Chapter 26

Continuous symmetry factorization

Hard work builds character.
— V.I. Warshavski, Private Investigator

TRACE FORMULAS relate short time dynamics (unstable periodic orbits) to long time invariant state space densities (natural measure). Higher dimensional dynamics requires inclusion of higher-dimensional compact invariant sets, such as partially hyperbolic invariant tori, into trace formulas. A trace formula for a partially hyperbolic $(N + 1)$-dimensional compact manifold invariant under N global continuous symmetries is derived here. In this extension of 'periodic orbit' theory there are no or very few periodic orbits - the relative periodic orbits that the trace formula has support on are almost never eventually periodic.

The classical trace formula for smooth continuous time flows

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$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta A_p - s T_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}$$

relates the spectrum of the evolution operator

$$\mathcal{L}(x', x) = \delta\left(x' - f^t(x)\right) e^{\beta A(x,t)} \tag{26.1}$$

to the unstable periodic orbits p of the flow $f^t(x)$. This formula (and the associated spectral determinants and cycle expansions) is valid for fully hyperbolic flows.

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Here we derive the corresponding formula for dynamics invariant under a compact group of symmetry transformations. In what follows, a familiarity with basic group-theoretic notions is assumed, with the definitions relegated to appendix A10.1.

fast track:
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26.1 Compact groups

All the group theory that we shall need here is given by

> *The Peter-Weyl Theorem, and its corollaries:* A compact Lie group G is completely reducible, its representations are fully reducible (just as in the finite group representation theory), every compact Lie group is a closed subgroup of $U(n)$ for some n, and every continuous, unitary, irreducible representation of a compact Lie group is finite dimensional.

The theory of semisimple Lie groups is elegant, perhaps too elegant. In what follows, we serve group theoretic nuggets in need-to-know portions, offering a pedestrian route through a series of simple examples of familiar aspects of group theory and Fourier analysis, and a high, cyclist road in the text proper.

But main idea is this: the character $\chi^{(m)}(\theta)$ of the Frobenius-Weyl representation theory is a generalization to all compact continuous Lie groups of the weight $e^{i\theta m}$ in the Fourier decomposition of a smooth function on a circle into eigenmodes of translation. mth Fourier component fits m node function around the circle; $(m_1, m_2, \ldots, m_N)$ representation of a compact Lie group fits a corresponding multi-mode function onto the smooth manifold swept out by the action of the group. So a basis for a d-dimensional representation $(m_1, m_2, \ldots, m_N)$ of an N-dimensional compact Lie group is a set of d linearly independent eigenfunctions on the N-dimensional compact group manifold, with m_1, m_2, $\ldots$, m_N 'nodes' along the N directions needed to span the manifold. For a circle this is Fourier analysis; for a sphere these are spherical harmonics, and the Peter-Weyl theorem states that analogous expansion exists for every compact Lie group. We will never need to construct these explicitly.

exercise 26.2

26.1.1 Group representations

Let q_a be a vector in d-dimensional vector space V, and G be a group of linear transformations

$$q'_a = D(g)_a{}^b q_b \,, \qquad a, b = 1, 2, \ldots, d \,, \quad g \in G$$

(repeated indices summed throughout this chapter). The $[d\times d]$ matrices $D(g)$ form a representation of the group G. Vectors in the dual space $\overline{q}$ transform as

$$q'^a = D(g)^a{}_b \, q^b \,.$$

Tensors transform as

$$h'_{ab}{}^c = D(g)_a{}^f D(g)_b{}^e D(g)^c{}_d \, h_{fe}{}^d \,.$$

A function H is an invariant function if (and only if) for any transformation $g \in G$ and for any set of vectors $\overline{q}, \overline{r}, s, \ldots$

$$H(D(g)^{\dagger}\overline{q}, D(g)^{\dagger}\overline{r}, \ldots D(g)s) = H(\overline{q}, \overline{r}, \ldots, s)\,. \tag{26.2}$$

Unitary transformations connected to the identity can be generated by sequences of infinitesimal transformations

$$D(g)_a{}^b \simeq \delta_a^b + i\epsilon_i (T_i)_a^b \qquad \epsilon_i \in \mathbb{R}\,, \quad T_i \text{ hermitian}\,,$$

and $|\epsilon_i| \ll 1$. (More generally, one also needs to study invariance under discrete coordinate transformations (see chapter 25).

