Group Theory

Birdtracks, Lie's, and Exceptional Groups

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3.2 DEFINING SPACE, TENSORS, REPS

Definition. In what follows V will always denote the *defining* n-dimensional complex vector representation space, that is to say the initial, "elementary multiplet" space within which we commence our deliberations. Along with the defining vector representation space V comes the *dual* n-dimensional vector representation space $\bar{V}$. We shall denote the corresponding element of $\bar{V}$ by raising the index, as in (3.3), so the components of defining space vectors, resp. dual vectors, are distinguished by lower, resp. upper indices:

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_n)\,, \qquad \mathbf{x} \in V \\ \bar{x} &= (x^1, x^2, \ldots, x^n)\,, \qquad \bar{\mathbf{x}} \in \bar{V}\,. \end{aligned} \tag{3.10}$$

Definition. Let $\mathcal{G}$ be a group of transformations acting linearly on V, with the action of a group element $g \in \mathcal{G}$ on a vector $x \in V$ given by an $[n\times n]$ matrix G

$$x'_a = G_a{}^b x_b \qquad a, b = 1, 2, \ldots, n\,. \tag{3.11}$$

We shall refer to $G_a{}^b$ as the *defining rep* of the group $\mathcal{G}$. The action of $g \in \mathcal{G}$ on a vector $\bar{q} \in \bar{V}$ is given by the *dual rep* $[n\times n]$ matrix $G^\dagger$:

$$x'^a = x^b (G^\dagger)_b{}^a = G^a{}_b x^b\,. \tag{3.12}$$

In the applications considered here, the group $\mathcal{G}$ will almost always be assumed to be a subgroup of the *unitary group*, in which case $G^{-1} = G^\dagger$, and † indicates hermitian conjugation:

$$(G^\dagger)_a{}^b = (G_b{}^a)^* = G^b{}_a\,. \tag{3.13}$$

Definition. A *tensor* $x \in V^p \otimes \bar{V}^q$ transforms under the action of $g \in \mathcal{G}$ as

$$x'^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p} = G^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}{}, {}^{d_p \ldots d_1}_{c_q \ldots c_2 c_1}\, x^{c_1 c_2 \ldots c_q}_{d_1 \ldots d_p}\,, \tag{3.14}$$

where the $V^p \otimes \bar{V}^q$ *tensor rep* of $g \in \mathcal{G}$ is defined by the group acting on all indices of x.

$$G^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q}{}, {}^{d_q \ldots d_1}_{c_p \ldots c_2 c_1} \equiv G^{a_1}{}_{c_1} G^{a_2}{}_{c_2} \ldots G^{a_p}{}_{c_p} G_{b_q}{}^{d_q} \ldots G_{b_2}{}^{d_2 1} G_{b_1}{}^{d_1}\,. \tag{3.15}$$

Tensors can be combined into other tensors by

(a) *addition:*

$$z^{ab\ldots c}_{d\ldots e} = \alpha x^{ab\ldots c}_{d\ldots e} + \beta y^{ab\ldots c}_{d\ldots e}\,, \qquad \alpha, \beta \in \mathbb{C}\,, \tag{3.16}$$

(b) *product:*

$$z^{abcd}_{efg} = x^{abc}_{e} y^{d}_{fg}\,, \tag{3.17}$$

(c) *contraction:* Setting an upper and a lower index equal and summing over all of its values yields a tensor $z \in V^{p-1} \otimes \bar{V}^{q-1}$ without these indices:

$$z^{bc\ldots d}_{e\ldots f} = x^{abc\ldots d}_{e\ldots af}\,, \qquad z^{ad}_{e} = x^{abc}_{e} y^{d}_{cb}\,. \tag{3.18}$$

A tensor $x \in V^p \otimes \bar{V}^q$ transforms linearly under the action of g, so it can be considered a vector in the $d = n^{p+q}$-dimensional vector space $\tilde{V} = V^p \otimes \bar{V}^q$. We can replace the array of its indices by one collective index:

$$x_\alpha = x^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}\,. \tag{3.19}$$

One could be more explicit and give a table like

$$x_1 = x^{11\ldots1}_{1\ldots1}\ ,\ x_2 = x^{21\ldots1}_{1\ldots1}, \ldots,\ x_d = x^{nn\ldots n}_{n\ldots n}\ , \tag{3.20}$$

but that is unnecessary, as we shall use the compact index notation only as a shorthand.

Definition. *Hermitian conjugation* is effected by complex conjugation and index transposition:

$$(h^\dagger)^{ab}_{cde} \equiv (h^{edc}_{ba})^* \,. \tag{3.21}$$

Complex conjugation interchanges upper and lower indices, as in (3.10); transposition reverses their order. A matrix is *hermitian* if its elements satisfy

$$(\mathbf{M}^\dagger)^a_b = M^a_b \,. \tag{3.22}$$

For a hermitian matrix there is no need to keep track of the relative ordering of indices, as $M_b{}^a = (\mathbf{M}^\dagger)_b{}^a = M^a{}_b$.

Definition. The tensor dual to x_α defined by (3.19) has form

$$x^\alpha = x^{b_p\ldots b_1}_{a_q\ldots a_2 a_1} \,. \tag{3.23}$$

Combined, the above definitions lead to the hermitian conjugation rule for collective indices: a collective index is raised or lowered by interchanging the upper and lower indices and reversing their order:

$${}_\alpha = \left\{ \begin{array}{c} a_1 a_2 \ldots a_q \\ b_1 \ldots b_p \end{array} \right\} \quad \leftrightarrow \quad {}^\alpha = \left\{ \begin{array}{c} b_p \ldots b_1 \\ a_q \ldots a_2 a_1 \end{array} \right\} . \tag{3.24}$$

This transposition convention will be motivated further by the diagrammatic rules of section 4.1.

The tensor rep (3.15) can be treated as a $[d \times d]$ matrix

$$G_\alpha{}^\beta = G^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}{}_{,}{}^{d_p \ldots d_1}_{c_q \ldots c_2 c_1} \,, \tag{3.25}$$

and the tensor transformation (3.14) takes the usual matrix form

$$x'_\alpha = G_\alpha{}^\beta x_\beta \,. \tag{3.26}$$

3.3 INVARIANTS

Definition. The vector $q \in V$ is an *invariant vector* if for any transformation $g \in \mathcal{G}$

$$q = Gq \,. \tag{3.27}$$

Definition. A tensor $x \in V^p \otimes \bar{V}^q$ is an *invariant tensor* if for any $g \in G$

$$x^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q} = G^{a_1}{}_{c_1} G^{a_2}{}_{c_2} \ldots G_{b_1}{}^{d_1} \ldots G_{b_q}{}^{d_q} x^{c_1 c_2 \ldots c_p}_{d_1 \ldots d_q} \,. \tag{3.28}$$

We can state this more compactly by using the notation of (3.25)

$$x_\alpha = G_\alpha{}^\beta x_\beta \,. \tag{3.29}$$

Here we treat the tensor $x^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q}$ as a vector in $[d \times d]$-dimensional space, $d = n^{p+q}$.

