Appendix A2

Discrete symmetries of dynamics

BASIC GROUP-THEORETIC NOTIONS are recapitulated here: groups, irreducible representations, invariants. Our notation follows birdtracks.eu.

The key result is the construction of projection operators from invariant matrices. The basic idea is simple: a hermitian matrix can be diagonalized. If this matrix is an invariant matrix, it decomposes the reps of the group into direct sums of lower-dimensional reps. Most of computations to follow implement the spectral decomposition

$$\mathbf{M} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \cdots + \lambda_r \mathbf{P}_r \,,$$

which associates with each distinct root λ_i of invariant matrix $\mathbf{M}$ a projection operator (A2.20):

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} \,.$$

Sects. A2.3 and A2.4 develop Fourier analysis as an application of the general theory of invariance groups and their representations.

A2.1 Preliminaries and definitions

(A. Wirzba and P. Cvitanović)

We define *group*, *representation*, *symmetry of a dynamical system*, and *invariance*.

Group axioms. A group G is a set of elements $g_1, g_2, g_3, \ldots$ for which *composition* or *group multiplication* $g_2 \circ g_1$ (which we often abbreviate as $g_2 g_1$) of any two elements satisfies the following conditions:

1. If $g_1, g_2 \in G$, then $g_2 \circ g_1 \in G$.
2. The group multiplication is associative: $g_3 \circ (g_2 \circ g_1) = (g_3 \circ g_2) \circ g_1$.
3. The group G contains *identity* element e such that $g \circ e = e \circ g = g$ for every element $g \in G$.
4. For every element $g \in G$, there exists a unique $h == g^{-1} \in G$ such that $h \circ g = g \circ h = e$.

A *finite* group is a group with a finite number of elements

$$G = \{e, g_2, \ldots, g_{|G|}\} ,$$

where $|G|$, the number of elements, is the *order* of the group.

Groups are defined and classified as abstract objects by their multiplication tables (for finite groups) or Lie algebras (for Lie groups). What concerns us in applications is their *action* as groups of transformations on a given space, usually a vector space (see appendix **??**), but sometimes an affine space, or a more general manifold $\mathcal{M}$.

Repeated index summation. Throughout this text, the repeated pairs of upper/lower indices are always summed over

$$G_a{}^b x_b \equiv \sum_{b=1}^{n} G_a{}^b x_b , \qquad \text{(A2.1)}$$

unless explicitly stated otherwise.

General linear transformations. Let $GL(n, \mathbb{F})$ be the group of general linear transformations,

$$GL(n, \mathbb{F}) = \{g : \mathbb{F}^n \to \mathbb{F}^n \,|\, \det(g) \neq 0\} . \qquad \text{(A2.2)}$$

Under $GL(n, \mathbb{F})$ a basis set of V is mapped into another basis set by multiplication with a $[n\times n]$ matrix g with entries in field $\mathbb{F}$ ($\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$),

$$\mathbf{e}'^{\,a} = \mathbf{e}^b (g^{-1})_b{}^a .$$

As the vector $\mathbf{x}$ is what it is, regardless of a particular choice of basis, under this transformation its coordinates must transform as

$$x'_a = g_a{}^b x_b .$$

Standard rep. We shall refer to the set of $[n\times n]$ matrices g as a *standard rep* of $GL(n,\mathbb{F})$, and the space of all n-tuples $(x_1,x_2,\ldots,x_n)^\top$, $x_i\in\mathbb{F}$ on which these matrices act as the *standard representation space V*.

Under a general linear transformation $g\in GL(n,\mathbb{F})$, the row of basis vectors transforms by right multiplication as $\mathbf{e}'=\mathbf{e}\,g^{-1}$, and the column of x_a's transforms by left multiplication as $x'=gx$. Under left multiplication the column (row transposed) of basis vectors $\mathbf{e}^\top$ transforms as $\mathbf{e}'^\top=g^\dagger\mathbf{e}^\top$, where the *dual rep* $g^\dagger=(g^{-1})^\top$ is the transpose of the inverse of g. This observation motivates introduction of a *dual* representation space $\bar{V}$, the space on which $GL(n,\mathbb{F})$ acts via the dual rep $g^\dagger$.

Dual space. If V is a vector representation space, then the *dual space* $\bar{V}$ is the set of all linear forms on V over the field $\mathbb{F}$.

If $\{\mathbf{e}^{(1)},\cdots,\mathbf{e}^{(d)}\}$ is a (right) basis of V, then $\bar{V}$ is spanned by the *dual basis* (left basis) $\{\mathbf{e}_{(1)},\cdots,\mathbf{e}_{(d)}\}$, the set of n linear forms $\mathbf{e}_{(j)}$ such that

$$\mathbf{e}_{(i)}\cdot\mathbf{e}^{(j)}=\delta_i^j\,,$$

where δ_a^b is the Kronecker symbol, $\delta_a^b=1$ if $a=b$, and zero otherwise. The components of dual representation space vectors will here be distinguished by upper indices

$$(y^1,y^2,\ldots,y^n)\,. \tag{A2.3}$$

They transform under $GL(n,\mathbb{F})$ as

$$y'^a=(g^\dagger)_b{}^a y^b\,. \tag{A2.4}$$

For $GL(n,\mathbb{F})$ no complex conjugation is implied by the † notation; that interpretation applies only to unitary subgroups of $GL(n,\mathbb{C})$. g can be distinguished from $g^\dagger$ by meticulously keeping track of the relative ordering of the indices,

$$g_a^b\to g_a{}^b\,,\qquad (g^\dagger)_a^b\to g^b{}_a\,. \tag{A2.5}$$

Defining space, dual space. In what follows V will always denote the *defining* n-dimensional complex vector representation space, that is to say the initial, "elementary multiplet" space within which we commence our deliberations. Along with the defining vector representation space V comes the *dual* n-dimensional vector representation space $\bar{V}$. We shall denote the corresponding element of $\bar{V}$ by raising the index, as in (A2.3), so the components of defining space vectors, resp. dual vectors, are distinguished by lower, resp. upper indices:

$$\begin{aligned} x &= (x_1,x_2,\ldots,x_n)\,, &\quad \mathbf{x}\in V\\ \bar{x} &= (x^1,x^2,\ldots,x^n)\,, &\quad \bar{\mathbf{x}}\in\bar{V}\,. \end{aligned} \tag{A2.6}$$

Defining rep. Let G be a group of transformations acting linearly on V, with the action of a group element $g \in G$ on a vector $x \in V$ given by an $[n\times n]$ matrix g

$$x'_a = g_a{}^b x_b \qquad a, b = 1, 2, \ldots, n\,. \tag{A2.7}$$

We shall refer to $g_a{}^b$ as the *defining rep* of the group G. The action of $g \in G$ on a vector $\bar{q} \in \bar{V}$ is given by the *dual rep* $[n\times n]$ matrix $g^\dagger$:

$$x'^a = x^b (g^\dagger)_b{}^a = g^a{}_b x^b\,. \tag{A2.8}$$

In the applications considered here, the group G will almost always be assumed to be a subgroup of the *unitary group*, in which case $g^{-1} = g^\dagger$, and † indicates hermitian conjugation:

$$(g^\dagger)_a{}^b = (g_b{}^a)^* = g^b{}_a\,. \tag{A2.9}$$

Hermitian conjugation is effected by complex conjugation and index transposition: Complex conjugation interchanges upper and lower indices; transposition reverses their order. A matrix is *hermitian* if its elements satisfy

$$(\mathbf{M}^\dagger)^a_b = M^a_b\,. \tag{A2.10}$$

For a hermitian matrix there is no need to keep track of the relative ordering of indices, as $M_b{}^a = (\mathbf{M}^\dagger)_b{}^a = M^a{}_b$.

Invariant vectors. The vector $q \in V$ is an *invariant vector* if for any transformation $g \in G$

$$q = gq\,. \tag{A2.11}$$

If a bilinear form $\mathbf{M}(\bar{x}, y) = x^a M_a{}^b y_b$ is invariant for all $g \in G$, the matrix

$$M_a{}^b = g_a{}^c g^b{}_d M_c{}^d \tag{A2.12}$$

is an *invariant matrix*. Multiplying with $g_b{}^e$ and using the unitary condition (A2.9), we find that the invariant matrices *commute* with all transformations $g \in G$:

$$[g, \mathbf{M}] = 0\,. \tag{A2.13}$$

Invariants. We shall refer to an invariant relation between p vectors in V and q vectors in $\bar{V}$, which can be written as a homogeneous polynomial in terms of vector components, such as

$$H(x, y, \bar{z}, \bar{r}, \bar{s}) = h^{ab}{}_{cde} x_b y_a s^e r^d z^c\,, \tag{A2.14}$$

as an *invariant* in $V^q \otimes \bar{V}^p$ (repeated indices, as always, summed over). In this example, the coefficients $h^{ab}{}_{cde}$ are components of invariant tensor $h \in V^3 \otimes \bar{V}^2$.

Matrix representation of a group. Let us now map the abstract group G *homeomorphically* on a group of matrices $\mathbf{D}(G)$ acting on the vector space V, i.e., in such a way that the group properties, especially the group multiplication, are preserved:

1. Any $g \in G$ is mapped to a matrix $\mathbf{D}(g) \in \mathsf{D}(G)$.
2. The group product $g_2 \circ g_1 \in G$ is mapped onto the matrix product $\mathbf{D}(g_2 \circ g_1) = \mathbf{D}(g_2)\mathbf{D}(g_1)$.
3. The associativity follows from the associativity of matrix multiplication: $\mathbf{D}(g_3 \circ (g_2 \circ g_1)) = \mathbf{D}(g_3)(\mathbf{D}(g_2)\mathbf{D}(g_1)) = (\mathbf{D}(g_3)(\mathbf{D}(g_2))\mathbf{D}(g_1)$.
4. The identity element $e \in G$ is mapped onto the unit matrix $\mathsf{D}(e) = \mathbf{1}$ and the inverse element $g^{-1} \in G$ is mapped onto the inverse matrix $\mathbf{D}(g^{-1}) = [\mathbf{D}(g)]^{-1} \equiv \mathbf{D}^{-1}(g)$.