Consider a multilinear invariant function

$$H(\overline{q}, \overline{r}, \ldots, s) = h_{ab\ldots}{}^{\ldots c}\, q^a r^b \ldots s_c$$

In terms of the generators T_i, H is invariant if all generators “annihilate" it, $T_i \cdot h = 0$:

$$(T_i)_a^{a'} h_{a'b\ldots}{}^{c\ldots} + (T_i)_b^{b'} h_{ab'\ldots}{}^{c\ldots} - (T_i)_{c'}^{c} h_{ab\ldots}{}^{c'\ldots} + \ldots \quad = \quad 0\,. \tag{26.3}$$

example 26.1
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Vector space V is irreducible if the only invariant subspaces of V under the action of G are (0) and V. If every V on which G acts can be written as a direct sum of irreducible subspaces, then G is completely reducible.

26.1.2 Group integrals

Consider a group integral of form

$$\int dg\, D(g)_a{}^b D(g)^c{}_d\,, \tag{26.4}$$

where $D(g)_a{}^b$ is a unitary $[d \times d]$ matrix representation of $g \in G$, G a compact Lie group, $D(g)^c{}_d$ is the matrix representation of the action of g on the dual vector space,

$$D(g)^c{}_d = (D(g)^{\dagger})_d{}^c\,,$$

and the integration is over the entire range of $g \in G$, G a compact Lie group. For a finite group G with $|G|$ group elements the normalized measure is a discrete sum,

$$d\mu(x) \;=\; \frac{1}{|G|}\sum_g \delta(gx)\,.$$

For continuous groups, the integration measure dg is known as the Haar measure, and, given an explicit parametrization of the group manifold, is explicitly computable (see example 26.4 and example 26.5). However, we do need such explicit parametrizations, as the integral (26.4) over the entire group is defined by two requirements:

exercise 26.1

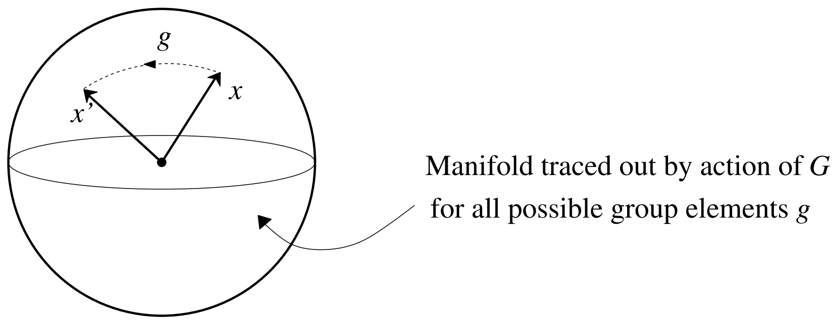
1. *Normalization:* The group average of an scalar quantity is the quantity itself,

$$\int dg = 1 \,. \tag{26.5}$$

2. *Orthonormality of irreducible representations.* How do we define

$$\int dg\, D(g)_a{}^b \,?$$

The action of $g \in G$ is to rotate a vector x_a into $x'_a = D(g)_a{}^b x_b$



The averaging smears x in all directions, hence the second integration rule

$$\int dg\, D(g)_a{}^b = 0 \,, \qquad \text{if } D(g) \text{ is } \textit{non-trivial} \text{ representation}\,, \tag{26.6}$$

simply states that the average over all rotations of a vector is zero.

A representation is trivial (a 'singlet') if $D(g) = 1$ for all group elements g. In this case no averaging is taking place, and the first integration rule (26.5) applies.

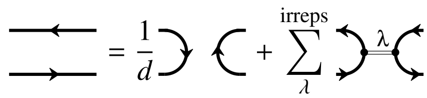
What happens if we average a bilinear combination of a pair of vectors x, y? There is no reason why such pair should average to zero; for example, we know that the scalar function $|x|^2 = \sum_a x_a x_a^* = x_a x^a$ is invariant under unitary transformations, so it cannot have a vanishing average. Therefore, in general

$$\int dg\, D(g)_a{}^b D(g)^c{}_d \neq 0 \,. \tag{26.7}$$

To get a feeling for what the right-hand side looks like, we recommend that you work out the examples.

example 26.2
p. 500

Now let $D(g)$ be any irreducible $[d\times d]$ rep. Irreducibility (known in this context as 'Schur's Lemma') means that any invariant $[d\times d]$ tensor A^a_b is proportional to δ^a_b. As the only bilinear invariant is δ^a_b, the Clebsch-Gordan series



(26.8)

contains one and only one singlet. Only the singlet survives the group averaging, and

$$\int dg\ D^{(\lambda)}(g)_a{}^d D^{(\lambda)}(g)^b{}_c = \frac{1}{d}\,\delta^d_c\delta^b_a\,. \tag{26.9}$$

is true for any $[d\times d]$ *irreducible* rep $D^{(\lambda)}(g)$.

