb{N}^T \\ 0 & \text{otherwise} \end{cases}$$

and

$$\tilde{f}_T(\xi) = \int_{\mathbb{R}^T} dx e^{i(x,\xi)} f_T(x)$$

From Poisson summation formula we have that

$$D_T = \frac{1}{T \mathcal{N}_T} \sum_{n \in \mathbb{N}^T} \tilde{f}_T(2\pi n) \tag{A24.43}$$

The quasilinear estimate for  $D_T$  amounts to considering the  $n = 0$  contribution to (A24.43):

$$D_T^{(q.l.)} = \int_{\mathcal{P}_T} dx (x_1 + x_2 + \dots + x_T)^2 \tag{A24.44}$$

The evaluation of (A24.44) requires introducing a coordinate transformation in symbolic space in which  $\mathcal{P}_T$  is transformed in a  $T$ -cube. This is equivalent to finding the inverse of the matrix  $A$ :

$$A \stackrel{\text{def}}{=} \begin{pmatrix} a_1 & a_2 & \dots & a_{T-1} & a_T \\ a_T & a_1 & \dots & a_{T-2} & a_{T-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_3 & a_4 & \dots & a_1 & a_2 \\ a_2 & a_3 & \dots & a_T & a_1 \end{pmatrix} . \tag{A24.45}$$

First of all let us observe that  $A$  is a circulant matrix, so that its determinant is the product of  $T$  factors, each of the form  $f(\epsilon_j) = a_1 + \epsilon_j a_2 + \dots + a_T \epsilon_j^{T-1}$ , where  $\epsilon_j$  is a  $T$ th root of unity. By using (A24.39) it is possible to see that

$$f(\epsilon_j) = \begin{cases} \frac{\epsilon_j^{n+1} B_n}{(\lambda \epsilon_j - 1)(1 - \lambda^{-1} \epsilon_j)} & T = 2n + 1 \\ \frac{\epsilon_j^n D_n}{(\lambda \epsilon_j - 1)(1 - \lambda^{-1} \epsilon_j)} & T = 2n \end{cases}$$

so that

$$|\det A| = \frac{(2e_T)^T}{\lambda^T + \lambda^{-T} - 2} \tag{A24.46}$$

By using the results coming from the former exercise we can finally express  $A^{-1}$

exercise A24.6

via

$$\tilde{C}A^{-1} = \frac{1}{B_n^T} \begin{pmatrix} \chi & -1 & \cdots & 0 & -1 \\ -1 & \chi & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \chi & -1 \\ -1 & 0 & \cdots & -1 & \chi \end{pmatrix} \quad (\text{A24.47})$$

where

$$\tilde{C} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_{n+1} \\ \mathbf{1}_n & \mathbf{0} \end{pmatrix} .$$

if  $T = 2n + 1$  and

$$\tilde{K}A^{-1} = \frac{1}{D_n^T} \begin{pmatrix} \chi & -1 & \cdots & 0 & -1 \\ -1 & \chi & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \chi & -1 \\ -1 & 0 & \cdots & -1 & \chi \end{pmatrix} \quad (\text{A24.48})$$

where

$$\tilde{K} = \begin{pmatrix} \mathbf{0} & \mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0} \end{pmatrix} .$$

if  $T = 2n$ . As a first check of quasilinear estimates let's compute the volume of  $\mathcal{P}_T$ :

$$\begin{aligned} \text{Vol}(\mathcal{P}_T) &= \int_{\mathcal{P}_T} dx_1 dx_2 \dots dx_T = \frac{1}{|\det A|} \int_{-e_T}^{e_T} \dots \int_{e_T}^{e_T} d\xi_1 \dots d\xi_T \\ &= \lambda^T + \lambda^{-T} - 2 \end{aligned} \quad (\text{A24.49})$$