We call this matrix group $\mathsf{D}(G)$ a linear or matrix *representation* of the group G in the *representation space V*. We emphasize here *'linear'* in order to distinguish the matrix representations from other representations that do not have to be linear, in general. Throughout this appendix we only consider linear representations.

If the dimensionality of V is d, we say the representation is an *d-dimensional representation*. We will often abbreviate the notation by writing matrices $\mathbf{D}(g) \in \mathsf{D}(G)$ as g, i.e., $x' = gx$ corresponds to the matrix operation $x'_i = \sum_{j=1}^{d} \mathbf{D}(g)_{ij} x_j$.

Character of a representation. The character of $\chi_\mu(g)$ of a d-dimensional representation $\mathbf{D}(g)$ of the group element $g \in G$ is defined as trace

$$\chi_\mu(g) = \mathrm{tr}\,\mathbf{D}(g) = \sum_{i=1}^{d} \mathbf{D}_{ii}(g)\,.$$

Note that $\chi(e) = d$, since $\mathbf{D}_{ij}(e) = \delta_{ij}$ for $1 \leq i, j \leq d$.

Faithful representations, factor group. If the mapping G on $\mathsf{D}(G)$ is an isomorphism, the representation is said to be *faithful*. In this case the order of the group of matrices $D(G)$ is equal to the order $|G|$ of the group. In general, however, there will be several elements $h \in G$ that will be mapped on the unit matrix $\mathbf{D}(h) = \mathbf{1}$. This property can be used to define a subgroup $H \subset G$ of the group G consisting of all elements $h \in G$ that are mapped to the unit matrix of a given representation. Then the representation is a faithful representation of the *factor group* G/H.

Equivalent representations, equivalence classes. A representation of a group is by no means unique. If the basis in the d-dimensional vector space V is changed, the matrices $\mathbf{D}(g)$ have to be replaced by their transformations $\mathbf{D}'(g)$, with the new matrices $\mathbf{D}'(g)$ and the old matrices $\mathbf{D}(g)$ are related by an equivalence transformation through a non-singular matrix $\mathbf{C}$

$$\mathbf{D}'(g) = \mathbf{C}\,\mathbf{D}(g)\,\mathbf{C}^{-1} \,.$$

The group of matrices $\mathbf{D}'(g)$ form a representation $\mathsf{D}'(G)$ equivalent to the representation $\mathsf{D}(G)$ of the group G. The equivalent representations have the same structure, although the matrices look different. Because of the cylic nature of the trace the character of equivalent representations is the same

$$\chi(g) = \sum_{i=1}^{n} \mathbf{D}'_{ii}(g) = \mathrm{tr}\,\mathbf{D}'(g) = \mathrm{tr}\left(\mathbf{C}\mathbf{D}(g)\mathbf{C}^{-1}\right) .$$

Definition: Character tables. Finding a transformation S which simultaneously block-diagonalizes the regular representation of each group element sounds difficult. However, suppose it can be achieved, and we obtain a set of irreps $D^{(\mu)}(g)$, then according to Schur's lemmas, $D^{(\mu)}(g)$ must satisfy a set of orthogonality rations:

$$\frac{d_\mu}{|G|} \sum_g D^{(\mu)}_{il}(g) D^{(\nu)}_{mj}(g^{-1}) = \delta_{\mu\nu}\delta_{ij}\delta_{lm} \,. \tag{A2.15}$$

Denote the trace of irrep $D^{(\mu)}_{il}$ as $\chi(\mu)$, and we call it the character of $D^{(\mu)}$. Properties of irreps can be derived from (A2.15), and we list them as follows:

1. The number of irreps is the same as the number of classes.
2. Dimensions of irreps satisfy $\sum_{\mu=1}^{r} d_\mu^2 = |G|$
3. orthonormal relation I : $\sum_i^r |K_i|\chi_i^{(\mu)}\chi_i^{(\nu)*} = |G|\delta_{\mu\nu}$.
   Here, the summation goes through all classes of this group, and $|K_i|$ is the number of elements in class i. This weight comes from the fact that elements in the same class have the same character.
4. orthonormal relation II : $\sum_\mu^r \chi_i^{(\mu)}\chi_j^{(\mu)*} = \frac{|G|}{|K_i|}\delta_{ij}$.

The characters for all classes and irreps of a finite group are conventionally arranged into a *character table*, a square array whose rows represent different classes and columns represent different irreps (as usual, 50% of authors, including ChaosBook, will use columns and rows instead). Rules 1 and 2 help determine the number of irreps and their dimensions. As matrix representation of class $\{e\}$ is always the identity matrix, the first column is always the dimension of the corresponding representation. All entries of the first row are always 1, because the symmetric irrep is always 1-dimensional. To compute the remaining entries, use properties 3, 4 and the class multiplication tables.

Definition: Projection operators. We have listed the properties of irreps and the techniques of constructing character table, but we still do not know how to construct the similarity transformation S which takes a regular representation into a block-diagonal form.

One of these invariant subspace is $|G|^{-1}\sum_g \rho(gx)$, which is the basis of the 1-d symmetric irrep A. For C_3, it is (25.38). But how to get others? We need to resort to the projection operator:

$$P_i^{(\mu)} = \frac{d_\mu}{|G|}\sum_g D_{ii}^{(\mu)}(g)U(g) \tag{A2.16}$$

It projects an arbitrary function into the i_{th} basis of irrep $D^{(\mu)}$ provided the diagonal elements of this representation $D_{ii}^{(\mu)}$ is known. $P_i^{(\mu)}\rho(x) = \rho_i^{(\mu)}$.

For 1-dimensional representations, this projection operator is known after we obtain the character table, since character of 1-d matrix is the matrix itself. But for 2-dimensional or higher dimensional representations, we need to know the diagonal elements $D_{ii}^{(\mu)}$ in order to get the bases of invariant subspaces.

Summing i in (A2.16) gives

$$P^{(\mu)} = \frac{d_\mu}{|G|}\sum_g \chi^{(\mu)}(g)U(g) \tag{A2.17}$$

This is also a projection operator which projects an arbitrary function onto the sum of basis of irrep $D^{(\mu)}$. We will use this operator to split the trace of evolution operator into sum over all different irreps.

A2.2 Invariants and reducibility

What follows is a bit dry, so we start with a motivational quote from Hermann Weyl on the "so-called first main theorem of invariant theory":

"*All invariants are expressible in terms of a finite number among them.* We cannot claim its validity for every group G; rather, it will be our chief task to investigate for each particular group whether a finite integrity basis exists or not; the answer, to be sure, will turn out affirmative in the most important cases."

It is easy to show that any rep of a finite group can be brought to unitary form, and the same is true of all compact Lie groups. Hence, in what follows, we specialize to unitary and hermitian matrices.

A2.2.1 Projection operators

For $\mathbf{M}$ a hermitian matrix, there exists a diagonalizing unitary matrix $\mathbf{C}$ such that

$$\mathbf{CMC}^\dagger = \left[\begin{array}{ccc|c|c} \lambda_1 & \dots & 0 & & \\ & \ddots & & 0 & 0 \\ 0 & \dots & \lambda_1 & & \\ \hline & 0 & & \begin{array}{cccc} \lambda_2 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \dots & \lambda_2 \end{array} & 0 \\ \hline & 0 & & 0 & \begin{array}{cc} \lambda_3 & \dots \\ \vdots & \ddots \end{array} \end{array}\right] . \tag{A2.18}$$

Here $\lambda_i \neq \lambda_j$ are the r distinct roots of the minimal *characteristic* (or *secular*) polynomial

$$\prod_{i=1}^{r}(\mathbf{M} - \lambda_i \mathbf{1}) = 0 . \tag{A2.19}$$

In the matrix $\mathbf{C}(\mathbf{M} - \lambda_2 \mathbf{1})\mathbf{C}^\dagger$ the eigenvalues corresponding to λ_2 are replaced by zeroes:

$$\left[\begin{array}{c|c|c} \begin{array}{cc} \lambda_1 - \lambda_2 & \\ & \lambda_1 - \lambda_2 \end{array} & & \\ \hline & \begin{array}{ccc} 0 & & \\ & \ddots & \\ & & 0 \end{array} & \\ \hline & & \begin{array}{ccc} \lambda_3 - \lambda_2 & & \\ & \lambda_3 - \lambda_2 & \\ & & \ddots \end{array} \end{array}\right] ,$$

and so on, so the product over all factors $(\mathbf{M} - \lambda_2 \mathbf{1})(\mathbf{M} - \lambda_3 \mathbf{1}) \dots$, with exception of the $(\mathbf{M} - \lambda_1 \mathbf{1})$ factor, has nonzero entries only in the subspace associated with λ_1:

$$\mathbf{C}\prod_{j\neq 1}(\mathbf{M} - \lambda_j \mathbf{1})\mathbf{C}^\dagger = \prod_{j\neq 1}(\lambda_1 - \lambda_j) \left[\begin{array}{c|c} \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} & 0 \\ \hline 0 & \begin{array}{cccc} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{array} \end{array}\right] .$$

Thus we can associate with each distinct root λ_i a *projection operator* $\mathbf{P}_i$,

$$\mathbf{P}_i = \prod_{j\neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} , \tag{A2.20}$$

which acts as identity on the ith subspace, and zero elsewhere. For example, the projection operator onto the λ_1 subspace is

$$\mathbf{P}_1 = \mathbf{C}^\dagger \left[\begin{array}{ccc|cccc} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \hline & & & 0 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{array} \right] \mathbf{C} \,. \tag{A2.21}$$

The diagonalization matrix $\mathbf{C}$ is deployed in the above only as a pedagogical device. The whole point of the projector operator formalism is that we *never* need to carry such explicit diagonalization; all we need are whatever invariant matrices $\mathbf{M}$ we find convenient, the algebraic relations they satisfy, and orthonormality and completeness of $\mathbf{P}_i$: The matrices $\mathbf{P}_i$ are *orthogonal*

$$\mathbf{P}_i \mathbf{P}_j = \delta_{ij} \mathbf{P}_j \,, \qquad \text{(no sum on } j) \,, \tag{A2.22}$$

and satisfy the *completeness relation*

$$\sum_{i=1}^{r} \mathbf{P}_i = \mathbf{1} \,. \tag{A2.23}$$

As $\operatorname{tr}(\mathbf{C}\mathbf{P}_i\mathbf{C}^\dagger) = \operatorname{tr}\mathbf{P}_i$, the dimension of the ith subspace is given by

$$d_i = \operatorname{tr}\mathbf{P}_i \,. \tag{A2.24}$$

It follows from the characteristic equation (A2.19) and the form of the projection operator (A2.20) that λ_i is the eigenvalue of $\mathbf{M}$ on $\mathbf{P}_i$ subspace:

$$\mathbf{M}\mathbf{P}_i = \lambda_i \mathbf{P}_i \,, \qquad \text{(no sum on } i) \,. \tag{A2.25}$$

Hence, any matrix polynomial $f(\mathbf{M})$ takes the scalar value $f(\lambda_i)$ on the $\mathbf{P}_i$ subspace

$$f(\mathbf{M})\mathbf{P}_i = f(\lambda_i)\mathbf{P}_i \,. \tag{A2.26}$$

This, of course, is the reason why one wants to work with irreducible reps: they reduce matrices and "operators" to pure numbers.