If we take $D^{(\mu)}(g)_\alpha{}^\beta$ and $D^{(\lambda)}(g)_d{}^c$ in inequivalent representations λ,μ (there is no matrix K such that $D^{(\lambda)}(g) = KD^{(\mu)}(g)K^{-1}$ for any $g\in G$), then there is no way of forming a singlet, and

$$\int dg\, D^{(\lambda)}(g)_a{}^d D^{(\mu)}(g)^\beta{}_\alpha = 0 \qquad \text{if} \qquad \lambda\neq\mu\,. \tag{26.10}$$

26.1.3 Characters

The trace of an irreducible $[d\times d]$ matrix representation λ of g is called the *character* of the representation:

$$\chi^{(\lambda)}(g) = \mathrm{tr}\, D^{(\lambda)}(g) = D^{(\lambda)}(g)_a{}^a\,. \tag{26.11}$$

The character of the conjugate representation is

$$\chi^{(\lambda)}(g^{-1}) = \mathrm{tr}\, D^{(\lambda)}(g)^\dagger = D^{(\lambda)}(g)^a{}_a = \chi^{(\lambda)}(g)^*\,. \tag{26.12}$$

Contracting (26.8) with two arbitrary invariant $[d\times d]$ tensors $h_d{}^a$ and $(f^\dagger)_b{}^c$, we obtain the *character orthonormality relation*

$$\int dg\ \chi^{(\lambda)}(hg)\chi^{(\mu)}(gf) = \delta_{\lambda\mu}\,\frac{1}{d_\lambda}\,\chi^{(\lambda)}(hf^\dagger) \tag{26.13}$$

The character orthonormality tells us that if two group invariant quantities share a $D^{(\lambda)}(g)D^{(\lambda)}(g^{-1})$ pair, the group averaging sews them into a single group invariant quantity. The replacement of $D^{(\lambda)}(g)_a{}^b$ by the character $\chi^{(\lambda)}(h^{-1}g)$ does not mean that the matrix structure is lost; $D^{(\lambda)}(g)_a{}^b$ can be recovered by differentiating

$$D(g)_a{}^b = \frac{d}{dh_b{}^a}\chi^{(\lambda)}(h^{-1}g)\,. \tag{26.14}$$

The essential group theory we shall need here is most compactly summarized by

The Group Orthogonality Theorem: Let D_μ, $D_{\mu'}$ be two irreducible matrix representations of a compact group G of dimensions d_μ, $d_{\mu'}$,

$$\int dg\; D^{(\mu)}(g)_a{}^b D^{(\mu')}(g^{-1})_{b'}{}^{a'} = \frac{1}{d_\mu}\delta_{\mu,\mu'}\delta_a^{a'}\delta_b^{b'}\,.$$

The new trace formula follows from the full reducibility of representations of a compact group G acting linearly on a vector space V, with irreducible representations labeled by sets of integers $\mu = (\mu_1, \cdots, \mu_N)$, and the vector space V decomposed into invariant subspaces V_μ. For a N-dimensional compact Lie group G the fundamental result is the Weyl full reducibility theorem, with projection operator onto the V_μ irreducible subspace given by

$$P_\mu = d_\mu \int_G g\; \chi^{(\mu)}(g^{-1}) U(g)\,. \tag{26.15}$$

The group elements $g = g(\theta_1, \ldots, \theta_N) = e^{i\theta\cdot T}$ are parameterized by N real numbers $\{\theta_1, \ldots, \theta_N\}$ of finite range, hence designation 'compact'. The N group generators T_a, $a = 1, \cdots, N$ close the Lie algebra of G.

26.1.4 Transformation operators, projection operators

Suppose we have an arbitrary function or set of functions. How do we obtain functions with desired symmetry properties? If f is an arbitrary function,

$$P^\alpha_{ij} f = \frac{d_\alpha}{|G|}\sum_G D^{(\alpha)}(-1) g f = F^\alpha_{ij}$$

which is either zero or a basis function for the ith row of irrep α: a function of symmetry species (α, i).

example 26.3 p. 501 example 26.4 p. 501 example 26.5 p. 501

The character χ is the trace $\chi^{(\mu)}(g) = \mathrm{tr}\, D_\mu(g) = \sum_{i=1}^{d_\mu} D_\mu(g)_{ii}$, where $D_\mu(g)$ is a $[d_\mu \times d_\mu]$-dimensional matrix representation of action of the group element g on the irreducible subspace V_μ. We will sometimes employ notation g as a shorthand for $D(g)$, i.e., by $x' = gx$ we mean the matrix operation $x'_i = \sum_{j=1}^{d} D(g)_{ij} x_j$, and by $f'(x) = g\, f(x) = f(gx)$, $f(x)$ a smooth function over the state space $x \in \mathcal{M}$, we mean $f'(x) = f(D(g)x)$.