In an analogous way we may compute the quasilinear estimate for  $\mathcal{N}_{T,k}$

exercise A24.7

$$\begin{aligned} \mathcal{N}_{T,k}^{(q.l.)} &= \int_{\mathcal{P}_T} dx_1 \dots dx_T \delta(x_1 + \dots + x_T - k) \\ &= \frac{\lambda^T + \lambda^{-T} - 2}{(2e_T)^T} \int_{-\infty}^{\infty} d\alpha e^{-2\pi i \alpha k} \int_{-e_T}^{e_T} \dots \int_{-e_T}^{e_T} d\xi_1 \dots d\xi_T e^{\frac{2\pi i \alpha \chi}{2e_T} (\xi_1 + \dots + \xi_T)} \\ &= \frac{2}{\pi \chi} (\lambda^T + \lambda^{-T} - 2) \int_0^{\infty} dy \cos\left(\frac{2qy}{\chi}\right) \left(\frac{\sin y}{y}\right)^T \end{aligned} \quad (\text{A24.50})$$

where we have used  $x_1 + \dots + x_T = (\chi/(2e_T))(\xi_1 + \dots + \xi_T)$  (cfr. (A24.47),(A24.48)).

We are now ready to evaluate the quasilinear estimate for the diffusion coefficient

$$D_T^{(q.l.)} = \frac{1}{\pi \chi T} \int_{-T\chi/2}^{T\chi/2} dz z^2 \int_0^{\infty} dy \cos\left(\frac{2zy}{\chi}\right) \left(\frac{\sin y}{y}\right)^T \quad (\text{A24.51})$$

(where the bounds on the jumping number again come easily from (A24.47),(A24.48)). By dropping terms vanishing as  $T \mapsto \infty$ , and using [10]

$$\int_0^\infty dx \left( \frac{\sin x}{x} \right)^n \frac{\sin(mx)}{x} = \frac{\pi}{2} \quad m \geq n$$

we can evaluate

$$D^{(q.l.)} = \frac{\chi^2}{24} \tag{A24.52}$$

which is the correct result [5] (and again for cat maps (A24.52) is not the quasilinear estimate but the exact value of the diffusion coefficient).

exercise A24.8

## Commentary

**Remark A24.1.** Who has talked about it? Maps of this kind have been extensively analyzed as examples of genuine Hamiltonian chaotic evolution: in particular they admit simple Markov partitions [2, 9], which lead to simple analytic expressions for topological zeta functions [11]. The linear code was introduced by Percival and Vivaldi [4, 15]. Measure polytopes are discussed in ref. [7]. The quasilinear estimate (A24.44) was given in ref. [5]. (A24.44) was evaluated in ref. [3, 16]. Circulant matrix are discussed in ref. [1]. The result (A24.52) agrees with the saw-tooth result of ref. [5]; for the cat maps (A24.52) is the exact value of the diffusion coefficient. This result was obtained, by using periodic orbits also in ref. [8], where Gaussian nature of the diffusion process is explicitly assumed.

**Remark A24.2.** Phase space. The cylinder phase is  $[-1/2, 1/2) \times \mathbb{R}$ : the map is originally defined on  $[-1/2, 1/2)^2$ , and is unfolded over the cylinder by symmetry requirements (24.21).

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## Exercises

**A24.1. Laplacian is a non-local operator.**

While the Laplacian is a simple tri-diagonal difference operator (A24.10), its inverse (the “free” propagator of statistical mechanics and quantum field theory) is a messier object. A way to compute is to start expanding propagator as a power series in the Laplacian

$$\frac{1}{m^2 \mathbf{1} - \square} = \frac{1}{m^2} \sum_{n=0}^{\infty} \frac{1}{m^{2n}} \square^n. \quad (\text{A24.53})$$

As  $\square$  is a finite matrix, the expansion is convergent for sufficiently large  $m^2$ . To get a feeling for what is involved in evaluating such series, show that  $\square^2$  is:

$$\square^2 = \frac{1}{a^4} \begin{bmatrix} 6 & -4 & 1 & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & & 1 & -4 & \ddots & & \\ & & & & & & 6 & -4 \\ -4 & 1 & & & & 1 & -4 & 6 \end{bmatrix}. \quad (\text{A24.54})$$

What  $\square^3, \square^4, \dots$  contributions look like is now clear; as we include higher and higher powers of the Laplacian, the propagator matrix fills up; while the *inverse* propagator is differential operator connecting only the nearest neighbors, the propagator is integral operator, connecting every lattice site to any other lattice site.