A2.2.2 Irreducible representations

Suppose there exist several linearly independent invariant $[d\times d]$ hermitian matrices $\mathbf{M}_1, \mathbf{M}_2, \ldots$, and that we have used $\mathbf{M}_1$ to decompose the d-dimensional vector space $V = V_1 \oplus V_2 \oplus \cdots$. Can $\mathbf{M}_2, \mathbf{M}_3, \ldots$ be used to further decompose V_i? Further decomposition is possible if, and only if, the invariant matrices commute:

$$[\mathbf{M}_1, \mathbf{M}_2] = 0 \,, \tag{A2.27}$$

or, equivalently, if projection operators $\mathbf{P}_j$ constructed from $\mathbf{M}_2$ commute with projection operators $\mathbf{P}_i$ constructed from $\mathbf{M}_1$,

$$\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_j\mathbf{P}_i\,. \qquad \text{(A2.28)}$$

Usually the simplest choices of independent invariant matrices do not commute. In that case, the projection operators $\mathbf{P}_i$ constructed from $\mathbf{M}_1$ can be used to project commuting pieces of $\mathbf{M}_2$:

$$\mathbf{M}_2^{(i)} = \mathbf{P}_i\mathbf{M}_2\mathbf{P}_i\,, \qquad \text{(no sum on } i)\,.$$

That $\mathbf{M}_2^{(i)}$ commutes with $\mathbf{M}_1$ follows from the orthogonality of $\mathbf{P}_i$:

$$[\mathbf{M}_2^{(i)}, \mathbf{M}_1] = \sum_j \lambda_j [\mathbf{M}_2^{(i)}, \mathbf{P}_j] = 0\,. \qquad \text{(A2.29)}$$

Now the characteristic equation for $\mathbf{M}_2^{(i)}$ (if nontrivial) can be used to decompose V_i subspace.

An invariant matrix $\mathbf{M}$ induces a decomposition only if its diagonalized form (A2.18) has more than one distinct eigenvalue; otherwise it is proportional to the unit matrix and commutes trivially with all group elements. A rep is said to be *irreducible* if all invariant matrices that can be constructed are proportional to the unit matrix.

According to (A2.13), an invariant matrix $\mathbf{M}$ commutes with group transformations $[G, \mathbf{M}] = 0$. Projection operators (A2.20) constructed from $\mathbf{M}$ are polynomials in $\mathbf{M}$, so they also commute with all $g \in \mathcal{G}$:

$$[G, \mathbf{P}_i] = 0 \qquad \text{(A2.30)}$$

Hence, a $[d{\times}d]$ matrix rep can be written as a direct sum of $[d_i{\times}d_i]$ matrix reps:

$$G = \mathbf{1}G\mathbf{1} = \sum_{i,j} \mathbf{P}_i G \mathbf{P}_j = \sum_i \mathbf{P}_i G \mathbf{P}_i = \sum_i G_i\,. \qquad \text{(A2.31)}$$

In the diagonalized rep (A2.21), the matrix $\mathbf{g}$ has a block diagonal form:

$$\mathbf{C}\mathbf{g}\mathbf{C}^\dagger = \begin{bmatrix} \mathbf{g}_1 & 0 & 0 \\ 0 & \mathbf{g}_2 & 0 \\ 0 & 0 & \ddots \end{bmatrix}, \qquad \mathbf{g} = \sum_i \mathbf{C}^i \mathbf{g}_i \mathbf{C}_i\,. \qquad \text{(A2.32)}$$

The rep $\mathbf{g}_i$ acts only on the d_i-dimensional subspace V_i consisting of vectors $\mathbf{P}_i q$, $q \in V$. In this way an invariant $[d{\times}d]$ hermitian matrix $\mathbf{M}$ with r distinct eigenvalues induces a decomposition of a d-dimensional vector space V into a direct sum of d_i-dimensional vector subspaces V_i:

$$V \xrightarrow{\mathbf{M}} V_1 \oplus V_2 \oplus \ldots \oplus V_r\,. \qquad \text{(A2.33)}$$

A2.3 Lattice derivatives

In order to set up continuum field-theoretic equations which describe the evolution of spatial variations of fields, we need to define *lattice derivatives*.

Consider a smooth function $\phi(x)$ evaluated on a d-dimensional lattice

$$\phi_\ell = \phi(x) , \qquad x = a\ell = \text{lattice point} , \quad \ell \in \mathbf{Z}^d , \tag{A2.34}$$

where a is the lattice spacing. Each set of values of $\phi(x)$ (a vector ϕ_ℓ) is a possible lattice configuration. Assume the lattice is hyper-cubic, and let $\hat{n}_\mu \in \{\hat{n}_1, \hat{n}_2, \cdots, \hat{n}_d\}$ be the unit lattice cell vectors pointing along the d positive directions. The *lattice derivative* is then

$$(\partial_\mu \phi)_\ell = \frac{\phi(x + a\hat{n}_\mu) - \phi(x)}{a} = \frac{\phi_{\ell+\hat{n}_\mu} - \phi_\ell}{a} . \tag{A2.35}$$

Anything else with the correct $a \to 0$ limit would do, but this is the simplest choice. We can rewrite the lattice derivative as a linear operator, by introducing the *stepping operator* in the direction μ

$$\left(\sigma_\mu\right)_{\ell j} = \delta_{\ell+\hat{n}_\mu, j} . \tag{A2.36}$$

As σ will play a central role in what follows, it pays to understand what it does.

In computer dicretizations, the lattice will be a finite d-dimensional hyper-cubic lattice

$$\phi_\ell = \phi(x) , \qquad x = a\ell = \text{lattice point} , \quad \ell \in (\mathbf{Z}/N)^d , \tag{A2.37}$$

where a is the lattice spacing and there are N^d points in all. For a hyper-cubic lattice the translations in different directions commute, $\sigma_\mu \sigma_\nu = \sigma_\nu \sigma_\mu$, so it is sufficient to understand the action of (A2.36) on a 1-dimensional lattice.

Let us write down σ for the 1-dimensional case in its full $[N\times N]$ matrix glory. Writing the finite lattice stepping operator (A2.36) as an 'upper shift' matrix,

$$\sigma = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & 0 & 1 \\ 0 & & & & & 0 \end{bmatrix} , \tag{A2.38}$$

is no good, as σ so defined is nilpotent, and after N steps the particle marches off the lattice edge, and nothing is left, $\sigma^N = 0$. A sensible way to approximate an infinite lattice by a finite one is to replace it by a lattice periodic in each $\hat{n}_\mu$ direction. On a *periodic lattice* every point is equally far from the 'boundary' $N/2$ steps away, the 'surface' effects are equally negligible for all points, and the stepping operator acts as a cyclic permutation matrix

$$\sigma = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{bmatrix}, \tag{A2.39}$$

with '1' in the lower left corner assuring periodicity.

Applied to the lattice configuration $\phi = (\phi_1, \phi_2, \cdots, \phi_N)$, the stepping operator translates the configuration by one site, $\sigma\phi = (\phi_2, \phi_3, \cdots, \phi_N, \phi_1)$. Its transpose translates the configuration the other way, so the transpose is also the inverse, $\sigma^{-1} = \sigma^T$. The partial lattice derivative (A2.35) can now be written as a multiplication by a matrix:

$$\partial_\mu \phi_\ell = \frac{1}{a}\left(\sigma_\mu - \mathbf{1}\right)_{\ell j} \phi_j \,.$$

In the 1-dimensional case the $[N\times N]$ matrix representation of the lattice derivative is:

$$\partial = \frac{1}{a}\begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ 1 & & & & & -1 \end{bmatrix}. \tag{A2.40}$$

To belabor the obvious: On a finite lattice of N points a derivative is simply a finite $[N\times N]$ matrix. Continuum field theory is a world in which the lattice is so fine that it looks smooth to us. Whenever someone calls something an "operator," think "matrix." For finite-dimensional spaces a linear operator *is* a matrix; things get subtler for infinite-dimensional spaces.