For an invariant scalar quantity the average over the group in (26.15) must be the quantity itself, so the group integral is weighted by the normalized Haar measure (unit group volume) $\int_G g = 1$, and d_μ is the multiplicity of degenerate eigenvalues in representation μ.

26.2 Continuous symmetries of dynamics

If action of every element g of a compact group G commutes with the flow $\dot{x} = v(x)$,

$$D(g)v(x) = v(D(g)x)\,,\ D(g)f^t(x) = f^t(D(g)x)\,,$$

G is a global symmetry of the dynamics. The finite time evolution operator (26.1) can be written as $\mathcal{L}^t = e^{t\mathcal{A}}$ in terms of the time-evolution generator (19.24)

$$\mathcal{A} = \lim_{\delta\tau\to 0^+} \frac{1}{\delta\tau}\left(\mathcal{L}^{\delta\tau} - I\right) = -\partial_i(v_i(x))\,. \tag{26.16}$$

The operator $e^{t\mathcal{A}}$ commutes with all symmetry transformations $e^{i\theta\cdot T}$. For a given state space point x together they sweep out a $(N+1)$-dimensional manifold of equivalent orbits.

As in (25.13), $\mathcal{L}(y,x)$ is invariant function. The irreducible eigenspaces of G are also eigenspaces of the dynamical evolution operator $\mathcal{L}^t$, with the decomposition of the evolution operator to irreducible subspaces, $\mathcal{L} = \sum_\mu \mathcal{L}_\mu$, following immediately by application of the projection operator (26.15):

$$\mathcal{L}^t_\mu(y,x) = d_\mu \int_G g\, \chi^{(\mu)}(g)\mathcal{L}^t(D_\mu(g^{-1})y, x)\,. \tag{26.17}$$

As G commutes with f^t, all eigenfunctions ρ of $\mathcal{L}^t$ must be invariant under G, $\rho(x) = \rho(gx)$. Infinitesimally, in terms of Lie algebra generators $\mathbf{T}_\phi\rho(x) = 0$.

26.2.1 Relative periodic orbits

Relative periodic orbits are orbits $x(t)$ in state space $\mathcal{M}$ which exactly recur

$$x(t) = D(g_p^r)x(t + rT_p) \tag{26.18}$$

for a fixed *relative period* T, its repeats $r = 1, 2, \cdots$, and a fixed group action $g \in G$ of $\mathcal{M}$. This group action is sometimes referred to as a 'phase', or a 'shift'. Relative periodic orbits are to periodic solutions what relative equilibria (traveling waves) are to equilibria (steady solutions).

For dynamical systems with continuous symmetries relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space. As almost any such orbit explores ergodically the manifold swept by action of G, they are sometimes referred to as 'quasiperiodic.' However, an orbit can be periodic if it satisfies a special symmetry. If $g^m = 1$ is of finite order m, then the corresponding orbit is periodic with period mT. If g is not of finite order k, orbits can be periodic only after the action of g.

chapter 25

In either case, we refer to the orbits in $\mathcal{M}$ satisfying (26.18) as *relative periodic orbits*.

26.2.2 Stability of relative periodic orbits

A infinitesimal group transformation maps globally a trajectory in a nearby trajectory, so we expect the initial point perturbations along to group manifold to be marginal, growing at rates slower than exponential. The argument is akin to (4.9), the proof of marginality of perturbations along the trajectory. Consider two nearby initial points separated by an infinitesimal group rotation $\delta\theta$: $\delta x_0 = f^{\delta\theta}(x_0) - x_0 = v(x_0)\delta\theta$. By the commutativity of the group with the flow, $f^{t+\delta t} = f^{\delta t+t}$. Expanding both sides of $f^t(f^{\delta t}(x_0)) = f^{\delta t}(f^t(x_0))$, keeping the leading term in δt, and using the definition of the Jacobian matrix (4.5), we observe that $J^t(x_0)$ transports the velocity vector at x_0 to the velocity vector at $x(t)$ at time t:

$$v(x(t)) = J^t(x_0)\, v(x_0)\,. \tag{26.19}$$

In nomenclature of page 83, the Jacobian matrix maps the initial, Lagrangian coordinate frame into the current, Eulerian coordinate frame.