This matrix can be evaluated as is, on the lattice, and sometime it is evaluated this way, but in case at hand a wonderful simplification follows from the observation that the lattice action is translationally invariant, exercise A24.2.

**A24.2. Lattice Laplacian diagonalized.** Insert the identity  $\sum \mathbf{P}^{(k)} = \mathbf{1}$  wherever you profitably can, and use the eigenvalue equation (A24.25) to convert shift  $\sigma$  matrices into scalars. If  $\mathbf{M}$  commutes with  $\sigma$ , then  $(\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) = \tilde{M}^{(k)} \delta_{kk'}$ , and the matrix  $\mathbf{M}$  acts as a multiplication by the scalar  $\tilde{M}^{(k)}$  on the  $k$ th subspace. Show that for the 1-dimensional version of the lattice Laplacian (A24.10) the projection on the  $k$ th subspace is

$$(\varphi_k^\dagger \cdot \square \cdot \varphi_{k'}) = \frac{2}{a^2} \left( \cos \left( \frac{2\pi}{N} k \right) - 1 \right) \delta_{kk'}. \quad (\text{A24.55})$$

In the  $k$ th subspace the propagator is simply a number, and, in contrast to the mess generated by (A24.53), there is nothing to evaluating:

$$\varphi_k^\dagger \cdot \frac{1}{m^2 \mathbf{1} - \square} \cdot \varphi_{k'} = \frac{\delta_{kk'}}{m^2 - \frac{2}{(ma)^2} (\cos 2\pi k/N - 1)},$$

(A24.56)

where  $k$  is a site in the  $N$ -dimensional dual lattice, and  $a = L/N$  is the lattice spacing.

**A24.3. Recursion relations.** Verify that the following recursion relations are satisfied

$$u_{k+2} = \chi u_{k+1} - u_k$$

where  $u_k = A_k, B_k, C_k, D_k$ .

**A24.4. Arnol'd cat map.** Show that for  $\chi = 3, A_k = F_{2k+1}, B_k = L_{2k+1}, C_k = L_{2k}$  and  $D_k = 5F_{2k}$ , where  $F_n$  and  $L_n$  are the Fibonacci and Lucas numbers..

**A24.5. Pruning rules for substrings of length 2.** Take  $K = 8$  and draw the region determined by (A24.38).

**A24.6. Diagonalization of A.** Show that  $A$  can be diagonalized by considering the auxiliary matrix  $U$

$$U \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \epsilon_0 & \epsilon_1 & \dots & \epsilon_{T-2} & \epsilon_{T-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \epsilon_0^{T-2} & \epsilon_1^{T-2} & \dots & \epsilon_{T-2}^{T-2} & \epsilon_{T-1}^{T-2} \\ \epsilon_0^{T-1} & \epsilon_1^{T-1} & \dots & \epsilon_{T-2}^{T-1} & \epsilon_{T-1}^{T-1} \end{pmatrix}.$$

In fact  $U^{-1}AU$  is a diagonal matrix (the diagonal elements coinciding with  $f(\epsilon_j)$ ).

**A24.7. Periodic points of cat maps.** Verify that (A24.49) is exactly the number of  $T$ -periodic points of the map when  $K$  is an integer.

**A24.8. Probability distribution.** Higher order moments can be computed easily for integer  $K$  (or generic  $K$  within the quasilinear approximation), by generalizations of (A24.51): show that the results prove that, given a period  $T$ , the distribution of periodic orbits with respect to their jumping number is asymptotically Gaussian, with parameter  $D^{(q,l)}$ .