If a bilinear form $m(\bar{x}, y) = x^a M_a{}^b y_b$ is invariant for all $g \in \mathcal{G}$, the matrix

$$M_a{}^b = G_a{}^c G^b{}_d M_c{}^d \tag{3.30}$$

is an *invariant matrix*. Multiplying with $G_b{}^e$ and using the unitary condition (3.13), we find that the invariant matrices *commute* with all transformations $g \in \mathcal{G}$:

$$[G, \mathbf{M}] = 0\,. \tag{3.31}$$

If we wish to treat a tensor with equal number of upper and lower indices as a matrix $\mathbf{M} : V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$,

$$M_\alpha{}^\beta = M^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}{}_{,\, c_q \ldots c_2 c_1}^{,\, d_p \ldots d_1}\,, \tag{3.32}$$

then the invariance condition (3.29) will take the commutator form (3.31). Our convention of separating the two sets of indices by a comma, and reversing the order of the indices to the right of the comma, is motivated by the diagrammatic notation introduced below (see (4.6)).

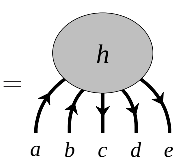
Definition. We shall refer to an invariant relation between p vectors in V and q vectors in $\bar{V}$, which can be written as a homogeneous polynomial in terms of vector components, such as

$$h(x, y, \bar{z}, \bar{r}, \bar{s}) = h^{ab}{}_{cde} x_b y_a s^e r^d z^c\,, \tag{3.33}$$

as an *invariant* in $V^q \otimes \bar{V}^p$ (repeated indices, as always, summed over). In this example, the coefficients $h^{ab}{}_{cde}$ are components of invariant tensor $h \in V^3 \otimes \bar{V}^2$, obeying the invariance condition (3.28).

Diagrammatic representation of tensors, such as

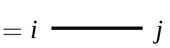
$$h^{ab}{}_{cde} = \tag{3.34}$$



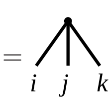
makes it easier to distinguish different types of invariant tensors. We shall explain in great detail our conventions for drawing tensors in section 4.1; sketching a few simple examples should suffice for the time being.

The standard example of a defining vector space is our 3-dimensional Euclidean space: $V = \bar{V}$ is the space of all 3-component real vectors $(n = 3)$, and examples of invariants are the length $L(x, x) = \delta_{ij} x_i x_j$ and the volume $V(x, y, z) = \epsilon_{ijk} x_i y_j z_k$. We draw the corresponding invariant tensors as

$$\delta_{ij} = \tag{3.35}$$



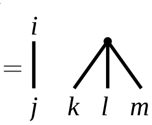
$$\epsilon_{ijk} =$$



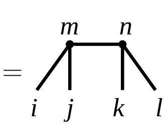
Definition. A *composed* invariant tensor can be written as a product and/or contraction of invariant tensors.

Examples of composed invariant tensors are

$$\delta_{ij}\epsilon_{klm} = \tag{3.36}$$



$$\epsilon_{ijm}\delta_{mn}\epsilon_{nkl} =$$

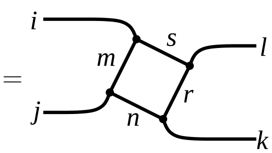


The first example corresponds to a product of the two invariants $L(x, y)V(z, r, s)$. The second involves an index *contraction*; we can write this as $V(x, y, \frac{d}{dz})V(z, r, s)$.

In order to proceed, we need to distinguish the "primitive" invariant tensors from the infinity of composed invariants. We begin by defining a finite basis for invariant tensors in $V^p \otimes \bar{V}^q$:

Definition. A *tree invariant* can be represented diagrammatically as a product of invariant tensors involving no loops of index contractions. We shall denote by $T = \{\mathbf{t}_0, \mathbf{t}_1 \ldots \mathbf{t}_{r-1}\}$ a (maximal) set of r linearly independent tree invariants $\mathbf{t}_\alpha \in V^p \otimes \bar{V}^q$. As any linear combination of $\mathbf{t}_\alpha$ can serve as a basis, we clearly have a great deal of freedom in making informed choices for the basis tensors.

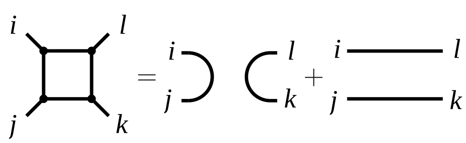
Example: Tensors (3.36) are tree invariants. The tensor

$$h_{ijkl} = \epsilon_{ims}\epsilon_{jnm}\epsilon_{krn}\epsilon_{\ell sr} =$$ 
, (3.37)

with intermediate indices m, n, r, s summed over, is not a tree invariant, as it involves a loop.

Definition. An invariant tensor is called a *primitive* invariant tensor if it cannot be expressed as a linear combination of tree invariants composed from lower-rank primitive invariant tensors. Let $\mathcal{P} = \{\mathbf{p}_1, \mathbf{p}_2, \ldots \mathbf{p}_k\}$ be the set of all primitives.

For example, the Kronecker delta and the Levi-Civita tensor (3.35) are the primitive invariant tensors of our 3-dimensional space. The loop contraction (3.37) is not a primitive, because by the Levi-Civita completeness relation (6.28) it reduces to a sum of tree contractions:

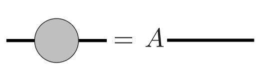

$$= \delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}\,, \qquad (3.38)$$

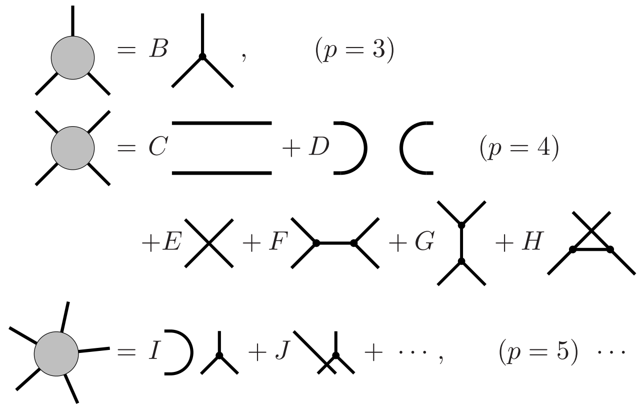
(The Levi-Civita tensor is discussed in section 6.3.)

Primitiveness assumption. Any invariant tensor $h \in V^p \otimes \bar{V}^q$ can be expressed as a linear sum over the tree invariants $T \subset V^q \otimes \bar{V}^p$:

$$h = \sum_{\alpha \in T} h^\alpha \mathbf{t}_\alpha\,. \qquad (3.39)$$

In contradistinction to arbitrary composite invariant tensors, the number of tree invariants for a fixed number of external indices is finite. For example, given bilinear and trilinear primitives $\mathcal{P} = \{\delta_{ij}, f_{ijk}\}$, any invariant tensor $h \in V^p$ (here denoted by a blob) must be expressible as


$$, \qquad (p = 2) \qquad (3.40)$$



(3.41)