A2.3.1 Lattice Laplacian

In the continuum, integration by parts moves ∂ around, $\int [dx] \phi^T \cdot \partial^2 \phi \to -\int [dx] \partial\phi^T \cdot \partial\phi$; on a lattice this amounts to a matrix transposition

$$\left[\left(\sigma_\mu - \mathbf{1}\right)\phi\right]^T \cdot \left[\left(\sigma_\mu - \mathbf{1}\right)\phi\right] = \phi^T \cdot \left(\sigma_\mu^{-1} - \mathbf{1}\right)\left(\sigma_\mu - \mathbf{1}\right)\phi \,.$$

If you are wondering where the "integration by parts" minus sign is, it is there in discrete case at well. It comes from the identity

$$\partial^T = \frac{1}{a}\left(\sigma^{-1} - \mathbf{1}\right) = -\sigma^{-1}\frac{1}{a}\left(\sigma - \mathbf{1}\right) = -\sigma^{-1}\partial \,.$$

The symmetric (self-adjoint) combination $\square = -\partial^T \partial$

$$\begin{aligned}\square &= -\frac{1}{a^2}\sum_{\mu=1}^{d}\left(\sigma_\mu^{-1}-\mathbf{1}\right)\left(\sigma_\mu-\mathbf{1}\right) = -\frac{2}{a^2}\sum_{\mu=1}^{d}\left(\mathbf{1}-\frac{1}{2}(\sigma_\mu^{-1}+\sigma_\mu)\right) \\ &= \frac{1}{a^2}\left(\mathbf{N}-2d\mathbf{1}\right)\end{aligned} \tag{A2.41}$$

is the *lattice Laplacian*. We shall show below that this Laplacian has the correct continuum limit. In the 1-dimensional case the $[N\times N]$ matrix representation of the lattice Laplacian is:

$$\square = \frac{1}{a^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & \ddots & \\ & & & & 1 \\ 1 & & & 1 & -2 \end{bmatrix}. \tag{A2.42}$$

The lattice Laplacian measures the second variation of a field ϕ_ℓ across three neighboring sites: it is spatially *non-local*. You can easily check that it does what the second derivative is supposed to do by applying it to a parabola restricted to the lattice, $\phi_\ell = \phi(a\ell)$, where $\phi(a\ell)$ is defined by the value of the continuum function $\phi(x) = x^2$ at the lattice point $x_\ell = a\ell$.

The Euclidean free scalar particle propagator can thus be written as

$$\Delta = \frac{1}{\mathbf{1} - \frac{a^2 h}{s}\square}. \tag{A2.43}$$

A2.3.2 Inverting the Laplacian

Evaluation of perturbative corrections requires that we come to grips with the "free" or "bare" propagator M. While the the Laplacian is a simple difference operator (A2.42), the propagator is a messier object. A way to compute is to start expanding the propagator M as a power series in the Laplacian

$$M = \frac{1}{m^2 - \square} = \frac{1}{m^2}\sum_{k=0}^{\infty}\frac{1}{m^{2k}}\square^k. \tag{A2.44}$$

As $\square$ is a finite matrix, the expansion is convergent for sufficiently large m^2. To get a feeling for what is involved in evaluating such series, evaluate $\square^2$ in the 1-dimensional case:

$$\square^2 = \frac{1}{a^4}\begin{bmatrix} 6 & -4 & 1 & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & 1 \\ 1 & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & \ddots & & & 1 \\ 1 & & & & & 6 & -4 \\ -4 & 1 & & & 1 & -4 & 6 \end{bmatrix}. \tag{A2.45}$$

What $\Box^3$, $\Box^4$, $\cdots$ contributions look like is now clear; as we include higher and higher powers of the Laplacian, the propagator matrix fills up; while the *inverse* propagator is differential operator connecting only the nearest neighbors, the propagator is integral, *non-local* operator, connecting every lattice site to any other lattice site. Due to the periodicity, these are all Toeplitz matrices, meaning that each successive row is a one-step cyclic shift of the preceding one. In statistical mechanics, M is the (bare) 2-point correlation. In quantum field theory, it is called a propagator.

These matrices can be evaluated as is, on the lattice, and sometime it is evaluated this way, but in case at hand a wonderful simplification follows from the observation that the lattice action is translationally invariant. We will show how this works in sect. A2.4.

A2.4 Periodic lattices

Our task now is to transform M into a form suitable to explicit evaluation.

Consider the effect of a lattice translation $\phi \to \sigma\phi$ on the matrix polynomial

$$S[\sigma\phi] = -\frac{1}{2}\phi^T\left(\sigma^T M^{-1}\sigma\right)\phi\,.$$

As M^{-1} is constructed from σ and its inverse, M^{-1} and σ commute, and $S[\phi]$ is invariant under translations,

$$S[\sigma\phi] = S[\phi] = -\frac{1}{2}\phi^T \cdot M^{-1}\cdot\phi\,. \tag{A2.46}$$

If a function defined on a vector space commutes with a linear operator σ, then the eigenvalues of σ can be used to decompose the ϕ vector space into invariant subspaces. For a hyper-cubic lattice the translations in different directions commute, $\sigma_\mu\sigma_\nu = \sigma_\nu\sigma_\mu$, so it is sufficient to understand the spectrum of the 1-dimensional stepping operator (A2.39). To develop a feeling for how this reduction to invariant subspaces works in practice, let us continue humbly, by expanding the scope of our deliberations to a lattice consisting of 2 points.

A2.4.1 A 2-point lattice diagonalized

The action of the stepping operator σ (A2.39) on a 2-point lattice $\phi = (\phi_0, \phi_1)$ is to permute the two lattice sites

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

As exchange repeated twice brings us back to the original configuration, $\sigma^2 = \mathbf{1}$, the characteristic polynomial of σ is

$$(\sigma + 1)(\sigma - 1) = 0\,,$$

with eigenvalues $\lambda_0 = 1, \lambda_1 = -1$. The symmetrization, antisymmetrization projection operators are

$$P_0 = \frac{\sigma - \lambda_1 \mathbf{1}}{\lambda_0 - \lambda_1} = \frac{1}{2}(\mathbf{1} + \sigma) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{A2.47}$$

$$P_1 = \frac{\sigma - \mathbf{1}}{-1 - 1} = \frac{1}{2}(\mathbf{1} - \sigma) = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{A2.48}$$

Noting that $P_0 + P_1 = \mathbf{1}$, we can project a lattice configuration ϕ onto the two normalized eigenvectors of σ:

$$\phi = \mathbf{1}\,\phi = P_0 \cdot \phi + P_1 \cdot \phi\,,$$

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{(\phi_0 + \phi_1)}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{(\phi_0 - \phi_1)}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{A2.49}$$

$$= \tilde{\phi}_0 \hat{n}_0 + \tilde{\phi}_1 \hat{n}_1\,. \tag{A2.50}$$

As $P_0 P_1 = 0$, the symmetric and the antisymmetric configurations transform separately under any linear transformation constructed from σ and its powers.

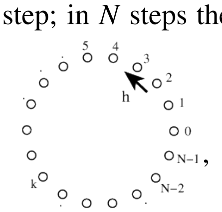
In this way the characteristic equation $\sigma^2 = \mathbf{1}$ enables us to reduce the 2-dimensional lattice configuration to two 1-dimensional ones, on which the value of the stepping operator σ is a number, $\lambda \in \{1, -1\}$, and the normalized eigenvectors are $\hat{n}_0 = \frac{1}{\sqrt{2}}(1, 1)$, $\hat{n}_1 = \frac{1}{\sqrt{2}}(1, -1)$. As we shall now see, $(\tilde{\phi}_0, \tilde{\phi}_1)$ is the 2-site periodic lattice discrete Fourier transform of the field (ϕ_1, ϕ_2).

A2.5 Discrete Fourier transforms

Let us generalize this reduction to a 1-dimensional periodic lattice with N sites.

Each application of σ translates the lattice one step; in N steps the lattice is back in the original configuration

$$\sigma^N = \mathbf{1}$$



so the eigenvalues of σ are the N distinct N-th roots of unity

$$\sigma^N - \mathbf{1} = \prod_{k=0}^{N-1} (\sigma - \omega^k \mathbf{1}) = 0\,, \qquad \omega = e^{i\frac{2\pi}{N}}\,. \tag{A2.51}$$

As the eigenvalues are all distinct and N in number, the space is decomposed into N 1-dimensional subspaces. The general theory (expounded in appendix A2.2) associates with the k-th eigenvalue of σ a projection operator that projects a configuration ϕ onto k-th eigenvector of σ,

$$P_k = \prod_{j\neq k} \frac{\sigma - \lambda_j \mathbf{1}}{\lambda_k - \lambda_j} . \tag{A2.52}$$

A factor $(\sigma - \lambda_j \mathbf{1})$ kills the j-th eigenvector φ_j component of an arbitrary vector in expansion $\phi = \cdots + \tilde{\phi}_j \varphi_j + \cdots$. The above product kills everything but the eigen-direction φ_k, and the factor $\prod_{j\neq k}(\lambda_k - \lambda_j)$ ensures that P_k is normalized as a projection operator. The set of the projection operators is complete,

$$\sum_k P_k = \mathbf{1} , \tag{A2.53}$$

and orthonormal

$$P_k P_j = \delta_{kj} P_k \qquad \text{(no sum on } k) . \tag{A2.54}$$

Constructing explicit eigenvectors is usually not a the best way to fritter one's youth away, as choice of basis is largely arbitrary, and all of the content of the theory is in projection operators (see appendix A2.2). However, in case at hand the eigenvectors are so simple that we can construct the solutions of the eigenvalue condition

$$\sigma \varphi_k \quad = \quad \omega^k \varphi_k \tag{A2.55}$$

by hand:

$$\frac{1}{\sqrt{N}} \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(N-1)k} \end{bmatrix} = \omega^k \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ \omega^k \\ \omega^{2k} \\ \omega^{3k} \\ \vdots \\ \omega^{(N-1)k} \end{bmatrix}$$

The $1/\sqrt{N}$ factor is chosen in order that φ_k be normalized complex unit vectors

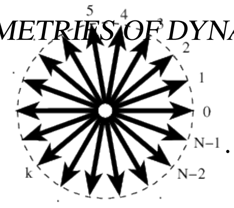
$$\begin{aligned} \varphi_k^\dagger \cdot \varphi_k &= \frac{1}{N} \sum_{k=0}^{N-1} 1 \; = \; 1 \, , \qquad \text{(no sum on } k) \\ \varphi_k^\dagger &= \frac{1}{\sqrt{N}} \left(1, \omega^{-k}, \omega^{-2k}, \cdots, \omega^{-(N-1)k}\right) . \end{aligned} \tag{A2.56}$$

The eigenvectors are orthonormal

$$\varphi_k^\dagger \cdot \varphi_j = \delta_{kj} , \tag{A2.57}$$

as the explicit evaluation of $\varphi_k^\dagger \cdot \varphi_j$ yields the *Kronecker delta function for a periodic lattice*

$$\delta_{kj} = \frac{1}{N}\sum_{\ell=0}^{N-1} e^{i\frac{2\pi}{N}(k-j)\ell}$$



. (A2.58)

The sum is over the N unit vectors pointing at a uniform distribution of points on the complex unit circle; they cancel each other unless $k = j \pmod N$, in which case each term in the sum equals 1.

