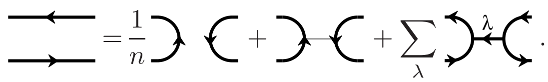
Consider the decomposition of $V \otimes \bar{V}$. The corresponding projection operators satisfy the completeness relation (4.20):

$$\mathbf{1} = \frac{1}{n} T + \mathbf{P}_A + \sum_{\lambda \neq A} \mathbf{P}_\lambda$$

$$\delta_d^a \delta_b^c = \frac{1}{n} \delta_b^a \delta_d^c + (\mathbf{P}_A)_b^a{}_,{}^c_d + \sum_{\lambda \neq A} (\mathbf{P}_\lambda)_b^a{}_,{}^c_d$$


(4.27)

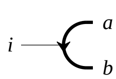
If δ_λ^μ is the only primitive invariant tensor, then $V \otimes \bar{V}$ decomposes into two subspaces, and there are no other irreducible reps. However, if there are further primitive invariant tensors, $V \otimes \bar{V}$ decomposes into more irreducible reps, indicated by the sum over λ. Examples will abound in what follows. The singlet projection operator T/n always figures in this expansion, as $\delta_b^a{}_,{}^c_d$ is always one of the invariant matrices (see the example worked out in section 2.2). Furthermore, the infinitesimal generators D_b^a must belong to at least one of the irreducible subspaces of $V \otimes \bar{V}$.

This subspace is called the *adjoint* space, and its special role warrants introduction of special notation. We shall refer to this vector space by letter A, in distinction to the defining space V of (3.10). We shall denote its dimension by N, label its tensor indices by $i, j, k \ldots$, denote the corresponding Kronecker delta by a thin, straight line,

$$\delta_{ij} = i \text{ ——— } j\,, \qquad i, j = 1, 2, \ldots, N\,, \tag{4.28}$$

and the corresponding clebsches by

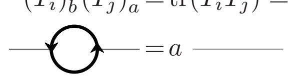
$$(C_A)_i{}_,{}^a_b = \frac{1}{\sqrt{a}} (T_i)_b^a = $$



$$a, b = 1, 2, \ldots, n \qquad i = 1, 2, \ldots, N\,.$$

Matrices T_i are called the *generators* of infinitesimal transformations. Here a is an (uninteresting) overall normalization fixed by the orthogonality condition (4.19):

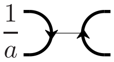
$$(T_i)_b^a (T_j)_a^b = \mathrm{tr}(T_i T_j) = a\, \delta_{ij}$$


(4.29)

The scale of T_i is not set, as any overall rescaling can be absorbed into the normalization a. For our purposes it will be most convenient to use $a = 1$ as the normalization convention. Other normalizations are commonplace. For example, $SU(2)$ Pauli matrices $T_i = \frac{1}{2}\sigma_i$ and $SU(n)$ Gell-Mann [137] matrices $T_i = \frac{1}{2}\lambda_i$ are conventionally normalized by fixing $a = 1/2$:

$$\mathrm{tr}(T_i T_j) = \frac{1}{2} \delta_{ij}\,. \tag{4.30}$$

The projector relation (4.18) expresses the adjoint rep projection operators in terms of the generators:

$$(\mathbf{P}_A)_b^a{}_{,}{}^c_d = \frac{1}{a}(T_i)_b^a (T_i)_d^c = \frac{1}{a}$$  . (4.31)

Clearly, the adjoint subspace is always included in the sum (4.27) (there must exist some allowed infinitesimal generators D_a^b, or otherwise there is no group to describe), but how do we determine the corresponding projection operator?

