Repeated index summation. Throughout this text, the repeated pairs of upper/lower indices are always summed over

$$G_a{}^b x_b \equiv \sum_{b=1}^{n} G_a{}^b x_b \,, \tag{3.1}$$

unless explicitly stated otherwise.

Let $GL(n, \mathbb{F})$ be the group of general linear transformations,

$$GL(n, \mathbb{F}) = \{G : \mathbb{F}^n \to \mathbb{F}^n \mid \det(G) \neq 0\} \,. \tag{3.2}$$

Under $GL(n, \mathbb{F})$ a basis set of V is mapped into another basis set by multiplication with a $[n \times n]$ matrix G with entries in $\mathbb{F}$,

$$\mathbf{e}'^{\,a} = \mathbf{e}^b (G^{-1})_b{}^a \,.$$

As the vector $\mathbf{x}$ is what it is, regardless of a particular choice of basis, under this transformation its coordinates must transform as

$$x'_a = G_a{}^b x_b \,.$$

Definition. We shall refer to the set of $[n \times n]$ matrices G as a *standard rep* of $GL(n, \mathbb{F})$, and the space of all n-tuples $(x_1, x_2, \ldots, x_n)^t$, $x_i \in \mathbb{F}$ on which these matrices act as the *standard representation space* V.

Under a general linear transformation $G \in GL(n, \mathbb{F})$, the row of basis vectors transforms by right multiplication as $\mathbf{e}' = \mathbf{e}\, G^{-1}$, and the column of x_a's transforms by left multiplication as $x' = Gx$. Under left multiplication the column (row transposed) of basis vectors $\mathbf{e}^t$ transforms as $\mathbf{e}'^t = G^\dagger \mathbf{e}^t$, where the *dual rep* $G^\dagger = (G^{-1})^t$ is the transpose of the inverse of G. This observation motivates introduction of a *dual* representation space $\bar{V}$, the space on which $GL(n, \mathbb{F})$ acts via the dual rep $G^\dagger$.

Definition. If V is a vector representation space, then the *dual space* $\bar{V}$ is the set of all linear forms on V over the field $\mathbb{F}$.

If $\{\mathbf{e}^1, \cdots, \mathbf{e}^n\}$ is a basis of V, then $\bar{V}$ is spanned by the *dual basis* $\{\mathbf{f}_1, \cdots, \mathbf{f}_n\}$, the set of n linear forms $\mathbf{f}_a$ such that

$$\mathbf{f}_a(\mathbf{e}^b) = \delta_a^b \,,$$

where δ_a^b is the Kronecker symbol, $\delta_a^b = 1$ if $a = b$, and zero otherwise. The components of dual representation space vectors will here be distinguished by upper indices

$$(y^1, y^2, \ldots, y^n) \,. \tag{3.3}$$

They transform under $GL(n, \mathbb{F})$ as

$$y'^a = (G^\dagger)_b{}^a y^b \,. \tag{3.4}$$

For $GL(n, \mathbb{F})$ no complex conjugation is implied by the † notation; that interpretation applies only to unitary subgroups of $GL(n, \mathbb{C})$. G can be distinguished from $G^\dagger$ by meticulously keeping track of the relative ordering of the indices,

$$G_a^b \to G_a{}^b \,, \qquad (G^\dagger)_a^b \to G^b{}_a \,. \tag{3.5}$$

If the tensorial object acts as a matrix, in general the order of indices must be indicated.

As a plethora of vector spaces, indices and dual spaces looms large in our immediate future, it pays to streamline the notation now, by singling out one vector space as "defining" and indicating the dual vector space by raised indices.

The next two sections introduce the three key notions in our construction of invariance groups: *defining rep*, section 3.2 (see also comments on page 23); *invariants*, section 3.4; and *primitiveness assumption*, page 21. Chapter 4 introduces diagrammatic notation, the computational tool essential to understanding all computations to come. As these concepts can be understood only in relation to one another, not singly, and an exposition of necessity progresses linearly, the reader is asked to be patient, in the hope that the questions that naturally arise upon first reading will be addressed in due course.

3.2 DEFINING SPACE, TENSORS, REPS

Definition. In what follows V will always denote the *defining* n-dimensional complex vector representation space, that is to say the initial, "elementary multiplet" space within which we commence our deliberations. Along with the defining vector representation space V comes the *dual* n-dimensional vector representation space $\bar{V}$. We shall denote the corresponding element of $\bar{V}$ by raising the index, as in (3.3), so the components of defining space vectors, resp. dual vectors, are distinguished by lower, resp. upper indices:

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_n)\,, \qquad \mathbf{x} \in V \\ \bar{x} &= (x^1, x^2, \ldots, x^n)\,, \qquad \bar{\mathbf{x}} \in \bar{V}\,. \end{aligned} \tag{3.10}$$

Definition. Let $\mathcal{G}$ be a group of transformations acting linearly on V, with the action of a group element $g \in \mathcal{G}$ on a vector $x \in V$ given by an $[n \times n]$ matrix G

$$x'_a = G_a{}^b x_b \qquad a, b = 1, 2, \ldots, n\,. \tag{3.11}$$

We shall refer to $G_a{}^b$ as the *defining rep* of the group $\mathcal{G}$. The action of $g \in \mathcal{G}$ on a vector $\bar{q} \in \bar{V}$ is given by the *dual rep* $[n \times n]$ matrix $G^\dagger$:

$$x'^a = x^b (G^\dagger)_b{}^a = G^a{}_b x^b\,. \tag{3.12}$$

In the applications considered here, the group $\mathcal{G}$ will almost always be assumed to be a subgroup of the *unitary group*, in which case $G^{-1} = G^\dagger$, and † indicates hermitian conjugation:

$$(G^\dagger)_a{}^b = (G_b{}^a)^* = G^b{}_a\,. \tag{3.13}$$

Definition. A *tensor* $x \in V^p \otimes \bar{V}^q$ transforms under the action of $g \in \mathcal{G}$ as

$$x'^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p} = G^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}{}^{,\, d_p \ldots d_1}_{\,c_q \ldots c_2 c_1}\, x^{c_1 c_2 \ldots c_q}_{d_1 \ldots d_p}\,, \tag{3.14}$$

where the $V^p \otimes \bar{V}^q$ *tensor rep* of $g \in \mathcal{G}$ is defined by the group acting on all indices of x.