The projection operators can be expressed in terms of the eigenvectors (A2.55), (A2.56) as

$$(P_k)_{\ell\ell'} = (\varphi_k)_\ell(\varphi_k^\dagger)_{\ell'} = \frac{1}{N}e^{i\frac{2\pi}{N}(\ell-\ell')k}\,, \qquad \text{(no sum on } k). \tag{A2.59}$$

The completeness (A2.53) follows from (A2.58), and the orthonormality (A2.54) from (A2.57).

$\tilde{\phi}_k$, the projection of the ϕ configuration on the k-th subspace is given by

$$\begin{aligned} (P_k \cdot \phi)_\ell &= \tilde{\phi}_k\,(\varphi_k)_\ell\,, \qquad \text{(no sum on } k) \\ \tilde{\phi}_k &= \varphi_k^\dagger \cdot \phi = \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1} e^{-i\frac{2\pi}{N}k\ell}\phi_\ell \end{aligned} \tag{A2.60}$$

We recognize $\tilde{\phi}_k$ as the *discrete Fourier transform* of ϕ_ℓ. Hopefully rediscovering it this way helps you a little toward understanding why Fourier transforms are full of $e^{ix\cdot p}$ factors (they are eigenvalues of the generator of translations) and when are they the natural set of basis functions (only if the theory is translationally invariant).

A2.5.1 Fourier transform of the propagator

Now insert the identity $\sum P_k = \mathbf{1}$ wherever profitable:

$$\mathbf{M} = \mathbf{1}\mathbf{M}\mathbf{1} = \sum_{kk'} P_k\mathbf{M}P_{k'} = \sum_{kk'}\varphi_k(\varphi_k^\dagger\cdot\mathbf{M}\cdot\varphi_{k'})\varphi_{k'}^\dagger\,.$$

The matrix

$$\tilde{M}_{kk'} = (\varphi_k^\dagger\cdot\mathbf{M}\cdot\varphi_{k'}) \tag{A2.61}$$

is the Fourier space representation of $\mathbf{M}$. According to (A2.57) the matrix $U_{k\ell} = (\varphi_k)_\ell = \frac{1}{\sqrt{N}}e^{i\frac{2\pi}{N}k\ell}$ is a unitary matrix, so the Fourier transform is a linear, unitary transformation, $UU^\dagger = \sum P_k = \mathbf{1}$, with Jacobian $\det U = 1$. The form of the invariant function (A2.46) does not change under $\phi \to \tilde{\phi}_k$ transformation, and from the formal point of view, it does not matter whether we compute in the Fourier space or in the configuration space that we started out with. For example, the trace of $\mathbf{M}$ is the trace in either representation

$$\begin{aligned}\operatorname{tr}\mathbf{M} &= \sum_{\ell} M_{\ell\ell} = \sum_{kk'}\sum_{\ell}(P_k\mathbf{M}P_{k'})_{\ell\ell} \\ &= \sum_{kk'}\sum_{\ell}(\varphi_k)_\ell(\varphi_k^\dagger\cdot\mathbf{M}\cdot\varphi_{k'})(\varphi_{k'}^\dagger)_\ell = \sum_{kk'}\delta_{kk'}\tilde{M}_{kk'} = \operatorname{tr}\tilde{\mathbf{M}}\,.\end{aligned}$$

From this it follows that $\operatorname{tr}\mathbf{M}^n = \operatorname{tr}\tilde{\mathbf{M}}^n$, and from the tr ln = ln tr relation that $\det\mathbf{M} = \det\tilde{\mathbf{M}}$. In fact, any scalar combination of ϕ's, J's and couplings, such as the partition function $Z[J]$, has exactly the same form in the configuration and the Fourier space.

OK, a dizzying quantity of indices. But what's the payback?

A2.5.2 Lattice Laplacian diagonalized

Now use the eigenvalue equation (A2.55) to convert σ matrices into scalars. If $\mathbf{M}$ *commutes* with σ, then $(\varphi_k^\dagger\cdot\mathbf{M}\cdot\varphi_{k'}) = \tilde{M}_k\delta_{kk'}$, and the matrix $\mathbf{M}$ acts as a multiplication by the scalar $\tilde{M}_k$ on the kth subspace. For example, for the 1-dimensional version of the lattice Laplacian (A2.41) the projection on the k-th subspace is

$$\begin{aligned}(\varphi_k^\dagger\cdot\Box\cdot\varphi_{k'}) &= \frac{2}{a^2}\left(\frac{1}{2}(\omega^{-k}+\omega^k)-1\right)(\varphi_k^\dagger\cdot\varphi_{k'}) \\ &= \frac{2}{a^2}\left(\cos\left(\frac{2\pi}{N}k\right)-1\right)\delta_{kk'} \end{aligned} \qquad \text{(A2.62)}$$

In the k-th subspace the bare propagator is simply a number, and, in contrast to the mess generated by (A2.44), there is nothing to inverting M^{-1}:

$$(\varphi_{\mathbf{k}}^\dagger\cdot M\cdot\varphi_{\mathbf{k}'}) = (\tilde{G}_0)_{\mathbf{k}}\delta_{\mathbf{k}\mathbf{k}'} = \frac{1}{\beta}\frac{\delta_{\mathbf{k}\mathbf{k}'}}{m_0'^2-\frac{2c}{a^2}\sum_{\mu=1}^{d}\left(\cos\left(\frac{2\pi}{N}k_\mu\right)-1\right)}\,, \qquad \text{(A2.63)}$$

where $\mathbf{k} = (k_1,k_2,\cdots,k_\mu)$ is a d-dimensional vector in the N^d-dimensional dual lattice.

Going back to the partition function and sticking in the factors of **1** into the bilinear part of the interaction, we replace the spatial J_ℓ by its Fourier transform $\tilde{J}_k$, and the spatial propagator $(M)_{\ell\ell'}$ by the diagonalized Fourier transformed $(\tilde{G}_0)_k$

$$J^T\cdot M\cdot J = \sum_{k,k'}(J^T\cdot\varphi_k)(\varphi_k^\dagger\cdot M\cdot\varphi_{k'})(\varphi_{k'}^\dagger\cdot J) = \sum_k \tilde{J}_k^\dagger(\tilde{G}_0)_k\tilde{J}_k\,. \qquad \text{(A2.64)}$$

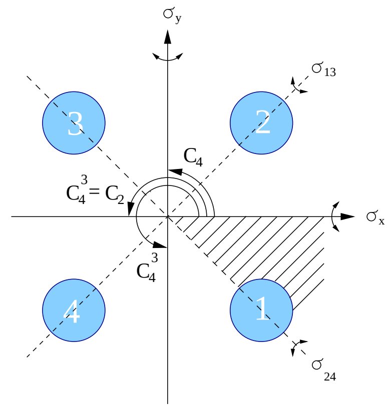


Figure A2.1: Symmetries of four disks on a square. A fundamental domain indicated by the shaded wedge.

A2.6 C_{4v} factorization

If an N-disk arrangement has C_N symmetry, and the disk visitation sequence is given by disk labels $\{\epsilon_1\epsilon_2\epsilon_3\ldots\}$, only the relative increments $\rho_i = \epsilon_{i+1} - \epsilon_i \mod N$ matter. Symmetries under reflections across axes increase the group to C_{Nv} and add relations between symbols: $\{\epsilon_i\}$ and $\{N - \epsilon_i\}$ differ only by a reflection. As a consequence of this reflection increments become decrements until the next reflection and vice versa. Consider four equal disks placed on the vertices of a square (figure A2.1). The symmetry group consists of the identity $\mathbf{e}$, the two reflections σ_x, σ_y across x, y axes, the two diagonal reflections σ_{13}, σ_{24}, and the three rotations C_4, C_2 and C_4^3 by angles $\pi/2$, π and $3\pi/2$. We start by exploiting the C_4 subgroup symmetry in order to replace the absolute labels $\epsilon_i \in \{1,2,3,4\}$ by relative increments $\rho_i \in \{1,2,3\}$. By reflection across diagonals, an increment by 3 is equivalent to an increment by 1 and a reflection; this new symbol will be called $\underline{1}$. Our convention will be to first perform the increment and then to change the orientation due to the reflection. As an example, consider the fundamental domain cycle 112. Taking the disk 1 → disk 2 segment as the starting segment, this symbol string is mapped into the disk visitation sequence $1_{+1}2_{+1}3_{+2}1\cdots = \overline{123}$, where the subscript indicates the increments (or decrements) between neighboring symbols; the period of the cycle $\overline{112}$ is thus 3 in both the fundamental domain and the full space. Similarly, the cycle $\overline{\underline{1}12}$ will be mapped into $1_{+1}2_{-1}1_{-2}3_{-1}2_{+1}3_{+2}1 = \overline{121323}$ (note that the fundamental domain symbol $\underline{1}$ corresponds to a flip in orientation after the second and fifth symbols); this time the period in the full space is twice that of the fundamental domain. In particular, the fundamental domain fixed points correspond to the following 4-disk cycles:

4-disk		reduced
12	↔	$\underline{1}$
1234	↔	1
13	↔	2

Conversions for all periodic orbits of reduced symbol period less than 5 are listed in table A2.1.

Table A2.1: C_{4v} correspondence between the ternary fundamental domain prime cycles $\tilde{p}$ and the full 4-disk {1,2,3,4} labeled cycles p, together with the C_{4v} transformation that maps the end point of the $\tilde{p}$ cycle into an irreducible segment of the p cycle. For typographical convenience, the symbol $\underline{1}$ of sect. A2.6 has been replaced by 0, so that the ternary alphabet is $\{0, 1, 2\}$. The degeneracy of the p cycle is $m_p = 8n_{\tilde{p}}/n_p$. Orbit $\overline{2}$ is the sole boundary orbit, invariant both under a rotation by π and a reflection across a diagonal. The two pairs of cycles marked by (a) and (b) are related by time reversal, but cannot be mapped into each other by C_{4v} transformations.