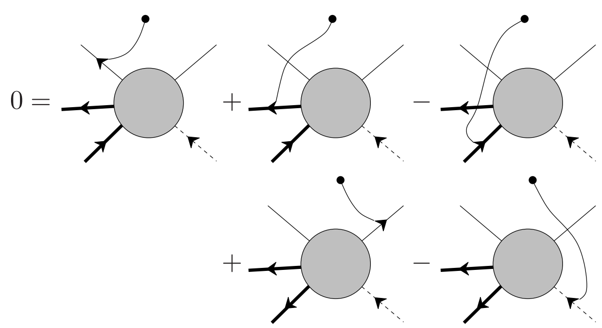
The adjoint projection operator is singled out by the requirement that the group transformations do not affect the invariant quantities. (Remember, the group is *defined* as the totality of all transformations that leave the invariants invariant.) For every invariant tensor q, the infinitesimal transformations $G = 1 + iD$ must satisfy the invariance condition (3.27). Parametrizing D as a projection of an arbitrary hermitian matrix $H \in V \otimes \bar{V}$ into the adjoint space, $D = \mathbf{P}_A H \in V \otimes \bar{V}$,

$$D_b^a = \frac{1}{a}(T_i)_b^a \epsilon_i \,, \qquad \epsilon_i = \frac{1}{a}\,\mathrm{tr}(T_i H)\,, \tag{4.32}$$

we obtain the *invariance condition*, which the *generators* must satisfy: they *annihilate* invariant tensors:

$$T_i q = 0\,. \tag{4.33}$$

The invariance conditions take a particularly suggestive form in the diagrammatic notation. Equation (4.33) amounts to the insertion of a generator into all external legs of the diagram corresponding to the invariant tensor q:

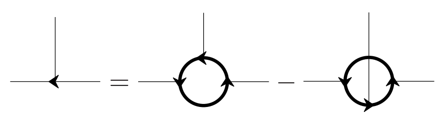


(4.36)

The insertions on the lines going into the diagram carry a minus sign relative to the insertions on the outgoing lines.

4.5 LIE ALGEBRA

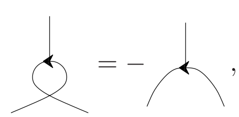
As the simplest example of computation of the generators of infinitesimal transformations acting on spaces other than the defining space, consider the adjoint rep. Using (4.40) on the $V \otimes \bar{V} \to A$ adjoint rep clebsches (*i.e.*, generators T_i), we obtain



(4.42)

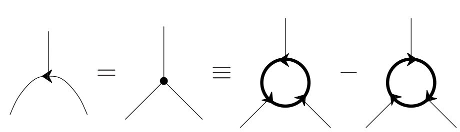
$$(T_i)_{jk} = (T_i)_a^c (T_k)_c^b (T_j)_b^a - (T_i)_a^c (T_j)_c^b (T_k)_b^a \, .$$

Our convention is always to assume that the generators T_i have been chosen hermitian. That means that ϵ_i in the expansion (4.32) is real; A is a real vector space, there is no distinction between upper and lower indices, and there is no need for arrows on the adjoint rep lines (4.28). However, the arrow on the adjoint rep generator (4.42) is necessary to define correctly the overall sign. If we interchange the two legs, the right-hand side changes sign:



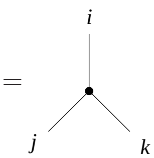
(4.43)

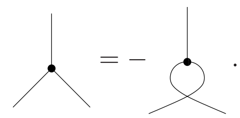
(the generators for real reps are always antisymmetric). This arrow has no absolute meaning; its direction is *defined* by (4.42). Actually, as the right-hand side of (4.42) is antisymmetric under interchange of any two legs, it is convenient to replace the arrow in the vertex by a more symmetric symbol, such as a dot:



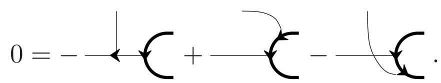
$$(T_i)_{jk} \equiv -iC_{ijk} = -\,\mathrm{tr}[T_i, T_j]T_k\,, \tag{4.44}$$

and replace the adjoint rep generators $(T_i)_{jk}$ by the fully antisymmetric structure constants iC_{ijk}. The factor i ensures their reality (in the case of hermitian generators T_i), and we keep track of the overall signs by always reading indices *counterclockwise* around a vertex:

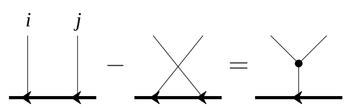
$-\,iC_{ijk} =$  (4.45)

 (4.46)

As all other clebsches, the generators must satisfy the invariance conditions (4.39):

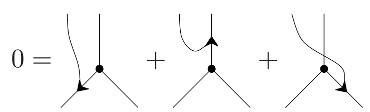


Redrawing this a little and replacing the adjoint rep generators (4.44) by the structure constants, we find that the generators obey the *Lie algebra* commutation relation

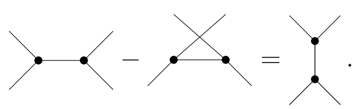


$$T_iT_j - T_jT_i = iC_{ijk}T_k\,. \tag{4.47}$$

In other words, the Lie algebra is simply a statement that T_i, the generators of invariance transformations, are themselves invariant tensors. The invariance condition for structure constants C_{ijk} is likewise



Rewriting this with the dot-vertex (4.44), we obtain

 (4.48)

This is the Lie algebra commutator for the adjoint rep generators, known as the *Jacobi relation* for the structure constants

$$C_{ijm}C_{mkl} - C_{ljm}C_{mki} = C_{iml}C_{jkm}\,. \tag{4.49}$$

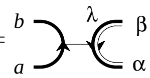
Hence, the Jacobi relation is also an invariance statement, this time the statement that the structure constants are invariant tensors.

Sign convention for C_{ijk}. A word of caution about using (4.47): vertex C_{ijk} is an oriented vertex. If the arrows are reversed (matrices T_i, T_j multiplied in reverse order), the right-hand side acquires an overall minus sign.

4.6 OTHER FORMS OF LIE ALGEBRA COMMUTATORS

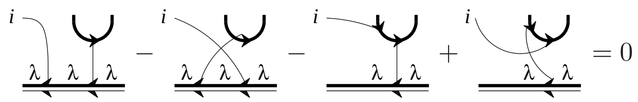
In our calculations we shall never need explicit generators; we shall instead use the projection operators for the adjoint rep. For rep λ they have the form

$$(\mathbf{P}_A)_b^a{}_{,\alpha}^{\beta} =$$



$$a, b = 1, 2, \ldots, n \qquad \alpha, \beta = 1, \ldots, d_\lambda\,. \tag{4.50}$$

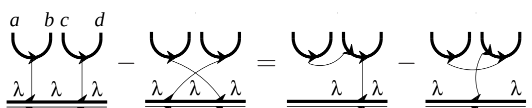
The invariance condition (4.36) for a projection operator is



(4.51)

Contracting with $(T_i)_b^a$ and defining $[d_\lambda \times d_\lambda]$ matrices $(T_b^a)_\alpha^\beta \equiv (\mathbf{P}_A)_b^a{}_{,\alpha}^{\beta}$, we obtain

$$[T_b^a, T_d^c] = (\mathbf{P}_A)_b^a{}_{,e}^{c} T_d^e - T_e^c (\mathbf{P}_A)_b^a{}_{,d}^{e}$$



(4.52)

This is a common way of stating the Lie algebra conditions for the generators in an arbitrary rep λ. For example, for $U(n)$ the adjoint projection operator is simply a unit matrix (any hermitian matrix is a generator of unitary transformation; *cf.* chapter 9), and the right-hand side of (4.52) is given by

$$U(n), SU(n): \qquad [T_b^a, T_d^c] = \delta_b^c T_d^a - T_b^c \delta_d^a\,. \tag{4.53}$$

For the orthogonal groups the generators of rotations are antisymmetric matrices, and the adjoint projection operator antisymmetrizes generator indices:

$$SO(n): \qquad [T_{ab}, T_{cd}] = \frac{1}{2} \left\{ \begin{array}{c} g_{ac}T_{bd} - g_{ad}T_{bc} \\ -g_{bc}T_{ad} + g_{bd}T_{ac} \end{array} \right\}. \tag{4.54}$$

Apart from the normalization convention, these are the familiar Lorentz group commutation relations.