However, already at this stage we see that if the orbit is periodic, $g_p x(T_p) = x(0)$, at any point along cycle p the velocity v is an eigenvector of the Jacobian matrix $J_p = J^{T_p}$ with an eigenvalue of unit magnitude,

$$J_p(x)\, v(x) = v(x)\,, \qquad x \in \mathcal{M}_p\,. \tag{26.20}$$

Two successive points along the cycle separated by δx_0 have the same separation after a completed period $\delta x(T_p) = g_p \delta x_0$, hence eigenvalue of magnitude 1.

26.3 Symmetry reduced trace formula for flows

As any pair of nearby points on a periodic orbit returns to itself exactly at each cycle period, the eigenvalue of the Jacobian matrix corresponding to the eigenvector along the flow necessarily equals unity for all periodic orbits. In presence of N-dimensional symmetry Lie group G, further N eigenvalues equal unity. Hence the trace integral $\mathrm{tr}\,\mathcal{L}^t$ requires a separate treatment for the direction along the flow and for the N group transformation directions.

To evaluate the contribution of a prime cycle p of period T_p, restrict the integration to an infinitesimally thin manifold $\mathcal{M}_p$ enveloping the cycle and all of its rotations by G, pick a point on the cycle, and choose a local coordinate system with a longitudinal coordinate $dx_\parallel$ along the direction of the flow, N coordinates dx_G along the invariant manifold swept by p under the action of the symmetry group G, and $(d-N-1)$ transverse coordinates $x_\perp$.

$$\mathrm{tr}\,_p\mathcal{L}^t_\mu = d_\mu \int_G g\, \chi^{(\mu)}(g) \int_{\mathcal{M}_p} dx_\perp dx_\parallel dx_G\, \delta\Big(x - D(g))f^t(x)\Big)\,. \tag{26.21}$$

The integral along the longitudinal, time-evolution coordinate was computed in (21.16). Eliminating the time dependence by Laplace transform one obtains

$$\int_0^\infty e^{-st} \oint_p dx_\parallel\, \delta_\parallel(x_\parallel - f^t(x_\parallel)) \;=\; T_p \sum_{r=1}^\infty e^{-sT_p r}\,. \tag{26.22}$$

example 26.6
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The μ subspace group integral is simple:

$$\int_G g\, \chi^{(\mu)}(g) \int_{\mathcal{M}_p} dx_G\, \delta\Big(x_G - D_\mu(g) f^{rT_p}(x_G)\Big) = \chi^{(\mu)}(g_p^r). \tag{26.23}$$

For the remaining transverse coordinates the Jacobian matrix is defined in a $(N{+}1)$-dimensional surface of section $\mathcal{P}$ of constant $(x_\parallel, x_G)$. Linearization of the periodic flow transverse to the orbit yields

$$\int_{\mathcal{P}} dx_\perp \delta\Big(x_\perp - D_\mu(g_p^r) f^{rT_p}(x_\perp)\Big) = \frac{1}{\left|\det\left(\mathbf{1} - \hat{M}_p^r\right)\right|}\,, \tag{26.24}$$

where $\hat{M}_p = D_\mu(g_p) M_p$ is the p-cycle $[(d-1-N)\times(d-1-N)]$ symmetry reduced Jacobian matrix, computed on the reduced surface of section and rotated by g_p. As in (21.5), we assume hyperbolicity, i.e., that the magnitudes of all transverse eigenvalues are bounded away from unity.

The *classical symmetry reduced trace formula for flows* follows by substituting (26.22) - (26.24) into (26.21):

$$\sum_{\beta=0}^{\infty} \frac{1}{s - s_{\mu,\beta}} = d_\mu \sum_p T_p \sum_{r=1}^{\infty} \chi^{(\mu)}(g_p^r) \frac{e^{r(\beta A_p - s T_p)}}{\left|\det\left(\mathbf{1} - \hat{M}_p^r\right)\right|}\,. \tag{26.25}$$

(we can restore $e^{\beta A_p}$ from (26.1) provided that the observable $a(x)$ also commutes with G.) The sum is over all prime *relative periodic orbits* p and their repeats, orbits in state space which satisfy

$$x(t) = D(g_p) x(t + T_p) \tag{26.26}$$

for a fixed *relative period* T_p and a fixed *shift* g_p.

The $\mu = (0, 0, \cdots, 0)$ subspace is the one of most relevance to chaotic dynamics, as its leading eigenfunction, with the fewest nodes and the slowest decay rate, corresponds to the natural measure observed in the long time dynamics.

In contrast to the case of continuous symmetries, where relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space, for discrete symmetries all relative periodic orbits are eventually periodic.

Résumé

One of the goals of nonlinear dynamics is to describe the long time evolution of ensembles of trajectories, when individual trajectories are exponentially unstable.