**A24.9. Deterministic diffusion, zig-zag map.**

To illustrate the main idea of chapter 24, tracking of a globally diffusing orbit by the associated confined orbit restricted to the fundamental cell, we consider a class of simple 1-dimensional dynamical systems, chains of piecewise linear maps, where all transport coefficients can be evaluated analytically. The translational symmetry (24.21) relates the unbounded dynamics on the real line to the dynamics restricted to a “fundamental cell” -



in the present example the unit interval curled up into a circle. An example of such map is the sawtooth map

$$\hat{f}(x) = \begin{cases} \Lambda x & x \in [0, 1/4 + 1/4\Lambda] \\ -\Lambda x + (\Lambda + 1)/2 & x \in [1/4 + 1/4\Lambda, 3/4 - 1/4\Lambda] \\ \Lambda x + (1 - \Lambda) & x \in [3/4 - 1/4\Lambda, 1] \end{cases} \quad (\text{A24.57})$$

The corresponding circle map  $f(x)$  is obtained by modulo the integer part. The elementary cell map  $f(x)$  is sketched in figure ???. The map has the symmetry property

$$\hat{f}(\hat{x}) = -\hat{f}(-\hat{x}), \quad (\text{A24.58})$$

so that the dynamics has no drift, and all odd derivatives of the generating function (24.3) with respect to  $\beta$  evaluated at  $\beta = 0$  vanish.

The cycle weights are given by

$$t_p = z^{n_p} \frac{e^{\beta \hat{n}_p}}{|\Lambda_p|}. \quad (\text{A24.59})$$

The diffusion constant formula for 1-dimensional maps is

$$D = \frac{1}{2} \frac{\langle \hat{n}^2 \rangle_\zeta}{\langle n \rangle_\zeta} \quad (\text{A24.60})$$

where the ‘‘mean cycle time’’ is given by

$$\langle n \rangle_\zeta = z \frac{\partial}{\partial z} \frac{1}{\zeta(0, z)} \Big|_{z=1} = - \sum' (-1)^k \frac{n_{p_1} + \dots + n_{p_k}}{|\Lambda_{p_1} \dots \Lambda_{p_k}|}, \quad (\text{A24.61})$$

the mean cycle displacement squared by

$$\langle \hat{n}^2 \rangle_\zeta = \frac{\partial^2}{\partial \beta^2} \frac{1}{\zeta(\beta, 1)} \Big|_{\beta=0} = - \sum' (-1)^k \frac{(\hat{n}_{p_1} + \dots + \hat{n}_{p_k})^2}{|\Lambda_{p_1} \dots \Lambda_{p_k}|}, \quad (\text{A24.62})$$

and the sum is over all distinct non-repeating combinations of prime cycles. Most of results expected in this projects require no more than pencil and paper computations.

Implementing the symmetry factorization (24.18) is convenient, but not essential for this project, so if you find example 25.9 too long a read, skip the symmetrization.

A24.10. **The full shift sawtooth map.** Take the map (A24.57) and extend it to the real line. As in example of figure 24.3, denote by  $a$  the critical value of the map (the maximum height in the unit cell)

$$a = \hat{f}\left(\frac{1}{4} + \frac{1}{4\Lambda}\right) = \frac{\Lambda + 1}{4}. \quad (\text{A24.63})$$

Describe the symbolic dynamics that you obtain when  $a$  is an integer, and derive the formula for the diffusion constant:

$$D = \frac{(\Lambda^2 - 1)(\Lambda - 3)}{96\Lambda} \quad \text{for } \Lambda = 4a - 1, a \in \mathbb{Z}. \quad (\text{A24.64})$$

If you are going strong, derive also the formula for the half-integer  $a = (2k + 1)/2$ ,  $\Lambda = 4a + 1$  case and email it to predrag@nbi.dk. You will need to partition  $\mathcal{M}_2$  into the left and right half,  $\mathcal{M}_2 = \mathcal{M}_8 \cup \mathcal{M}_9$ , as in the derivation of (24.28). See exercise 24.1.