$$G^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q}{}^{,\, d_q \ldots d_1}_{\,c_p \ldots c_2 c_1} \equiv G^{a_1}{}_{c_1} G^{a_2}{}_{c_2} \ldots G^{a_p}{}_{c_p} G_{b_q}{}^{d_q} \ldots G_{b_2}{}^{d_2 1} G_{b_1}{}^{d_1}\,. \tag{3.15}$$

Tensors can be combined into other tensors by
(a) *addition:*

$$z^{ab\ldots c}_{d\ldots e} = \alpha x^{ab\ldots c}_{d\ldots e} + \beta y^{ab\ldots c}_{d\ldots e}\,, \qquad \alpha, \beta \in \mathbb{C}\,, \tag{3.16}$$

(b) *product:*

$$z^{abcd}_{efg} = x^{abc}_e y^d_{fg}\,, \tag{3.17}$$

(c) *contraction:* Setting an upper and a lower index equal and summing over all of its values yields a tensor $z \in V^{p-1} \otimes \bar{V}^{q-1}$ without these indices:

$$z^{bc\ldots d}_{e\ldots f} = x^{abc\ldots d}_{e\ldots af}\,, \qquad z^{ad}_e = x^{abc}_e y^d_{cb}\,. \tag{3.18}$$

A tensor $x \in V^p \otimes \bar{V}^q$ transforms linearly under the action of g, so it can be considered a vector in the $d = n^{p+q}$-dimensional vector space $\tilde{V} = V^p \otimes \bar{V}^q$. We can replace the array of its indices by one collective index:

$$x_\alpha = x^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}\,. \tag{3.19}$$

One could be more explicit and give a table like

$$x_1 = x^{11\ldots1}_{1\ldots1} \;,\; x_2 = x^{21\ldots1}_{1\ldots1}, \ldots, \; x_d = x^{nn\ldots n}_{n\ldots n} \,, \tag{3.20}$$

but that is unnecessary, as we shall use the compact index notation only as a shorthand.

Definition. *Hermitian conjugation* is effected by complex conjugation and index transposition:

$$(h^\dagger)^{ab}_{cde} \equiv (h^{edc}_{ba})^* \,. \tag{3.21}$$

Complex conjugation interchanges upper and lower indices, as in (3.10); transposition reverses their order. A matrix is *hermitian* if its elements satisfy

$$(\mathbf{M}^\dagger)^a_b = M^a_b \,. \tag{3.22}$$

For a hermitian matrix there is no need to keep track of the relative ordering of indices, as $M_b{}^a = (\mathbf{M}^\dagger)_b{}^a = M^a{}_b$.

Definition. The tensor dual to x_α defined by (3.19) has form

$$x^\alpha = x^{b_p\ldots b_1}_{a_q\ldots a_2 a_1} \,. \tag{3.23}$$

Combined, the above definitions lead to the hermitian conjugation rule for collective indices: a collective index is raised or lowered by interchanging the upper and lower indices and reversing their order:

$$\alpha = \left\{ \begin{array}{c} a_1 a_2 \ldots a_q \\ b_1 \ldots b_p \end{array} \right\} \quad \leftrightarrow \quad \alpha = \left\{ \begin{array}{c} b_p \ldots b_1 \\ a_q \ldots a_2 a_1 \end{array} \right\} \,. \tag{3.24}$$

This transposition convention will be motivated further by the diagrammatic rules of section 4.1.

The tensor rep (3.15) can be treated as a $[d \times d]$ matrix

$$G_\alpha{}^\beta = G^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}{}^{,\, d_p \ldots d_1}_{,\, c_q \ldots c_2 c_1} \,, \tag{3.25}$$

and the tensor transformation (3.14) takes the usual matrix form

$$x'_\alpha = G_\alpha{}^\beta x_\beta \,. \tag{3.26}$$

Tensor: ordering of upper only, lower only indices matters.
Matrix acting on a tensor: must separate "in", "out" indices (or groups of inidices).

If a bilinear form $m(\bar{x}, y) = x^a M_a{}^b y_b$ is invariant for all $g \in \mathcal{G}$, the matrix

$$M_a{}^b = G_a{}^c G^b{}_d M_c{}^d \tag{3.30}$$

is an *invariant matrix*. Multiplying with $G_b{}^e$ and using the unitary condition (3.13), we find that the invariant matrices *commute* with all transformations $g \in \mathcal{G}$:

$$[G, \mathbf{M}] = 0 \,. \tag{3.31}$$

If we wish to treat a tensor with equal number of upper and lower indices as a matrix $\mathbf{M} : V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$,

$$M_\alpha{}^\beta = M^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p}{}_{,}{}^{d_p \ldots d_1}_{c_q \ldots c_2 c_1} \,, \tag{3.32}$$

then the invariance condition (3.29) will take the commutator form (3.31). Our convention of separating the two sets of indices by a comma, and reversing the order of the indices to the right of the comma, is motivated by the diagrammatic notation introduced below (see (4.6)).