The main tool in this effort have been trace formulas because they make explicit the duality between individual short time trajectories, and long time invariant densities (natural measures, eigenfunctions of evolution operators). So far, the main successes have been in applications to low dimensional flows and iterated mappings, where the compact invariant sets of short-time dynamics are equilibria, periodic points and periodic orbits. Dynamics in higher dimensions requires extension of trace formulas to higher-dimensional compact invariant sets, such as partially hyperbolic invariant tori.

Here we have used a particularly simple direct product structure of a global symmetry that commutes with the flow to reduce the dynamics to a symmetry reduced (d–1 –N)-dimensional state space $\mathcal{M}/G$. The trace formulas do not require explicit construction (in general difficult), neither of the reduced state space, nor of the Haar measures.

Amusingly, in this extension of 'periodic orbit' theory from unstable 1-dimensional closed orbits to unstable (N + 1)-dimensional compact manifolds invariant under continuous symmetries, there are no or very few periodic orbits. Relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space, unless forced to do so by a discrete symmetry, so looking for periodic orbits in systems with continuous spatial symmetries is a fool's errand.

Restriction to compact Lie groups in derivation of the trace formula (26.25) was a matter of convenience, as the general case is more transparent than particular implementations (such as SO(2) and SO(3) rotations, with their explicit Haar measures and characters). This can be relaxed as the need arises - much powerful group theory developed since Cartan-Weyl era is at our disposal. For example, the time evolution is in general non-compact (a generic trajectory is an orbit of infinite length). Nevertheless, the trace formulas have support on compact invariant sets in $\mathcal{M}$, such as periodic orbits and (N+1)-dimensional manifolds generated from them by action of the global symmetry groups. Just as existence of a periodic orbit is a consequence of given dynamics, not any global symmetry, higher-dimensional flows beckon us on with nontrivial higher-dimensional compact invariant sets (for example, partially hyperbolic invariant tori) for whom the trace formulas are still to be written.

Commentary

Remark 26.1. Literature Here we need only basic results, on the level of any standard group theory textbook [11]. This material is covered in any introduction to linear algebra [9, 16, 19] We found Tinkham [25] the most enjoyable as a no-nonsense, the user friendliest introduction to the basic concepts. The construction of projection operators given here is taken from refs. [5–7]. Who wrote this down first we do not know, but we like Harter's exposition [12–14] best. Harter's theory of class algebras offers a more elegant and systematic way of constructing the maximal set of commuting invariant matrices $\mathbf{M}_i$ than the sketch offered here. Chapter 2. of ref. [1] offers a clear and pedagogical introduction to Lie groups of transformations. For the Group Orthogonality Theorem see, for example, [7, 26], or Google.

Remark 26.2. Full reducibility of semisimple Lie groups:

The study of integrals over compact Lie groups with respect to Haar measure is important in many areas of mathematics and physics, see Mehta [17]. In 1896-1897 Frobenius introduced notions of 'characters' and group 'representations', and proved the full reducibility of representations of finite groups. The characters $\chi^{(\mu)}(g)$ for all compact semisimple Lie groups were constructed and the full reducibility proven by Weyl [21], extending Cartan's local Lie algebra classification to a global theory of group representations. For the history of this period, see the excellent essay by Hawkins [15].

Diagrammatic notation for group theory is explained in the birdtracks.eu webbook.

Remark 26.3. A brief history of relativity: In context of semiclassical quantization Creagh and Littlejohn [3, 4] concentrate on the case when the continuous symmetry family of orbits includes a true periodic orbit (they use infinitesimal variation around true periodic orbit), not the symmetry reduced case considered here (where almost every relative periodic orbit of the symmetry-reduced dynamics is not a periodic orbit in the full space). They emphasize generalized surface of section dynamics. They refer to relative periodic orbits as 'generalized periodic orbits', with 'generalized period' $\mathbf{T}_p = (T_p, \mu_p)$. They mention, but do not go to irreducible reps of the symmetry groups, hence no 'classical symmetry reduced trace formula for flows' (26.25) in these papers. Instead, they explicitly compute group volumes. In addition to the reduced dynamics weight $|\det(1 - M_\perp)|$ they get $\partial\theta/\partial J$ which we do not have. The Berkeley group did it right for discrete symmetries [22, 23].