3.3.1 Algebra of invariants

Any invariant tensor of matrix form (3.32)

$$M_\alpha{}^\beta = M^{a_1 a_2 \dots a_q}_{b_1 \dots b_p}{}^{,\ d_p \dots d_1}_{,\ c_q \dots c_2 c_1}$$

that maps $V^q \otimes \bar{V}^p \to V^q \otimes \bar{V}^p$ can be expanded in the basis (3.39). In this case the basis tensors $\mathbf{t}_\alpha$ are themselves matrices in $V^q \otimes \bar{V}^p \to V^q \otimes \bar{V}^p$, and the matrix product of two basis elements is also an element of $V^q \otimes \bar{V}^p \to V^q \otimes \bar{V}^p$ and can be expanded in an r element basis:

$$\mathbf{t}_\alpha \mathbf{t}_\beta = \sum_{\mathbf{t} \in T} (\tau_\alpha)_\beta{}^\gamma \mathbf{t}_\gamma \,. \tag{3.42}$$

As the number of tree invariants composed from the primitives is finite, under matrix multiplication the bases $\mathbf{t}_\alpha$ form a finite r-dimensional algebra, with the coefficients $(\tau_\alpha)_\beta{}^\gamma$ giving their multiplication table. As in (3.7), the structure constants $(\tau_\alpha)_\beta{}^\gamma$ form a $[r \times r]$-dimensional matrix rep of $\mathbf{t}_\alpha$ acting on the vector $(\mathbf{e}, \mathbf{t}_1, \mathbf{t}_2, \cdots \mathbf{t}_{r-1})$. Given a basis, we can evaluate the matrices $\mathbf{e}_\beta{}^\gamma$, $(\tau_1)_\beta{}^\gamma$, $(\tau_2)_\beta{}^\gamma, \cdots (\tau_{r-1})_\beta{}^\gamma$ and their eigenvalues. For at least one of combinations of these matrices all eigenvalues will be distinct (or we have failed to choose a good basis). The projection operator technique of section 3.5 will enable us to exploit this fact to decompose the $V^q \otimes \bar{V}^p$ space into r irreducible subspaces.

This can be said in another way; the choice of basis $\{\mathbf{e}, \mathbf{t}_1, \mathbf{t}_2 \cdots \mathbf{t}_{r-1}\}$ is arbitrary, the only requirement being that the basis elements are linearly independent. Finding a $(\tau_\alpha)_\beta{}^\gamma$ with all eigenvalues distinct is all we need to construct an orthogonal basis $\{\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \cdots \mathbf{P}_{r-1}\}$, where the basis matrices $\mathbf{P}_i$ are the projection operators, to be constructed below in section 3.5. For an application of this algebra, see section 9.11.

Chapter Four

Diagrammatic notation

Some aspects of the representation theory of Lie groups are the subject of this monograph. However, it is not written in the conventional tensor notation but instead in terms of an equivalent diagrammatic notation. We shall refer to this style of carrying out group-theoretic calculations as *birdtracks* (and so do reputable journals [51]). The advantage of diagrammatic notation will become self-evident, I hope. Two of the principal benefits are that it eliminates “dummy indices,” and that it does not force group-theoretic expressions into the 1-dimensional tensor format (both being means whereby identical tensor expressions can be made to look totally different). In contradistinction to some of the existing literature in this manuscript I strive to keep the diagrammatic notation as simple and elegant as possible.

4.1 BIRDTRACKS

We shall often find it convenient to represent agglomerations of invariant tensors by birdtracks, a group-theoretical version of Feynman diagrams. Tensors will be represented by *vertices* and contractions by *propagators*.

Diagrammatic notation has several advantages over the tensor notation. Diagrams do not require dummy indices, so explicit labeling of such indices is unnecessary. More to the point, for a human eye it is easier to identify topologically identical diagrams than to recognize equivalence between the corresponding tensor expressions.

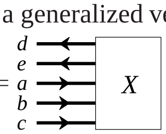
If readers find birdtrack notation abhorrent, they can surely derive all results of this monograph in more conventional algebraic notations. To give them a sense of how that goes, we have covered our tracks by algebra in the derivation of the E_7 family, chapter 20, where not a single birdtrack is drawn. It it is like speaking Italian without moving hands, if you are into that kind of thing.

In the birdtrack notation, the Kronecker delta is a propagator:

$$\delta^a_b = b \longleftarrow a\,. \tag{4.1}$$

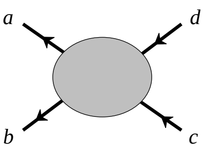
For a *real* defining space there is no distinction between V and $\bar{V}$, or up and down indices, and the lines do not carry arrows.

Any invariant tensor can be drawn as a generalized vertex:

$$X_\alpha = X^{abc}_{de} =$$  . (4.2)

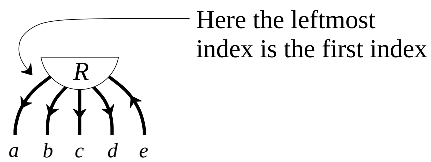
Whether the vertex is drawn as a box or a circle or a dot is a matter of taste. The orientation of propagators and vertices in the plane of the drawing is likewise irrelevant. The only rules are as follows:

1. Arrows point *away from the upper* indices and *toward the lower* indices; the line flow is “downward,” from upper to lower indices:

$$h_{ab}^{cd} =$$ 

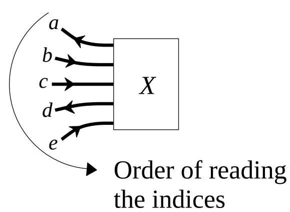
. (4.3)

2. Diagrammatic notation must indicate which in (out) arrow corresponds to the first upper (lower) index of the tensor (unless the tensor is cyclically symmetric);

$$R_{abcd}^{e} =$$ 

. (4.4)

3. The indices are read in the *counterclockwise* order around the vertex:

$$X_{ad}^{bce} =$$ 

. (4.5)

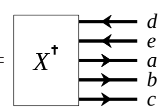
(The upper and the lower indices are read separately in the counterclockwise order; their relative ordering does not matter.)

In the examples of this section we index the external lines for the reader’s convenience, but indices can always be omitted. An internal line implies a summation over corresponding indices, and for external lines the equivalent points on each diagram represent the same index in all terms of a diagrammatic equation.

Hermitian conjugation (3.21) does two things:

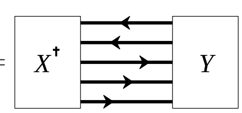
1. It exchanges the upper and the lower indices, *i.e.*, it reverses the directions of the arrows.

2. It reverses the order of the indices, *i.e.*, it transposes a diagram into its mirror image. For example, $X^\dagger$, the tensor conjugate to (4.5), is drawn as

$$X^\alpha = X_{cba}^{ed} =$$ 

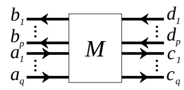
, (4.6)

and a contraction of tensors $X^\dagger$ and Y is drawn as

$$X^\alpha Y_\alpha = X_{a_q \ldots a_2 a_1}^{b_p \ldots b_1} Y_{b_1 \ldots b_p}^{a_1 a_2 \ldots a_q} =$$ 

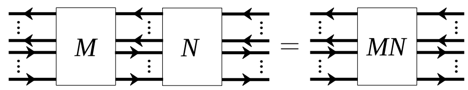
. (4.7)

In sections. 3.1–3.2 and here we define the hermitian conjugation and (3.32) matrices $\mathbf{M} : V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$ in the multi-index notation



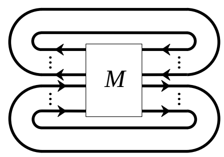
(4.8)

in such a way that the matrix multiplication



(4.9)

and the trace of a matrix



(4.10)

can be drawn in the plane. Notation in which all internal lines are maximally crossed at each multiplication [319] is equally correct, but less pleasing to the eye.