$\tilde{p}$	p	$\mathbf{h}_{\tilde{p}}$
0	1 2	σ_x
1	1 2 3 4	C_4
2	1 3	C_2, σ_{13}
01	12 14	σ_{24}
02	12 43	σ_y
12	12 41 34 23	C_4^3
001	121 232 343 414	C_4
002	121 343	C_2
011	121 434	σ_y
012	121 323	σ_{13}
021	124 324	σ_{13}
022	124 213	σ_x
112	123	e
122	124 231 342 413	C_4

$\tilde{p}$	p	$\mathbf{h}_{\tilde{p}}$
0001	1212 1414	σ_{24}
0002	1212 4343	σ_y
0011	1212 3434	C_2
0012	1212 4141 3434 2323	C_4^3
0021 (a)	1213 4142 3431 2324	C_4^3
0022	1213	e
0102 (a)	1214 2321 3432 4143	C_4
0111	1214 3234	σ_{13}
0112 (b)	1214 2123	σ_x
0121 (b)	1213 2124	σ_x
0122	1213 1413	σ_{24}
0211	1243 2134	σ_x
0212	1243 1423	σ_{24}
0221	1242 1424	σ_{24}
0222	1242 4313	σ_y
1112	1234 2341 3412 4123	C_4
1122	1231 3413	C_2
1222	1242 4131 3424 2313	C_4^3

This symbolic dynamics is closely related to the group-theoretic structure of the dynamics: the global 4-disk trajectory can be generated by mapping the fundamental domain trajectories onto the full 4-disk space by the accumulated product of the C_{4v} group elements $g_1 = C$, $g_2 = C^2$, $g_{\underline{1}} = \sigma_{diag}C = \sigma_{axis}$, where C is a rotation by $\pi/2$. In the $\overline{\underline{1}12}$ example worked out above, this yields $g_{\underline{1}12} = g_2 g_1 g_{\underline{1}} = C^2 C \sigma_{axis} = \sigma_{diag}$, listed in the last column of table A2.1. Our convention is to multiply group elements in the reverse order with respect to the symbol sequence. We need these group elements for our next step, the dynamical zeta function factorizations.

The C_{4v} group has four 1-dimensional representations, either symmetric (A_1) or antisymmetric (A_2) under both types of reflections, or symmetric under one and antisymmetric under the other (B_1, B_2), and a degenerate pair of 2-dimensional representations E. Substituting the C_{4v} characters

C_{4v}	A_1	A_2	B_1	B_2	E
e	1	1	1	1	2
C_2	1	1	1	1	-2
C_4, C_4^3	1	1	-1	-1	0
σ_{axes}	1	-1	1	-1	0
σ_{diag}	1	-1	-1	1	0

into (25.20) we obtain:

$$
\begin{array}{rcclllll}
h_{\tilde{p}} & & & A_1 & A_2 & B_1 & B_2 & E \\
e: & (1-t_{\tilde{p}})^8 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}})^4 \\
C_2: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}})^4 \\
C_4, C_4^3: & (1-t_{\tilde{p}}^4)^2 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}^2)^2 \\
\sigma_{axes}: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}^2)^2 \\
\sigma_{diag}: & (1-t_{\tilde{p}}^2)^4 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}^2)^2
\end{array}
$$

The possible irreducible segment group elements $\mathbf{h}_{\tilde{p}}$ are listed in the first column; σ_{axes} denotes a reflection across either the x-axis or the y-axis, and σ_{diag} denotes a reflection across a diagonal (see figure A2.1). In addition, degenerate pairs of boundary orbits can run along the symmetry lines in the full space, with the fundamental domain group theory weights $\mathbf{h}_p = (C_2 + \sigma_x)/2$ (axes) and $\mathbf{h}_p = (C_2 + \sigma_{13})/2$ (diagonals) respectively:

$$
\begin{array}{rccl}
 & & & \quad A_1 \qquad\quad A_2 \qquad\quad B_1 \qquad\quad B_2 \qquad\quad E \\
\text{axes:} & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1+t_{\tilde{p}})^2 \\
\text{diagonals:} & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}})(1-0t_{\tilde{p}})(1-0t_{\tilde{p}})(1-t_{\tilde{p}})(1+t_{\tilde{p}})^2
\end{array}
\qquad \text{(A2.65)}
$$

(we have assumed that $t_{\tilde{p}}$ does not change sign under reflections across symmetry axes). For the 4-disk arrangement considered here only the diagonal orbits $\overline{13}$, $\overline{24}$ occur; they correspond to the $\overline{2}$ fixed point in the fundamental domain.

The A_1 subspace in C_{4v} cycle expansion is given by

$$
\begin{aligned}
1/\zeta_{A_1} &= (1-t_0)(1-t_1)(1-t_2)(1-t_{01})(1-t_{02})(1-t_{12}) \\
&\quad (1-t_{001})(1-t_{002})(1-t_{011})(1-t_{012})(1-t_{021})(1-t_{022})(1-t_{112}) \\
&\quad (1-t_{122})(1-t_{0001})(1-t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\
&= 1-t_0-t_1-t_2-(t_{01}-t_0t_1)-(t_{02}-t_0t_2)-(t_{12}-t_1t_2) \\
&\quad -(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})-(t_{011}-t_1t_{01}) \\
&\quad -(t_{022}-t_2t_{02})-(t_{112}-t_1t_{12})-(t_{122}-t_2t_{12}) \\
&\quad -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots
\end{aligned}
\qquad \text{(A2.66)}
$$

(for typographical convenience, $\underline{1}$ is replaced by 0 in the remainder of this section). For 1-dimensional representations, the characters can be read off the symbol strings: $\chi_{A_2}(\mathbf{h}_{\tilde{\mathbf{p}}}) = (-1)^{n_0}$, $\chi_{B_1}(\mathbf{h}_{\tilde{\mathbf{p}}}) = (-1)^{n_1}$, $\chi_{B_2}(\mathbf{h}_{\tilde{\mathbf{p}}}) = (-1)^{n_0+n_1}$, where n_0 and n_1 are the number of times symbols 0, 1 appear in the $\tilde{p}$ symbol string. For B_2 all

t_p with an odd total number of 0's and 1's change sign:

$$\begin{aligned} 1/\zeta_{B_2} &= (1+t_0)(1+t_1)(1-t_2)(1-t_{01})(1+t_{02})(1+t_{12}) \\ &\quad (1+t_{001})(1-t_{002})(1+t_{011})(1-t_{012})(1-t_{021})(1+t_{022})(1-t_{112}) \\ &\quad (1+t_{122})(1-t_{0001})(1+t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\ &= 1+t_0+t_1-t_2-(t_{01}-t_0t_1)+(t_{02}-t_0t_2)+(t_{12}-t_1t_2) \\ &\quad +(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})+(t_{011}-t_1t_{01}) \\ &\quad +(t_{022}-t_2t_{02})-(t_{112}-t_1t_{12})+(t_{122}-t_2t_{12}) \\ &\quad -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots \end{aligned} \tag{A2.67}$$

The form of the remaining cycle expansions depends crucially on the special role played by the boundary orbits: by (A2.65) the orbit t_2 does not contribute to A_2 and B_1,

$$\begin{aligned} 1/\zeta_{A_2} &= (1+t_0)(1-t_1)(1+t_{01})(1+t_{02})(1-t_{12}) \\ &\quad (1-t_{001})(1-t_{002})(1+t_{011})(1+t_{012})(1+t_{021})(1+t_{022})(1-t_{112}) \\ &\quad (1-t_{122})(1+t_{0001})(1+t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\ &= 1+t_0-t_1+(t_{01}-t_0t_1)+t_{02}-t_{12} \\ &\quad -(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})+(t_{011}-t_1t_{01}) \\ &\quad +t_{022}-t_{122}-(t_{112}-t_1t_{12})+(t_{012}+t_{021}-t_0t_{12}-t_1t_{02})\ldots \end{aligned} \tag{A2.68}$$

and

$$\begin{aligned} 1/\zeta_{B_1} &= (1-t_0)(1+t_1)(1+t_{01})(1-t_{02})(1+t_{12}) \\ &\quad (1+t_{001})(1-t_{002})(1-t_{011})(1+t_{012})(1+t_{021})(1-t_{022})(1-t_{112}) \\ &\quad (1+t_{122})(1+t_{0001})(1-t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\ &= 1-t_0+t_1+(t_{01}-t_0t_1)-t_{02}+t_{12} \\ &\quad +(t_{001}-t_0t_{01})-(t_{002}-t_0t_{02})-(t_{011}-t_1t_{01}) \\ &\quad -t_{022}+t_{122}-(t_{112}-t_1t_{12})+(t_{012}+t_{021}-t_0t_{12}-t_1t_{02})\ldots \end{aligned} \tag{A2.69}$$

In the above we have assumed that t_2 does not change sign under C_{4v} reflections. For the mixed-symmetry subspace E the curvature expansion is given by

$$\begin{aligned} 1/\zeta_E &= 1+t_2+(-t_0{}^2+t_1{}^2)+(2t_{002}-t_2t_0{}^2-2t_{112}+t_2t_1{}^2) \\ &\quad +(2t_{0011}-2t_{0022}+2t_2t_{002}-t_{01}{}^2-t_{02}{}^2+2t_{1122}-2t_2t_{112} \\ &\quad +t_{12}{}^2-t_0{}^2t_1{}^2)+(2t_{00002}-2t_{00112}+2t_2t_{0011}-2t_{00121}-2t_{00211} \\ &\quad +2t_{00222}-2t_2t_{0022}+2t_{01012}+2t_{01021}-2t_{01102}-t_2t_{01}{}^2+2t_{02022} \\ &\quad -t_2t_{02}{}^2+2t_{11112}-2t_{11222}+2t_2t_{1122}-2t_{12122}+t_2t_{12}{}^2-t_2t_0{}^2t_1{}^2 \\ &\quad +2t_{002}(-t_0{}^2+t_1{}^2)-2t_{112}(-t_0{}^2+t_1{}^2)) \end{aligned} \tag{A2.70}$$

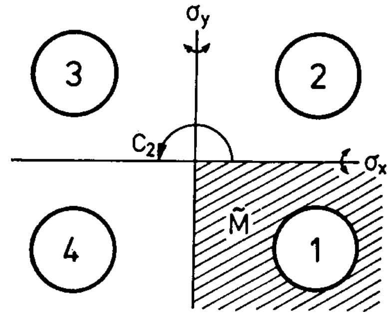


Figure A2.2: Symmetries of four disks on a rectangle. A fundamental domain indicated by the shaded wedge.