**Sawtooth map subshifts of finite type.** We now work out an example when the partition is Markov, although the slope is not an integer number. The key step is that of having a partition where intervals are mapped onto unions of intervals. Consider for example the case in which  $\Lambda = 4a - 1$ , where  $1 \leq a \leq 2$ . A first partition is constructed from seven intervals, which we label  $\{\mathcal{M}_1, \mathcal{M}_4, \mathcal{M}_5, \mathcal{M}_2, \mathcal{M}_6, \mathcal{M}_7, \mathcal{M}_3\}$ , with the alphabet ordered as the intervals are laid out along the unit interval. In general the critical value  $a$  will not correspond to an interval border, but now we choose  $a$  such that the critical point is mapped onto the right border of  $\mathcal{M}_1$ , as in figure ?? (a). The critical value of  $f()$  is  $f(\frac{\Lambda+1}{4\Lambda}) = a - 1 = (\Lambda - 3)/4$ . Equating this with the right border of  $\mathcal{M}_1$ ,  $x = 1/\Lambda$ , we obtain a quadratic equation with the expanding solution  $\Lambda = 4$ . We have that  $f(\mathcal{M}_4) = f(\mathcal{M}_5) = \mathcal{M}_1$ , so the transition matrix (17.1) is given by

$$\phi' = T\phi = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_4 \\ \phi_5 \\ \phi_2 \\ \phi_6 \\ \phi_7 \\ \phi_3 \end{bmatrix} \quad (\text{A24.65})$$

and the dynamics is unrestricted in the alphabet

$$\{1, \underline{41}, \underline{51}, 2, \underline{63}, \underline{73}, 3, \}$$

One could diagonalize (A24.65) on the computer, but, as we saw in chapter 17, the transition graph figure ?? (b) corresponding to figure ?? (a) offers more insight into the dynamics. The dynamical zeta function

$$\begin{aligned} 1/\zeta &= 1 - (t_1 + t_2 + t_3) - 2(t_{14} + t_{37}) \\ 1/\zeta &= 1 - 3\frac{z}{\Lambda} - 4 \cosh \beta \frac{z^2}{\Lambda^2}. \end{aligned} \quad (\text{A24.66})$$

follows from the loop expansion (18.13) of sect. 18.3. The material flow conservation sect. 23.4 and the symmetry factorization (24.18) yield

$$0 = \frac{1}{\zeta(0, 1)} = \left(1 + \frac{1}{\Lambda}\right) \left(1 - \frac{4}{\Lambda}\right)$$

which indeed is satisfied by the given value of  $\Lambda$ . Conversely, we can use the desired Markov partition topology to write down the corresponding dynamical zeta function, and use the  $1/\zeta(0, 1) = 0$  condition to fix  $\Lambda$ . For more complicated transition matrices the factorization (24.18) is very helpful in reducing the order of the polynomial condition that fixes  $\Lambda$ .

The diffusion constant follows from (24.19) and (A24.60)

$$\langle n \rangle_\zeta = -\left(1 + \frac{1}{\Lambda}\right)\left(-\frac{4}{\Lambda}\right), \quad \langle \hat{n}^2 \rangle_\zeta = \frac{4}{\Lambda^2}$$

$$D = \frac{1}{2} \frac{1}{\Lambda + 1} = \frac{1}{10}$$

Think up other non-integer values of the parameter for which the symbolic dynamics is given in terms of Markov partitions: in particular consider the cases illustrated in figure ?? and determine for what value of the parameter  $a$  each of them is realized. Work out the transition graph, symmetrization factorization and the diffusion constant, and check the material flow conservation for each case. Derive the diffusion constants listed in table ?. It is not clear why the final answers tend to be so simple. Numerically, the case of figure ?? (c) appears to yield the maximal diffusion constant. Does it? Is there an argument that it should be so?

The seven cases considered here (see table ??, figure ?? and (A24.64)) are the 7 simplest complete Markov partitions, the criterion being that the critical points map onto partition boundary points. This is, for example, what happens for unimodal tent map; if the critical point is preperiodic to an unstable cycle, the grammar is complete. The simplest example is the case in which the tent map critical point is preperiodic to a unimodal map 3-cycle, in which case the grammar is of golden mean type, with `_00_` substring prohibited (see figure 17.7). In case at hand, the “critical” point is the junction of branches 4 and 5 (symmetry automatically takes care of the other critical point, at the junction of branches 6 and 7), and for the cases considered the critical point maps into the endpoint of each of the seven branches.