4.2 CLEBSCH-GORDAN COEFFICIENTS

Consider the product

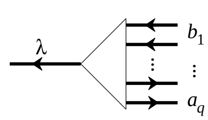
$$\begin{pmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & \\ & & & & & & & 0 \\ & & & & & & & & \ddots \end{pmatrix} C \tag{4.11}$$

of the two terms in the diagonal representation of a projection operator. This matrix has nonzero entries only in the d_λ rows of subspace V_λ. We collect them in a $[d_\lambda \times d]$ rectangular matrix $(C_\lambda)^\alpha_\sigma$, $\alpha = 1, 2, \ldots d$, $\sigma = 1, 2, \ldots d_\lambda$:

$$C_\lambda = \underbrace{\left.\begin{pmatrix} (C_\lambda)^1_1 & \ldots & (C_\lambda)^d_1 \\ \vdots & & \vdots \\ & & (C_\lambda)^d_{d_\lambda} \end{pmatrix}\right\} d_\lambda}_{d} \,. \tag{4.12}$$

The index α in $(C_\lambda)^\alpha_\sigma$ stands for all tensor indices associated with the $d = n^{p+q}$-dimensional tensor space $V^p \otimes \bar{V}^q$. In the birdtrack notation these indices are explicit:

$$(C_\lambda)_{\sigma,}{}^{b_p \ldots b_1}_{a_q \ldots a_2 a_1} =$$



. (4.13)

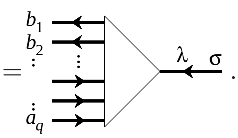
Such rectangular arrays are called *Clebsch-Gordan coefficients* (hereafter referred to as *clebsches* for short). They are explicit mappings $V \rightarrow V_\lambda$. The conjugate mapping $V_\lambda \rightarrow \bar{V}$ is provided by the product

$$C^\dagger \begin{pmatrix} \boxed{\begin{matrix} 0 & \\ & 0 \end{matrix}} & & \\ & \boxed{\begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix}} & \\ & & \boxed{\begin{matrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{matrix}} \end{pmatrix}, \tag{4.14}$$

which defines the $[d \times d_\lambda]$ rectangular matrix $(C^\lambda)_\alpha^\sigma$, $\alpha = 1, 2, \ldots d$, $\sigma = 1, 2, \ldots d_\lambda$:

$$C^\lambda = \underbrace{\left. \begin{pmatrix} (C^\lambda)_1^1 & \ldots & (C^\lambda)_1^{d_\lambda} \\ \vdots & & \vdots \\ & & (C^\lambda)_d^{d_\lambda} \end{pmatrix} \right\} d}_{d_\lambda}$$

$$(C^\lambda)_{b_1 \ldots b_p}^{a_1 a_2 \ldots a_q}, {}^\sigma = \tag{4.15}$$



The two rectangular Clebsch-Gordan matrices C^λ and C_λ are related by hermitian conjugation.

The tensors, which we have considered in section 3.10, transform as tensor products of the defining rep (3.14). In general, tensors transform as tensor products of various reps, with indices running over the corresponding rep dimensions:

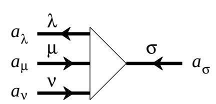
$$x_{a_1 a_2 \ldots a_p}^{a_{p+1} \ldots a_{p+q}} \qquad \text{where} \quad \begin{aligned} a_1 &= 1, 2, \ldots, d_1 \\ a_2 &= 1, 2, \ldots, d_2 \\ &\vdots \\ a_{p+q} &= 1, 2, \ldots, d_{p+q} \,. \end{aligned} \tag{4.16}$$

The action of the transformation g on the index a_k is given by the $[d_k \times d_k]$ matrix rep G_k.

Clebsches are notoriously index overpopulated, as they require a rep label and a tensor index for each rep in the tensor product. Diagrammatic notation alleviates this index plague in either of two ways:

1. One can indicate a rep label on each line:

$$C_{a_\lambda}^{a_\mu a_\nu}, {}^{a_\sigma} = \tag{4.17}$$

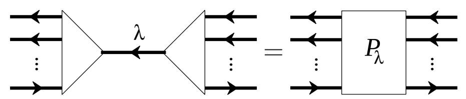


(An index, if written, is written at the end of a line; a rep label is written above the line.)

2. One can draw the propagators (Kronecker deltas) for different reps with different kinds of lines. For example, we shall usually draw the adjoint rep with a thin line.

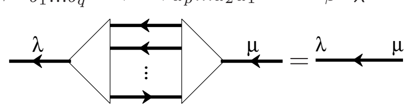
By the definition of clebsches (3.49), the λ rep projection operator can be written out in terms of Clebsch-Gordan matrices $C^\lambda C_\lambda$:

$$C^\lambda C_\lambda = \mathbf{P}_\lambda\,, \qquad \text{(no sum on } i\text{)}$$
$$(C^\lambda)_{b_1\ldots b_q}^{a_1a_2\ldots a_p}{}^{,\,\alpha}\,(C_\lambda)_{\alpha,}{}_{c_p\ldots c_2c_1}^{d_q\ldots d_1} = (\mathbf{P}_\lambda)_{b_1\ldots b_q}^{a_1a_2\ldots d_p}{}_{,}{}_{c_p\ldots c_2c_1}^{d_q\ldots d_1} \tag{4.18}$$



A specific choice of clebsches is quite arbitrary. All relevant properties of projection operators (orthogonality, completeness, dimensionality) are independent of the explicit form of the diagonalization transformation C. Any set of C_λ is acceptable as long as it satisfies the orthogonality and completeness conditions. From (4.11) and (4.14) it follows that C_λ are *orthonormal*:

$$C_\lambda C^\mu = \delta_\lambda^\mu \mathbf{1}\,,$$
$$(C_\lambda)_{\beta,}{}_{b_1\ldots b_q}^{a_1a_2\ldots a_p}(C^\mu)_{a_p\ldots a_2a_1}^{b_q\ldots b_1}{}_{,}{}^{\alpha} = \delta_\beta^\alpha \delta_\lambda^\mu$$

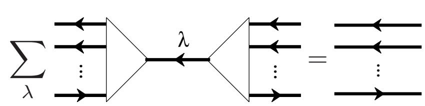


(4.19)

Here $\mathbf{1}$ is the $[d_\lambda \times d_\lambda]$ unit matrix, and C_λ's are multiplied as $[d_\lambda \times d]$ rectangular matrices.

The *completeness relation* (3.51)

$$\sum_\lambda C^\lambda C_\lambda = \mathbf{1}\,, \qquad ([d \times d] \text{ unit matrix})\,,$$
$$\sum_\lambda (C^\lambda)_{b_1\ldots b_q}^{a_1a_2\ldots a_p}{}^{,\,\alpha}(C_\lambda)_{\alpha,}{}_{c_p\ldots c_2c_1}^{d_q\ldots d_1} = \delta_{c_1}^{a_1}\delta_{c_2}^{a_2}\ldots\delta_{b_q}^{d_q}$$



(4.20)

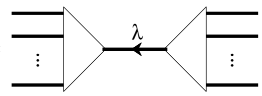
$$C^\lambda \mathbf{P}_\mu = \delta_\lambda^\mu C^\lambda\,,$$
$$\mathbf{P}_\lambda C^\mu = \delta_\lambda^\mu C^\mu\,, \qquad \text{(no sum on } \lambda, \mu)\,, \tag{4.21}$$

follows immediately from (3.50) and (4.19).