Here we follow Creagh [2], and in the axially-symmetric case ref. [20]. Creagh refers to relative periodic orbits as 'pseudoperiodic' orbits. Ref. [20] refers to relative periodic orbits as 'reduced periodic' orbits, and to the corresponding orbits in the full state space as 'quasiperiodic'. Creagh remarks at the very end of his paper to his formula (6.4) as the "pleasing result that the quantally reduced spectrum is determined by the classically reduced periodic orbits in the usual way." Ref. [10] discusses a trace formula in symmetry-reduced space. Muratore-Ginanneschi [18] gives an elegant discussion of 'zero-modes' in the path integral formulation, but does not go to irreps either. Ref. [24] applies the method to the problems of noninteracting identical particles.

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26.4 Examples

Example 26.1. ***Lie algebra.*** *As one does not want the rules to change at every step, the generators T_i are themselves invariant tensors,*

$$(T_i)_b^{\ a} = D(g)^a{}_{a'} D(g)_b{}^{b'} D^{(A)}(g)_{ii'} (T_{i'})_{b'}^{\ a'} \,, \tag{26.27}$$

where $D^{(A)}(g)_{ij}$ is the adjoint $[N \times N]$ matrix representation of $g \in G$. For infinitesimal transformations, $D(g)_a{}^b \simeq \delta_a^b + i\epsilon_i (T_i)_a^b$. The $[d \times d]$ matrices T_i are in general non-commuting, and from (26.3) it follows that they close N-element Lie algebra

$$T_i T_j - T_j T_i = i C_{ijk} T_k \qquad i, j, k = 1, 2, ..., N \,,$$

where C_{ijk} are the structure constants.

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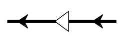
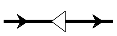
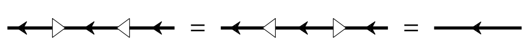
Example 26.2. ***A group integral for $SU(n)$ $\overline{V} \times V$ space.*** *Let $D(g)$ be the defining $[n \times n]$ matrix representation of $SU(n)$. The defining representation is non-trivial, so it averages to zero by (26.6). The first non-vanishing average involves $D(g)^\dagger$, the matrix representation of the action of g on the conjugate vector space. To avoid dealing with the multitude of dummy indices, we resort to diagrammatic notation:*

$D(g)_a{}^\ell = a$  l , $D(g)^a{}_\ell = a$  l . (26.28)

For G the arrows and the triangle point the same way, while for $G^\dagger$ they point the opposite way. Unitarity $D(g)^\dagger D(g) = 1$ is given by

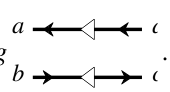
$$D(g)^c{}_a D(g)_c{}^b \; = \; D(g)_a{}^c D(g)^b{}_c \; = \; \delta_a^b \,,$$

or, diagramatically:

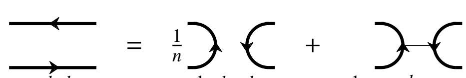


(26.29)

In this notation, the $D(g)D(g)^\dagger$ integral (26.7) to be evaluated is

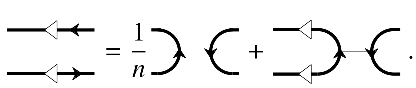
$\int dg$  (26.30)

For $SU(n)$ the $V \otimes \overline{V}$ tensors decompose into the singlet and the adjoint rep



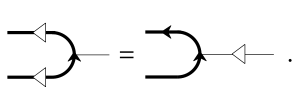
$$\delta_a^d \delta_c^b \quad = \quad \frac{1}{n} \delta_a^b \delta_c^d \quad + \quad \frac{1}{a} (T_i)_a^b (T_i)_c^d \,.$$

We multiply (26.30) with the above decomposition of the identity. The unitarity relation (26.29) eliminates G's from the singlet:



(26.31)

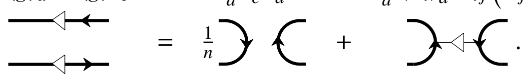
The generators T_i are invariant tensors, and transform under G according to (26.27). Multiplying by G_{ii}^{-1}, we obtain



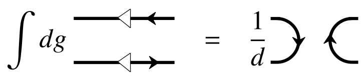
(26.32)

Hence, the pair $GG^\dagger$ in the defining representation can be traded in for a single G in the adjoint rep

$$D(g)_a{}^d D(g)^b{}_c = \frac{1}{d}\delta^d_c\delta^b_a + \frac{1}{a}(T_i)^b_a G_{ij}\left(T_j\right)^d_c$$



The adjoint representation G_{ij} is non-trivial, so it gets averaged to zero by (26.6). Only the singlet survives



$$\int dg\ D(g)_a{}^d D(g)^b{}_c = \frac{1}{d}\delta^d_c\delta^b_a\,. \qquad (26.33)$$

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Example 26.3. ***Irreducible representations of the $SO(2)^N$ abelian group:*** *(Example 12.1 continued) All irreducible representations of the $SO(2)^N$ abelian group acting on torus T^N are 1-dimensional and labeled by N integers $\mu = (m_1, \cdots, m_N)$. The character of μ representation is*

$$\chi^{(\mu)}(g) = e^{-i\mu\cdot\phi}$$

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Example 26.4. ***Haar measure for $SO(2)$:***

The normalized Haar measure is $dg = d\phi/(2\pi)$.