One can fill out parameter  $a$  axis arbitrarily densely with such points - each of the 7 primary intervals can be subdivided into 7 intervals obtained by 2-nd iterate of the map, and for the critical point mapping into any of those in 2 steps the grammar (and the corresponding cycle expansion) is finite, and so on.

A24.12. **Sawtooth map diffusion coefficient, numerically.** (optional:)

Attempt a numerical evaluation of

$$D = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{1}{n} \langle \hat{x}_n^2 \rangle. \tag{A24.67}$$

Study the convergence by comparing your numerical results to the exact answers derived above. Is it better to use few initial  $\hat{x}$  and average for long times, or to use many initial  $\hat{x}$  for shorter times? Or should one fit the distribution of  $\hat{x}^2$  with a Gaussian and get the  $D$  this way? Try to plot dependence of  $D$  on  $\Lambda$ ; perhaps blow up a small region to show that the dependence of  $D$  on the parameter  $\Lambda$  is fractal. Compare with figure 24.5 and figures in refs. [6, 12–14, 17].

A24.13. **Sawtooth  $D$  is a nonuniform function of the parameters.** (optional:)

The dependence of  $D$  on the map parameter  $\Lambda$  is rather unexpected - even though for larger  $\Lambda$  more points are mapped outside the unit cell in one iteration, the diffusion constant does not necessarily grow. An interpretation of this lack of monotonicity would be interesting.

You can also try applying periodic orbit theory to the sawtooth map (A24.57) for a random “generic” value of the parameter  $\Lambda$ , for example  $\Lambda = 6$ . The idea is to bracket this value of  $\Lambda$  by the nearby ones, for which higher and higher iterates of the critical value  $a = (\Lambda + 1)/4$  fall onto the partition boundaries, compute the exact diffusion constant for each such approximate Markov partition, and study their convergence toward the value of  $D$  for  $\Lambda = 6$ . Judging how difficult such problem is already for a tent map (see sect. 18.5 and appendix A18.1), this is too ambitious for a week-long exam.

A24.14. **Deterministic diffusion, sawtooth map.**

To illustrate the main idea of chapter 24, tracking of a globally diffusing orbit by the associated confined orbit restricted to the fundamental cell, we consider in more detail the class of simple 1-dimensional dynamical systems, chains of piecewise linear maps (24.20). The translational symmetry (24.21) relates the unbounded dynamics on the real line to the dynamics restricted to a “fundamental cell” - in the present example the unit interval curled up into a circle. The corresponding circle map  $f(x)$  is obtained by modulo the integer part. The elementary cell map  $f(x)$  is sketched in figure 24.3. The map has the symmetry property

$$\hat{f}(\hat{x}) = -\hat{f}(-\hat{x}), \tag{A24.68}$$

so that the dynamics has no drift, and all odd derivatives of the generating function (24.3) with respect to  $\beta$  evaluated at  $\beta = 0$  vanish.

The cycle weights are given by

$$t_p = z^{n_p} \frac{e^{\beta \hat{n}_p}}{|\Lambda_p|}. \tag{A24.69}$$

The diffusion constant formula for 1-dimensional maps

is

$$D = \frac{1}{2} \frac{\langle \hat{n}^2 \rangle_\zeta}{\langle n \rangle_\zeta} \quad (\text{A24.70})$$

where the “mean cycle time” is given by

$$\langle n \rangle_\zeta = z \frac{\partial}{\partial z} \frac{1}{\zeta(0, z)} \Big|_{z=1} = - \sum' (-1)^k \frac{n_{p_1} + \cdots + n_{p_k}}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|}, \quad (\text{A24.71})$$

the mean cycle displacement squared by

$$\langle \hat{n}^2 \rangle_\zeta = \frac{\partial^2}{\partial \beta^2} \frac{1}{\zeta(\beta, 1)} \Big|_{\beta=0} = - \sum' (-1)^k \frac{(\hat{n}_{p_1} + \cdots + \hat{n}_{p_k})^2}{|\Lambda_{p_1} \cdots \Lambda_{p_k}|}, \quad (\text{A24.72})$$

and the sum is over all distinct non-repeating combinations of prime cycles. Most of results expected in this projects require no more than pencil and paper computations.