4.3 ZERO- AND ONE-DIMENSIONAL SUBSPACES

If a projection operator projects onto a zero-dimensional subspace, it must vanish identically:

$$d_\lambda = 0 \qquad \Rightarrow \qquad \mathbf{P}_\lambda =$$



$$= 0 \,. \tag{4.22}$$

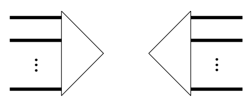
This follows from (3.49); d_λ is the number of 1's on the diagonal on the right-hand side. For $d_\lambda = 0$ the right-hand side vanishes. The general form of $\mathbf{P}_\lambda$ is

$$\mathbf{P}_\lambda = \sum_{k=1}^{r} c_k \mathbf{M}_k \,, \tag{4.23}$$

where $\mathbf{M}_k$ are the invariant matrices used in construction of the projector operators, and c_k are numerical coefficients. Vanishing of $\mathbf{P}_\lambda$ therefore implies a relation among invariant matrices $\mathbf{M}_k$.

If a projection operator projects onto a 1-dimensional subspace, its expression, in terms of the clebsches (4.18), involves no summation, so we can omit the intermediate line

$$d_\lambda = 1 \quad \Rightarrow \quad \mathbf{P}_\lambda =$$



$$= (C^\lambda)_{b_1 \ldots b_q}^{a_1 a_2 \ldots a_p} (C_\lambda)_{c_p \ldots c_2 c_1}^{d_q \ldots d_1} \,. \tag{4.24}$$

For any subgroup of $SU(n)$, the reps are unitary, with unit determinant. On the 1-dimensional spaces, the group acts trivially, $G = 1$. Hence, if $d_\lambda = 1$, the clebsch C_λ in (4.24) is an invariant tensor in $V^p \otimes \bar{V}^q$.

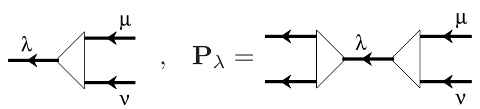
Chapter Five

Recouplings

Clebsches discussed in section 4.2 project a tensor in $V^p \otimes \bar{V}^q$ onto a subspace λ. In practice one usually reduces a tensor step by step, decomposing a 2-particle state at each step. While there is some arbitrariness in the order in which these reductions are carried out, the final result is invariant and highly elegant: any group-theoretical invariant quantity can be expressed in terms of Wigner 3- and 6-j coefficients.

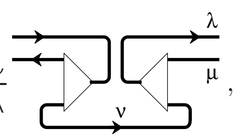
5.1 COUPLINGS AND RECOUPLINGS

We denote the clebsches for $\mu \otimes \nu \to \lambda$ by



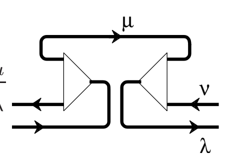
(5.1)

Here λ, μ, ν are rep labels, and the corresponding tensor indices are suppressed. Furthermore, if μ and ν are irreducible reps, the same clebsches can be used to project $\mu \otimes \bar{\lambda} \to \bar{\nu}$

$$\mathbf{P}_\nu = \frac{d_\nu}{d_\lambda}$$


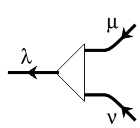
(5.2)

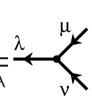
and $\nu \otimes \bar{\lambda} \to \bar{\mu}$

$$\mathbf{P}_\mu = \frac{d_\mu}{d_\lambda}$$


(5.3)

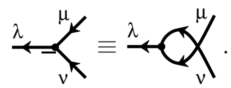
Here the normalization factors come from $P^2 = P$ condition. In order to draw the projection operators in a more symmetric way, we replace clebsches by 3-vertices:



$$\equiv \frac{1}{\sqrt{a_\lambda}}$$


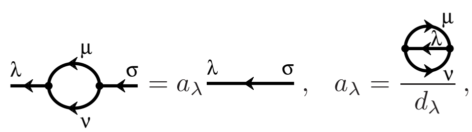
(5.4)

In this definition one has to keep track of the ordering of the lines around the vertex. If in some context the birdtracks look better with two legs interchanged, one can use Yutsis's notation [359]:



(5.5)

While all sensible clebsches are normalized by the orthonormality relation (4.19), in practice no two authors ever use the same normalization for 3-vertices (in other guises known as 3-j coefficients, Gell-Mann λ matrices, Cartan roots, Dirac γ matrices, *etc.*). For this reason we shall usually not fix the normalization

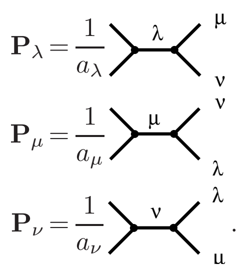


(5.6)

leaving the reader the option of substituting his or her favorite choice (such as $a = \frac{1}{2}$ if the 3-vertex stands for Gell-Mann $\frac{1}{2}\lambda_i$, *etc.*).

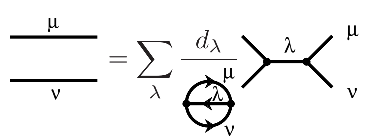
To streamline the discussion, we shall drop the arrows and most of the rep labels in the remainder of this chapter — they can always easily be reinstated.

The above three projection operators now take a more symmetric form:



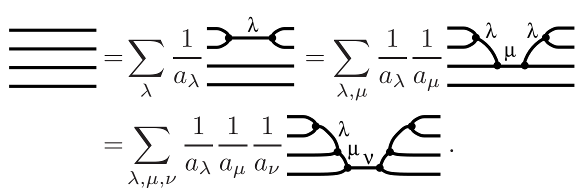
(5.7)

In terms of 3-vertices, the completeness relation (4.20) is



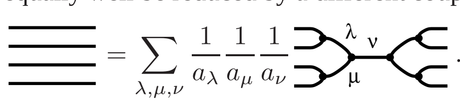
(5.8)

Any tensor can be decomposed by successive applications of the completeness relation:



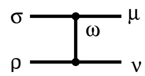
(5.9)

Hence, if we know clebsches for $\lambda \otimes \mu \to \nu$, we can also construct clebsches for $\lambda \otimes \mu \otimes \nu \otimes \ldots \to \rho$. However, there is no unique way of building up the clebsches; the above state can equally well be reduced by a different coupling scheme

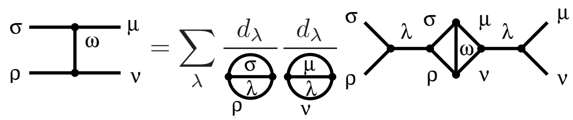


(5.10)

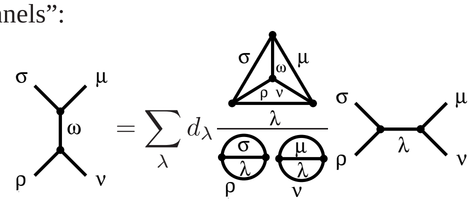
Consider now a process in which a particle in the rep μ interacts with a particle in the rep ν by exchanging a particle in the rep ω:


. (5.11)

The final particles are in reps ρ and σ. To evaluate the contribution of this exchange to the spectroscopic levels of the μ-ν particles system, we insert the Clebsch-Gordan series (5.8) twice, and eliminate one of the sums by the orthonormality relation (5.6):

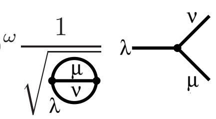

. (5.12)

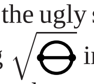
By assumption λ is an irrep, so we have a recoupling relation between the exchanges in "s" and "t channels":

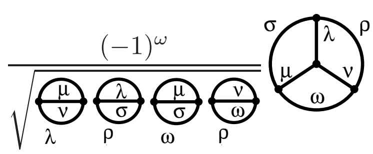

. (5.13)

We shall refer to  as 3-j coefficients and  as 6-j coefficients, and commit ourselves to no particular normalization convention.