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Example 26.5. ***Haar measure for $SO(3)$:***

$$SO(3): \qquad dg = \frac{1}{2\pi^2}\sin^2(\phi/2)d\Omega_e d\phi$$

with $d\Omega_e$ solid angle surface element for unit vector e.

$$8\pi^2 = \int_{SO(3)} dg$$

For details, see ref. [8].

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Example 26.6. *Trace group integral for $SO(2)$:* *Parameterize rotations on a circle by $\phi \in [0, 2\pi)$. The normalized Haar measure is $ddg = d\phi/2\pi$, and a trajectory point advanced by time t and shifted by ϕ can be denoted $x(t, \phi)$. The character is $e^{-i\mu\phi}$. For a circle this is just Fourier analysis, for a general compact semisimple Lie group Weyl's generalization of Fourier analysis. Consider projection on the μth subspace of the integral along the rotational direction*

$$I_G = \int_G g\, \chi^{(\mu)}(g) \oint dx_G\, \delta_G(x(t)_G - (D(g)x(0))_G).$$

Coordinate x_G is the set of points swept by $[0, 2\pi]$ rotation of a point $x_0 = x_G(0,0)$, so it is natural to parametrize it by the rotation angle ϕ': $x_G = x(0, \phi')$, and rewrite the circle integral as

$$\oint dx_G\, \delta\big(x_G - D_\mu(LieEl) x_G(t, 0)\big) = \int_0^{2\pi} d\phi' \frac{dx}{d\phi}(0, \phi')\, \delta\big(x(0, \phi') - x(t, \phi' + \phi)\big)\,.$$

Inverting the order of integrations,

$$I_G = \int_0^{2\pi} d\phi' \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{-i\mu\phi} \frac{dx}{d\phi}(0, \phi')\, \delta\big(x(0, \phi') - x(t, \phi' + \phi)\big)\,.$$

The integral is novanishing for smallest ϕ_p for which $x(0, \phi') = x(t, \phi' + \phi_p)$, and for all its repeats. Expand the argument of δ function in each such neighborhood $\phi' = \phi_p + \phi''$.

$$\begin{aligned} x(t, \phi' + \phi_p + \phi'') &= x(t, \phi' + \phi_p) + \phi'' \frac{dx}{d\phi}(t, \phi' + \phi_p) + \cdots \\ &= x(t, \phi') + \phi'' \frac{dx}{d\phi}(t, \phi') + \cdots\,. \end{aligned}$$

substituting back yields

$$\begin{aligned} I_G &= \int_0^{2\pi} \frac{d\phi'}{2\pi} \sum_{r=1}^{\infty} e^{-i\mu\phi_p r} \frac{dx(0, \phi')}{d\phi} \int_{-\epsilon}^{\epsilon} d\phi''\, e^{-i\mu\phi''}\, \delta(\phi'' \frac{dx}{d\phi}(0, \phi')) \\ &= \sum_{r=1}^{\infty} e^{-i\mu\phi_p r}\,. \end{aligned}$$

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Exercises

26.1. **Haar measure for SU(2).** SU(2) acts on vectors in $\mathbb{C}^2$, and preserves their absolute value, hence its action can be parameterized by a 3-sphere S_3, and multiplication can be viewed as an orthogonal transformation of S_3. This is a special case of the formula $N = n^2 - 1$ for the dimension of SU(n). Show that the invariant Haar measure on SU(2)

$$\int_{\mathrm{SU}(2)} f(g)dg = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{\pi} \sin^2\phi_1 \sin\phi_2 d\theta d\phi_1 d\phi_2 f(\theta, \phi_1, \phi_2)$$

is a normalized surface measure on S_3.

26.2. **Relative periodic orbits for circles, bagels and spheres:** (a) Show that relative periodic orbits for a point scattering specularly in a circular billiard are single scattering arcs. Compute their stability. Compute the spectrum.

(b) Show that relative periodic orbits for a point scattering specularly in the plane that slices symmetrically upper half of a bagel (floating tire, torus) are single scattering arcs. Compute their stability. Compute the spectrum. Compute the escape rate.

(c) Show that relative periodic orbits for a point scattering specularly within a sphere billiard are single scattering arcs. Compute their stability, spectrum.