A quick test of the $\zeta = \zeta_{A_1}\zeta_{A_2}\zeta_{B_1}\zeta_{B_2}\zeta_E^2$ factorization is afforded by the topological polynomial; substituting $t_p = z^{n_p}$ into the expansion yields

$$1/\zeta_{A_1} = 1 - 3z\ , \quad 1/\zeta_{A_2} = 1/\zeta_{B_1} = 1\ , \quad 1/\zeta_{B_2} = 1/\zeta_E = 1 + z\ ,$$

in agreement with (18.46).

exercise 23.8

A2.7 C_{2v} factorization

An arrangement of four identical disks on the vertices of a rectangle has C_{2v} symmetry, see figure A2.2. C_{2v} consists of $\{e, \sigma_x, \sigma_y, C_2\}$, i.e., the reflections across the symmetry axes and a rotation by π.

This system affords a rather easy visualization of the conversion of a 4-disk dynamics into a fundamental domain symbolic dynamics. An orbit leaving the fundamental domain through one of the axis may be folded back by a reflection on that axis; with these symmetry operations $g_0 = \sigma_x$ and $g_1 = \sigma_y$ we associate labels 1 and 0, respectively. Orbits going to the diagonally opposed disk cross the boundaries of the fundamental domain twice; the product of these two reflections is just $C_2 = \sigma_x\sigma_y$, to which we assign the label 2. For example, a ternary string $0\,0\,1\,0\,2\,0\,1\ldots$ is converted into $12143123\ldots$, and the associated group-theory weight is given by $\ldots g_1 g_0 g_2 g_0 g_1 g_0 g_0$.

Short ternary cycles and the corresponding 4-disk cycles are listed in table A2.2. Note that already at length three there is a pair of cycles (012 = 143 and 021 = 142) related by time reversal, but *not* by any C_{2v} symmetries.

The above is the complete description of the symbolic dynamics for 4 sufficiently separated equal disks placed at corners of a rectangle. However, if the fundamental domain requires further partitioning, the ternary description is insufficient. For example, in the stadium billiard fundamental domain one has to distinguish between bounces off the straight and the curved sections of the billiard wall; in that case five symbols suffice for constructing the covering symbolic dynamics.

The group C_{2v} has four 1-dimensional representations, distinguished by their behavior under axis reflections. The A_1 representation is symmetric with respect

Table A2.2: C_{2v} correspondence between the ternary $\{0, 1, 2\}$ fundamental domain prime cycles $\tilde{p}$ and the full 4-disk {1,2,3,4} cycles p, together with the C_{2v} transformation that maps the end point of the $\tilde{p}$ cycle into an irreducible segment of the p cycle. The degeneracy of the p cycle is $m_p = 4n_{\tilde{p}}/n_p$. Note that the 012 and 021 cycles are related by time reversal, but cannot be mapped into each other by C_{2v} transformations. The full space orbit listed here is generated from the symmetry reduced code by the rules given in sect. A2.7, starting from disk 1.

$\tilde{p}$	p	**g**
0	1 4	σ_y
1	1 2	σ_x
2	1 3	C_2
01	14 32	C_2
02	14 23	σ_x
12	12 43	σ_y
001	141 232	σ_x
002	141 323	C_2
011	143 412	σ_y
012	143	e
021	142	e
022	142 413	σ_y
112	121 343	C_2
122	124 213	σ_x

$\tilde{p}$	p	**g**
0001	1414 3232	C_2
0002	1414 2323	σ_x
0011	1412	e
0012	1412 4143	σ_y
0021	1413 4142	σ_y
0022	1413	e
0102	1432 4123	σ_y
0111	1434 3212	C_2
0112	1434 2343	σ_x
0121	1431 2342	σ_x
0122	1431 3213	C_2
0211	1421 2312	σ_x
0212	1421 3243	C_2
0221	1424 3242	C_2
0222	1424 2313	σ_x
1112	1212 4343	σ_y
1122	1213	e
1222	1242 4313	σ_y

to both reflections; the A_2 representation is antisymmetric with respect to both. The B_1 and B_2 representations are symmetric under one and antisymmetric under the other reflection. The character table is

C_{2v}	A_1	A_2	B_1	B_2
e	1	1	1	1
C_2	1	1	-1	-1
σ_x	1	-1	1	-1
σ_y	1	-1	-1	1

Substituted into the factorized determinant (25.19), the contributions of periodic orbits split as follows

$$\begin{array}{lllllll}
g_{\tilde{p}} & & & A_1 & A_2 & B_1 & B_2 \\
e: & (1-t_{\tilde{p}})^4 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) \\
C_2: & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1-t_{\tilde{p}}) \\
\sigma_x: & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) \\
\sigma_y: & (1-t_{\tilde{p}}^2)^2 & = & (1-t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1+t_{\tilde{p}}) & (1-t_{\tilde{p}})
\end{array}$$

Cycle expansions follow by substituting cycles and their group theory factors from table A2.2. For A_1 all characters are +1, and the corresponding cycle expansion is given in (A2.66). Similarly, the totally antisymmetric subspace factorization A_2

is given by (A2.67), the B_2 factorization of C_{4v}. For B_1 all t_p with an odd total number of 0's and 2's change sign:

$$\begin{aligned} 1/\zeta_{B_1} &= (1+t_0)(1-t_1)(1+t_2)(1+t_{01})(1-t_{02})(1+t_{12}) \\ &\quad (1-t_{001})(1+t_{002})(1+t_{011})(1-t_{012})(1-t_{021})(1+t_{022})(1+t_{112}) \\ &\quad (1-t_{122})(1+t_{0001})(1-t_{0002})(1-t_{0011})(1+t_{0012})(1+t_{0021})\ldots \\ &= 1+t_0-t_1+t_2+(t_{01}-t_0t_1)-(t_{02}-t_0t_2)+(t_{12}-t_1t_2) \\ &\quad -(t_{001}-t_0t_{01})+(t_{002}-t_0t_{02})+(t_{011}-t_1t_{01}) \\ &\quad +(t_{022}-t_2t_{02})+(t_{112}-t_1t_{12})-(t_{122}-t_2t_{12}) \\ &\quad -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots \end{aligned} \tag{A2.71}$$

For B_2 all t_p with an odd total number of 1's and 2's change sign:

$$\begin{aligned} 1/\zeta_{B_2} &= (1-t_0)(1+t_1)(1+t_2)(1+t_{01})(1+t_{02})(1-t_{12}) \\ &\quad (1+t_{001})(1+t_{002})(1-t_{011})(1-t_{012})(1-t_{021})(1-t_{022})(1+t_{112}) \\ &\quad (1+t_{122})(1+t_{0001})(1+t_{0002})(1-t_{0011})(1-t_{0012})(1-t_{0021})\ldots \\ &= 1-t_0+t_1+t_2+(t_{01}-t_0t_1)+(t_{02}-t_0t_2)-(t_{12}-t_1t_2) \\ &\quad +(t_{001}-t_0t_{01})+(t_{002}-t_0t_{02})-(t_{011}-t_1t_{01}) \\ &\quad -(t_{022}-t_2t_{02})+(t_{112}-t_1t_{12})+(t_{122}-t_2t_{12}) \\ &\quad -(t_{012}+t_{021}+t_0t_1t_2-t_0t_{12}-t_1t_{02}-t_2t_{01})\ldots \end{aligned} \tag{A2.72}$$

Note that all of the above cycle expansions group long orbits together with their pseudo-orbit shadows, so that the shadowing arguments for convergence still apply.

The topological polynomial factorizes as

$$\frac{1}{\zeta_{A_1}} = 1-3z \quad , \quad \frac{1}{\zeta_{A_2}} = \frac{1}{\zeta_{B_1}} = \frac{1}{\zeta_{B_2}} = 1+z,$$

consistent with the 4-disk factorization (18.46).

A2.8 Hénon map symmetries

We note here a few simple symmetries of the Hénon map (3.17). For $b \neq 0$ the Hénon map is reversible: the backward iteration of (3.18) is given by

$$x_{n-1} = -\frac{1}{b}(1-ax_n^2-x_{n+1})\,. \tag{A2.73}$$

Hence the time reversal amounts to $b \to 1/b$, $a \to a/b^2$ symmetry in the parameter plane, together with $x \to -x/b$ in the coordinate plane, and there is no need to explore the (a, b) parameter plane outside the strip $b \in \{-1, 1\}$. For $b = -1$ the map is orientation and area preserving ,

$$x_{n-1} = 1 - a x_n^2 - x_{n+1} , \tag{A2.74}$$

the backward and the forward iteration are the same, and the non–wandering set is symmetric across the $x_{n+1} = x_n$ diagonal. This is one of the simplest models of a Poincaré return map for a Hamiltonian flow. For the orientation reversing $b = 1$ case we have

$$x_{n-1} = 1 - a x_n^2 + x_{n+1} , \tag{A2.75}$$

and the non–wandering set is symmetric across the $x_{n+1} = -x_n$ diagonal.

Commentary

Remark A2.1. C_{4v} labeling conventions While there is a variety of labeling conventions [2, 3] for the reduced C_{4v} dynamics, we prefer the one introduced here because of its close relation to the group-theoretic structure of the dynamics: the global 4-disk trajectory can be generated by mapping the fundamental domain trajectories onto the full 4-disk space by the accumulated product of the C_{4v} group elements.