In atomic physics it is customary to absorb  into the 3-vertex and define a 3-j symbol [238, 287, 347]

$$\begin{pmatrix} \lambda & \mu & \nu \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{\omega}$$ 
. (5.14)

Here $\alpha = 1, 2, \ldots, d_\lambda$, *etc.*, are indices, λ, μ, ν rep labels and ω the phase convention. Fixing a phase convention is a waste of time, as the phases cancel in summed-over quantities. All the ugly square roots, one remembers from quantum mechanics, come from sticking  into 3-j symbols. Wigner [347] 6-j *symbols* are related to our 6-j *coefficients* by

$$\begin{Bmatrix} \lambda & \mu & \nu \\ \omega & \rho & \sigma \end{Bmatrix} =$$ 
. (5.15)

The name $3n$-j symbol comes from atomic physics, where a recoupling involves $3n$ angular momenta $j_1, j_2, \ldots, j_{3n}$ (see section 14.2).

Most of the textbook symmetries of and relations between 6-j symbols are obvious from looking at the corresponding diagrams; others follow quickly from completeness relations.

If we know the necessary 6-j's, we can compute the level splittings due to single particle exchanges. In the next section we shall show that a far stronger claim can be made: given the 3- and 6-j coefficients, we can compute *all* multiparticle matrix elements.

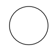
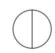
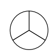
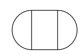
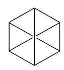
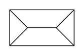
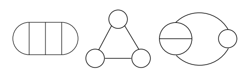
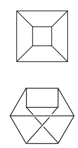
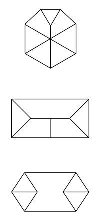
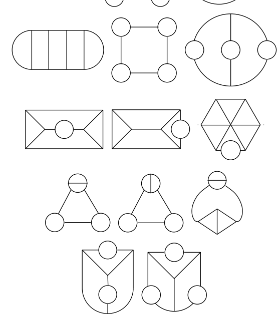
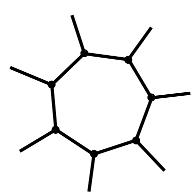
	Skeletons	Vertex insertions	Self-energy insertions	Total number
1-j				1
3j				1
6-j				2
9-j				5
12-j				16

Table 5.1 Topologically distinct types of Wigner $3n$-j coefficients, enumerated by drawing all possible graphs, eliminating the topologically equivalent ones by hand. Lines meeting in any 3-vertex correspond to any three irreducible representations with a nonvanishing Clebsch-Gordan coefficient, so in general these graphs cannot be reduced to simpler graphs by means of such as the Lie algebra (4.47) and Jacobi identity (4.48).

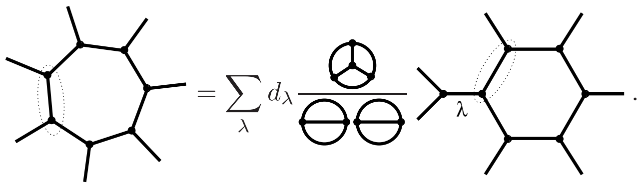
5.2 WIGNER $3n$-j COEFFICIENTS

An arbitrary higher-order contribution to a 2-particle scattering process will give a complicated matrix element. The corresponding energy levels, crosssections, *etc.*, are expressed in terms of scalars obtained by contracting all tensor indices; diagrammatically they look like “vacuum bubbles,” with $3n$ internal lines. The topologically distinct vacuum bubbles in low orders are given in table 5.1.

In group-theoretic literature, these diagrams are called $3n$-j symbols, and are studied in considerable detail. Fortunately, any $3n$-j symbol that contains as a subdiagram a loop with, let us say, seven vertices,

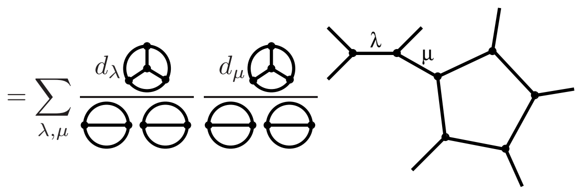
 ,

can be expressed in terms of 6-j coefficients. Replace the dotted pair of vertices by the cross-channel sum (5.13):



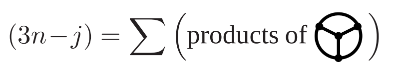
(5.16)

Now the loop has six vertices. Repeating the replacement for the next pair of vertices, we obtain a loop of length five:



(5.17)

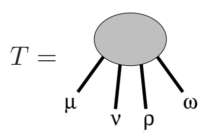
Repeating this process we can eliminate the loop altogether, producing 5-vertex-trees times bunches of 6-j coefficients. In this way we have expressed the original $3n$-j coefficients in terms of $3(n$-1)-j coefficients and 6-j coefficients. Repeating the process for the $3(n$-1)-j coefficients, we eventually arrive at the result that



(5.18)

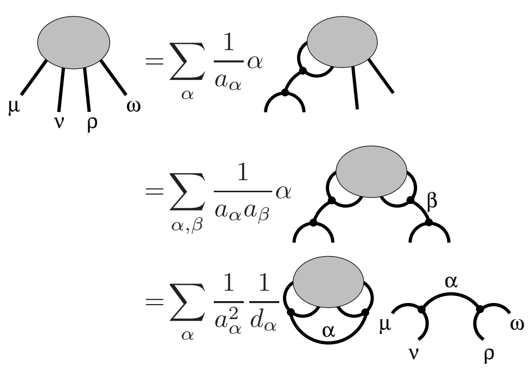
5.3 WIGNER-ECKART THEOREM

For concreteness, consider an arbitrary invariant tensor with four indices:



(5.19)

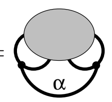
where μ, ν, ρ and ω are rep labels, and indices and line arrows are suppressed. Now insert repeatedly the completeness relation (5.8) to obtain



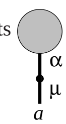
(5.20)

In the last line we have used the orthonormality of projection operators — as in (5.13) or (5.23).

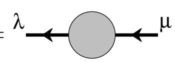
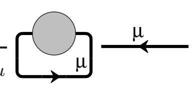
In this way any invariant tensor can be reduced to a sum over clebsches (*kinematics*) weighted by *reduced matrix elements*:

$$\langle T\rangle_\alpha =$$  . (5.21)

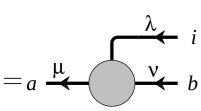
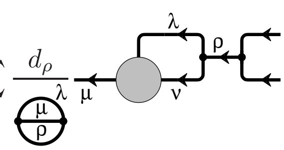
This theorem has many names, depending on how the indices are grouped. If T is a vector, then only the 1-dimensional reps (singlets) contribute

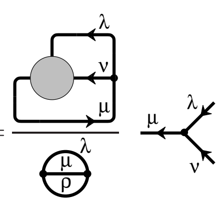
$$T_a = \sum_{\lambda}^{\text{singlets}}$$  . (5.22)

If T is a matrix, and the reps α, μ are irreducible, the theorem is called *Schur's Lemma* (for an irreducible rep an invariant matrix is either zero, or proportional to the unit matrix):

$$T_{a_\lambda}^{b_\mu} =$$  $$= \frac{1}{d_\mu}$$  $$\delta_{\lambda\mu}\,.$$ (5.23)

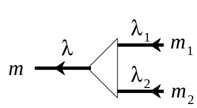
If T is an "invariant tensor operator," then the theorem is called the *Wigner-Eckart theorem* [347, 107]:

$$(T_i)_a^b =$$  $$= \sum_\rho$$ 

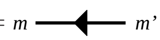
$$=$$  (5.24)

(assuming that μ appears only once in $\overline{\lambda}\otimes\mu$ Kronecker product). If T has many indices, as in our original example (5.19), the theorem is ascribed to Yutsis, Levinson, and Vanagas [359]. The content of all these theorems is that they reduce spectroscopic calculations to evaluation of "vacuum bubbles" or "reduced matrix elements" (5.21).

The rectangular matrices $(C_\lambda)_\sigma^\alpha$ from (3.27) do not look very much like the clebsches from the quantum mechanics textbooks; neither does the Wigner-Eckart theorem in its birdtrack version (5.24). The difference is merely a difference of notation. In the bra-ket formalism, a clebsch for $\lambda_1 \otimes \lambda_2 \to \lambda$ is written as

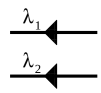
 $$= \langle \lambda_1\lambda_2\lambda m|\lambda_1 m_1\lambda_2 m_2\rangle\,.$$ (5.25)

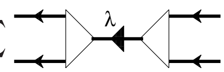
Representing the $[d_\lambda \times d_\lambda]$ rep of a group element g diagrammatically by a black triangle,

$$D^{\lambda}_{m,m'}(g) = $$ 

, (5.26)

we can write the Clebsch-Gordan series (3.49) as

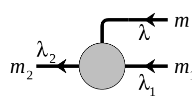


$= \sum_\lambda$ 

$$D^{\lambda_1}_{m_1 m_1'}(g) D^{\lambda_2}_{m_2 m_2'}(g) =$$

$$\sum_{\tilde\lambda,\tilde m,\tilde m_1} \langle \lambda_1 m_1 \lambda_2 m_2 | \lambda_1 \lambda_2 \tilde\lambda \tilde m \rangle D^{\tilde\lambda}_{\tilde m \tilde m_1}(g) \langle \lambda_1 \lambda_2 \tilde\lambda \tilde m_1 | \lambda_1 m_1' \lambda_2 m_2' \rangle \,.$$

An "invariant tensor operator" can be written as

$$\langle \lambda_2 m_2 | T^{\lambda}_{m} | \lambda_1 m_1 \rangle \; = $$ 

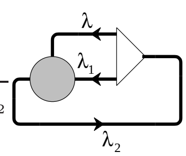
. (5.27)

In the bra-ket formalism, the Wigner-Eckart theorem (5.24) is written as

$$\langle \lambda_2 m_2 | T^{\lambda}_{m} | \lambda_1 m_1 \rangle = \langle \lambda \lambda_1 \lambda_2 m_2 | \lambda m \lambda_1 m_1 \rangle T(\lambda, \lambda_1 \lambda_2) \,, \tag{5.28}$$

where the reduced matrix element is given by

$$T(\lambda, \lambda_1 \lambda_2) = \frac{1}{d_{\lambda_2}} \sum_{n_1,n_2,n} \langle \lambda n \lambda_1 n_1 | \lambda \lambda_1 \lambda_2 n_2 \rangle \langle \lambda_2 n_2 | T^{\lambda}_{n} | \lambda_1 n_1 \rangle$$

$$= \frac{1}{d_{\lambda_2}}$$ 

. (5.29)

We do not find the bra-ket formalism convenient for the group-theoretic calculations that will be discussed here.

4.8 IRRELEVANCY OF CLEBSCHES

As was emphasized in section 4.2, an explicit choice of clebsches is highly arbitrary; it corresponds to a particular coordinatization of the d_λ-dimensional subspace V_λ. For computational purposes clebsches are largely irrelevant. Nothing that a physicist wants to compute depends on an explicit coordinatization. For example, in QCD the physically interesting objects are color singlets, and all color indices are summed over: one needs only an expression for the projection operators (4.31), not for the C_λ's separately.

Again, a nice example is the Lie algebra generators T_i. Explicit matrices are often constructed (Gell-Mann λ_i matrices, Cartan's canonical weights); however, in any singlet they always appear summed over the adjoint rep indices, as in (4.31). The summed combination of clebsches is just the adjoint rep projection operator, a very simple object compared with explicit T_i matrices ($\mathbf{P}_A$ is typically a combination of a few Kronecker deltas), and much simpler to use in explicit evaluations. As we shall show by many examples, all rep dimensions, casimirs, *etc.*. are computable once the projection operators for the reps involved are known. Explicit clebsches are superfluous from the computational point of view; we use them chiefly to state general theorems without recourse to any explicit realizations.

However, if one has to compute noninvariant quantities, such as subgroup embeddings, explicit clebsches might be very useful. Gell-Mann [137] invented λ_i matrices in order to embed $SU(2)$ of isospin into $SU(3)$ of the eightfold way. Cartan's canonical form for generators, summarized by Dynkin labels of a rep (table 7.6) is a very powerful tool in the study of symmetry-breaking chains [313, 126]. The same can be achieved with decomposition by invariant matrices (a nonvanishing expectation value for a direction in the defining space defines the little group of transformations in the remaining directions), but the tensorial technology in this context is underdeveloped compared to the canonical methods. And, as Stedman [318] rightly points out, if you need to check your calculations against the existing literature, keeping track of phase conventions is a necessity.

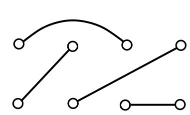
4.9 A BRIEF HISTORY OF BIRDTRACKS

> Ich wollte nicht eine abstracte Logik in Formeln darstellen, sondern einen Inhalt durch geschriebene Zeichen in genauerer und übersichtlicherer Weise zum Ausdruck bringen, als es durch Worte möglich ist.
>
> — Gottlob Frege

In this monograph, conventional subjects — symmetric group, Lie algebras (and, to a lesser extent, continuous Lie groups) — are presented in a somewhat unconventional way, in a flavor of diagrammatic notation that I refer to as "birdtracks." Similar diagrammatic notations have been invented many times before, and continue to be invented within new research areas. The earliest published example of diagrammatic notation as a language of computation, not a mere mnemonic device, appears to be F.L.G. Frege's 1879 *Begriffsschrift* [127], at its time a revolution that laid the foundation of modern logic. The idiosyncratic symbolism was not well received, ridiculed as "incorporating ideas from Japanese." Ruined by costs of typesetting, Frege died a bitter man, preoccupied by a deep hatred of the French, of Catholics, and of Jews.