Remark A2.2. C_{2v} symmetry C_{2v} is the symmetry of several systems studied in the literature, such as the stadium billiard [1], and the 2-dimensional anisotropic Kepler potential [4].

References

[1] L. A. Bunimovich, "On the ergodic properties of nowhere dispersing billiards", Commun. Math. Phys. **65**, 295–312 (1979).

[2] F. Christiansen, MA thesis (Univ. of Copenhagen, Copenhagen, 1989).

[3] B. Eckhardt and D. Wintgen, "Symbolic description of periodic orbits for the quadratic Zeeman effect", J. Phys. B **23**, 355–363 (1990).

[4] M. C. Gutzwiller, "The quantization of a classically ergodic system", Physica D **5**, 183–207 (1982).

[5] W. G. Harter, *Principles of Symmetry, Dynamics, and Spectroscopy* (Wiley, New York, 1993).

Exercises

A2.1. **Am I a group?** Show that multiplication table

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	b	f	c
b	b	d	e	f	c	a
c	c	b	f	e	a	d
d	d	f	c	a	e	b
f	f	c	a	d	b	e

describes a group. Or does it? (Hint: check whether this table satisfies the group axioms of appendix A2.1.)

From W.G. Harter [5]

A2.2. **Three coupled pendulums with a C_2 symmetry.** Consider 3 pendulums in a row: the 2 outer ones of the same mass m and length l, the one midway of same length but different mass M, with the tip coupled to the tips of the outer ones with springs of stiffness k. Assume displacements are small, $x_i/l \ll 1$.

(a) Show that the acceleration matrix $\ddot{\mathbf{x}} = -\mathbf{a}\,\mathbf{x}$ is

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = - \begin{bmatrix} a+b & -a & 0 \\ -c & 2c+b & -c \\ 0 & -a & a+b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

where $a = k/ml$, $c = k/Ml$ and $b = g/l$.

(b) Check that $[\mathbf{a}, \mathbf{R}] = 0$, i.e., that the dynamics is invariant under $C_2 = \{e, R\}$, where $\mathbf{R}$ interchanges the outer pendulums,

$$\mathbf{R} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(c) Construct the corresponding projection operators $\mathbf{P}_+$ and $\mathbf{P}_-$, and show that the 3-pendulum system decomposes into a 1-dimensional subspace, with eigenvalue $(\omega^{(-)})^2 = a + b$, and a 2-dimensional subspace, with acceleration matrix (trust your own algebra, if it strays from what is stated here)

$$\mathbf{a}^{(+)} = \begin{bmatrix} a+b & -\sqrt{2}a \\ -\sqrt{2}c & c+b \end{bmatrix}.$$

The exercise is simple enough that you can do it without using the symmetry, so: construct $\mathbf{P}_+, \mathbf{P}_-$ *first*, use them to reduce $\mathbf{a}$ to irreps, then proceed with computing remaining eigenvalues of $\mathbf{a}$.

(d) Does anything interesting happen if $M = m$?

The point of the above exercise is that almost always the symmetry reduction is only partial: a matrix representation of dimension d gets reduced to a set of subspaces whose dimensions $d^{(\alpha)}$ satisfy $\sum d^{(\alpha)} = d$. Beyond that, love many, trust few, and paddle your own canoe.

From W.G. Harter [5]

A2.3. **Lorenz system in polar coordinates: dynamics.** (continuation of exercise 11.4)

1. Show that (11.13) has two equilibria:

$$\begin{aligned} (r_0, z_0) &= (0, 0), \qquad \theta_0 \text{ undefined} \\ (r_1, \theta_1, z_1) &= (\sqrt{2b(\rho-1)}, \pi/4, \rho-1) \end{aligned} \qquad \text{(A2.176)}$$

2. Verify numerically that the eigenvalues and eigenvectors of the two equilibria are (we list here the precise numbers to help you check your programs):

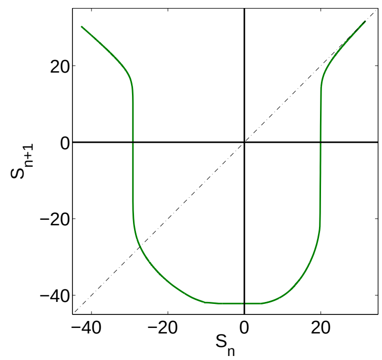
   EQ_1 = (0, 12, 27) **equilibrium:** (and its $C^{1/2}$-rotation EQ_2) has one stable real eigenvalue
   $\lambda^{(1)} = -13.854578$,
   and the unstable complex conjugate pair
   $\lambda^{(2,3)} = \mu^{(2)} \pm i\omega^{(2)} = 0.093956 \pm i10.194505$.
   The unstable eigenplane is defined by eigenvectors
   $\text{Re}\,\mathbf{e}^{(2)} = (-0.4955, -0.2010, -0.8450)$
   $\text{Im}\,\mathbf{e}^{(2)} = (0.5325, -0.8464, 0)$
   with period $T = 2\pi/\omega^{(2)} = 0.6163306$,
   radial expansion multiplier
   $\Lambda_r = \exp(2\pi\mu^{(2)}/\omega^{(2)}) = 1.059617$,
   and the contracting multiplier
   $\Lambda_c = \exp(2\pi\mu^{(1)}/\omega^{(2)}) \approx 1.95686 \times 10^{-4}$
   along the stable eigenvector of EQ_1,
   $\mathbf{e}^{(3)} = (0.8557, -0.3298, -0.3988)$.

   $EQ_0 = (0, 0, 0)$ **equilibrium:** The stable eigenvector $\mathbf{e}^{(1)} = (0, 0, 1)$ of EQ_0, has contraction rate
   $\lambda^{(2)} = -b = -2.666\ldots$.
   The other stable eigenvector is
   $\mathbf{e}^{(2)} = (-0.244001, -0.969775, 0)$, with contracting eigenvalue
   $\lambda^{(2)} = -22.8277$. The unstable eigenvector
   $\mathbf{e}^{(3)} = (-0.653049, 0.757316, 0)$ has eigenvalue
   $\lambda^{(3)} = 11.8277$.

3. Plot the Lorenz strange attractor both in the Lorenz coordinates figure 2.5, and in the doubled-polar angle coordinates (11.11) for the Lorenz parameter values $\sigma = 10$, $b = 8/3$, $\rho = 28$. Topologically, does it resemble the Lorenz butterfly, the

Rössler attractor, or neither? The Poincaré section of the Lorenz flow fixed by the z-axis and the equilibrium in the doubled polar angle representation, and the corresponding Poincaré return map $(s_n, s_n + 1)$ are plotted in figure 14.8.

4. Construct the Poincaré return map (s_n, s_{n+1}),



where s is arc-length measured along the unstable manifold of EQ_0, lower Poincaré section of figure 14.8 (b). Elucidate its relation to the Poincaré return map of figure 14.9. (plot by J. Halcrow)

5. Show that if a periodic orbit of the polar representation Lorenz is also periodic orbit of the Lorenz flow, their Floquet multipliers are the same. How do the Floquet multipliers of relative periodic orbits of the representations relate to each other?

6. What does the volume contraction formula (4.42) look like now? Interpret.

A2.4. **Laplacian is a non-local operator.**

While the Laplacian is a simple tri-diagonal difference operator (A2.42), its inverse (the "free" propagator of statistical mechanics and quantum field theory) is a messier object. A way to compute is to start expanding propagator as a power series in the Laplacian

$$\frac{1}{m^2\mathbf{1}-\Box} = \frac{1}{m^2}\sum_{n=0}^{\infty}\frac{1}{m^{2n}}\Box^n \,. \qquad \text{(A2.77)}$$

As $\Box$ is a finite matrix, the expansion is convergent for sufficiently large m^2. To get a feeling for what is involved in evaluating such series, show that $\Box^2$ is:

$$\Box^2 = \frac{1}{a^4}\begin{bmatrix} 6 & -4 & 1 & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & \ddots & & & \\ & & & & & 6 & -4 \\ -4 & 1 & & & 1 & -4 & 6 \end{bmatrix} . \qquad \text{(A2.78)}$$

What $\Box^3$, $\Box^4$, $\cdots$ contributions look like is now clear; as we include higher and higher powers of the Laplacian, the propagator matrix fills up; while the *inverse* propagator is differential operator connecting only the nearest neighbors, the propagator is integral operator, connecting every lattice site to any other lattice site.

This matrix can be evaluated as is, on the lattice, and sometime it is evaluated this way, but in case at hand a wonderful simplification follows from the observation that the lattice action is translationally invariant, exercise A2.5.

A2.5. **Lattice Laplacian diagonalized.** Insert the identity $\sum \mathbf{P}^{(k)} = \mathbf{1}$ wherever you profitably can, and use the eigenvalue equation (A2.55) to convert shift σ matrices into scalars. If $\mathbf{M}$ *commutes* with σ, then $(\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) = \tilde{M}^{(k)}\delta_{kk'}$, and the matrix $\mathbf{M}$ acts as a multiplication by the scalar $\tilde{M}^{(k)}$ on the kth subspace. Show that for the 1-dimensional version of the lattice Laplacian (A2.42) the projection on the kth subspace is

$$(\varphi_k^\dagger \cdot \Box \cdot \varphi_{k'}) = \frac{2}{a^2}\left(\cos\left(\frac{2\pi}{N}k\right)-1\right)\delta_{kk'} \,. \qquad \text{(A2.79)}$$

In the kth subspace the propagator is simply a number, and, in contrast to the mess generated by (A2.77), there is nothing to evaluating:

$$\varphi_k^\dagger \cdot \frac{1}{m^2\mathbf{1}-\Box} \cdot \varphi_{k'} = \frac{\delta_{kk'}}{m^2 - \frac{2}{(ma)^2}(\cos 2\pi k/N - 1)} \,, \qquad \text{(A2.80)}$$

where k is a site in the N-dimensional dual lattice, and $a = L/N$ is the lattice spacing.