According to Abdesselam and Chipalkatti [4], another precursor of diagrammatic methods was the invariant theory discrete combinatorial structures introduced by Cayley [50], Sylvester [322], and Clifford [61, 183], reintroduced in a modern, diagrammatic notation by Olver and Shakiban [265, 266].

In his 1841 fundamental paper [167] on the determinants today known as "Jacobians," Jacobi initiated the theory of irreps of the symmetric group $\mathcal{S}_k$. Schur used the $\mathcal{S}_k$ irreps to develop the representation theory of $GL(n;\mathbb{C})$ in his 1901 dissertation [307], and already by 1903 the Young tableaux [358, 339] (discussed here in chapter 9) came into use as a powerful tool for reduction of both $\mathcal{S}_k$ and $GL(n;\mathbb{C})$ representations. In quantum theory the group of choice [344] is the unitary group $U(n)$, rather than the general linear group $GL(n;\mathbb{C})$. Today this theory forms the core of the representation theory of both discrete and continuous groups, described in many excellent textbooks [238, 64, 350, 138, 26, 11, 317, 132, 133, 228]. Permutations and their compositions lend themselves naturally to diagrammatic representation developed here in chapter 6. In his extension of the $GL(n;\mathbb{C})$ Schur theory to representations of $SO(n)$, R. Brauer [31] introduced diagrammatic notation for δ_{ij} in order to represent "Brauer algebra" permutations, index contractions, and matrix multiplication diagrammatically, in the form developed here in chapter 10. His equation (39)



(send index 1 to 2, 2 to 4, contract ingoing $(3 \cdot 4)$, outgoing $(1 \cdot 3)$) is the earliest published proto-birdtrack I know about.

R. Penrose's papers are the first (known to me) to cast the Young projection operators into a diagrammatic form. In this monograph I use Penrose diagrammatic notation for symmetrization operators [281], Levi-Civita tensors [283], and "strand networks” [282]. For several specific, few-index tensor examples, diagrammatic Young projection operators were constructed by Canning [41], Mandula [227], and Stedman [319].

It is quite likely that since Sophus Lie’s days many have doodled birdtracks in private without publishing them, partially out of a sense of gravitas and in no insignificant part because preparing these doodles for publications is even today a painful thing. I have seen unpublished 1960s course notes of J. G. Belinfante [6, 19], very much like the birdtracks drawn here in chapters 6–9, and there are surely many other such doodles lost in the mists of time. But, citing Frege [128], “the comfort of the typesetter is certainly not the *summum bonum*,” and now that the typesetter is gone, it is perhaps time to move on.

The methods used here come down to us along two distinct lineages, one that can be traced to Wigner, and the other to Feynman.

Wigner’s 1930s theory, elegantly presented in his group theory monograph [347], is still the best book on what physics is to be extracted from symmetries, be it atomic, nuclear, statistical, many-body, or particle physics: all physical predictions (“spectroscopic levels”) are expressed in terms of Wigner’s $3n$-j coefficients, which can be evaluated by means of recursive or combinatorial algorithms. As explained here in chapter 5, decomposition (5.8) of tensor products into irreducible reps implies that any invariant number characterizing a physical system with a given symmetry corresponds to one or several “vacuum bubbles,” trivalent graphs (a graph in which every vertex joins three links) with no external legs, such as those listed in table 5.1.

Since the 1930s much of the group-theoretical work on atomic and nuclear physics had focused on explicit construction of clebsches for the rotation group $SO(3) \simeq SU(2)$. The first paper recasting Wigner’s theory in graphical form appears to be a 1956 paper by I. B. Levinson [213], further developed in the influential 1960 monograph by A. P. Yutsis (later A. Jucys), I. Levinson and V. Vanagas [359], published in English in 1962 (see also refs. [109, 33]). A recent contribution to this tradition is the book by G. E. Stedman [319], which covers a broad range of applications, including the methods introduced in the 1984 version of the present monograph [82]. The pedagogical work of computer graphics pioneer J. F. Blinn [25], who was inspired by Stedman’s book, also deserves mention.

The main drawback of such diagrammatic notations is lack of standardization, especially in the case of clebsches. In addition, the diagrammatic notations designed for atomic and nuclear spectroscopy are complicated by various phase conventions.

R. P. Feynman went public with Feynman diagrams on my second birthday, April 1, 1948, at the Pocono Conference. The idiosyncratic symbolism (Gleick [141] describes it as “chicken-wire diagrams”) was not well received by Bohr, Dirac, and Teller, leaving Feynman a despondent man [141, 308, 237]. The first Feynman diagram appeared in print in Dyson’s article [106, 309] on the equivalence of (at that time) the still unpublished Feynman theory and the theories of Schwinger and Tomonaga.

If diagrammatic notation is to succeed, it need be not only precise, but also beautiful. It is in this sense that this monograph belongs to the tradition of R. P. Feynman, whose sketches of the very first “Feynman diagrams” in his fundamental 1949 Q.E.D.

paper [119, 309] are beautiful to behold. Similarly, R. Penrose's [281, 282] way of drawing symmetrizers and antisymmetrizers, adopted here in chapter 6, is imbued with a very Penrose aesthetics, and even though the print is black and white, one senses that he had drawn them in color.

In developing the "birdtrack" notation in 1975 I was inspired by Feynman diagrams and by the elegance of Penrose's binors [281]. I liked G. 't Hooft's 1974 double-line notation for $U(n)$ gluon group-theory weights [163], and have introduced analogous notation for $SU(n)$, $SO(n)$ and $Sp(n)$ in my 1976 paper [73]. In an influential paper, M. Creutz [69] has applied such notation to the evaluation of $SU(n)$ lattice gauge integrals (described here in chapter 8). The challenge was to develop diagrammatic notation for the exceptional Lie algebras, and I succeeded [73], except for E_8, which came later.

In the quantum groups literature, graphs composed of vertices (4.44) are called *trivalent*. The Jacobi relation (4.48) in diagrammatic form was first published [73] in 1976; though it seems surprising, I have not found it in the earlier literature. This set of diagrams has since been given the moniker "IHX" by D. Bar-Natan [14]. In his Ph.D. thesis Bar-Natan has also renamed the Lie algebra commutator (4.47) the "STU relation," by analogy to Mandelstam's scattering cross-channel variables (s, t, u), and the full antisymmetry of structure constants (4.46) the "AS relation."

So why call this "birdtracks" and not "Feynman diagrams"? The difference is that here diagrams are not a mnemonic device, an aid in writing down an integral that is to be evaluated by other techniques. In our applications, explicit construction of clebsches would be superfluous, and we need no phase conventions. Here "birdtracks" are everything—unlike Feynman diagrams, here all calculations are carried out in terms of birdtracks, from start to finish. Left behind are blackboards and pages of squiggles of the kind that made Bernice Durand exclaim: "What are these birdtracks!?" and thus give them the name.